Math 240 – Linear Algebra/Matrices Unit

This unit has been developed for a freshman level engineering mathematics course and is intended as an introduction to matrices and some of their applications. Emphasis is placed on the introduction to some topics related to linear algebra (Gaussian elimination, binary operations, inverse, the determinant and Cramer's Rule). As such the rigor usually associated with a course in linear algebra has been replaced with a more “intuitive” approach. The aim is to expose the learner to these concepts. They are encouraged to continue their study of linear algebra in order to gain a deeper understanding of what is actually happening “under the hood.”

This compilation is a set of lecture notes and is not intended to be a text book replacement. No apologies are given for miss-spellings nor any offered for colloquial symbolism. The material is meant to be covered in three days, but probably would take four to do justice to the newness of the concepts involved. Webwork (Louisiana Tech's online homework system) has not, as of the writing of this text, been consulted. Alterations to examples and order of concept introduction may be warranted after that work is completed.

Math 240 Linear Algebra Section

Day One: Gauss Elimination:

We remember from previous algebra studies that we can use the "Addition Method" to solve a system of two equations and two unknowns as follows:

$$\begin{aligned} x-y&=1 \\ 2x+3y&=7 \end{aligned}$$

Multiply the top row by (-2) and add the rows together:

$$\begin{array}{r} -2(\quad x-y=1) \\ \underline{\quad 2x+3y=7} \\ 0x+5y=5 \end{array}$$

Divide the result by 5 giving: $y=1$.

Now by substituting back into the top equation, we have:

$$\begin{aligned} x-(1)&=1 \\ x&=2 \end{aligned}$$

Now, we could develop a short hand language to describe the process. For instance:

let $A=\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ which is a table (or matrix) of the coefficients for the variable terms

and $b=\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ which represents the constant terms. Since the variable terms are aligned in our equation, we can write a new system by combining, or augmenting the coefficient matrix with the constant matrix like this:

$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & 7 \end{array}\right] \end{array}$ multiplying the top row by (-2) and adding it to the second row gives:

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 5 & 5 \end{array}\right]$ and dividing the bottom equation by 5 gives

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right]$ which tells us that y = 1. By substituting this into the top equation, we have

$$\begin{aligned} x-(1)&=1 \\ x&=2 \end{aligned}$$

This method, called **Gaussian Elimination** is very helpful for larger systems. For example:

$\begin{aligned} x-2y-10z&=-6 \\ 2x-y+4z&=-3 \\ 3x\qquad +4z&=7 \end{aligned}$ in augmented matrix form we have:

$$\left[\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ 2 & -1 & 4 & -3 \\ 3 & 0 & 4 & 7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ 0 & 3 & 24 & 9 \\ 0 & 6 & 34 & 25 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ 0 & 3 & 24 & 9 \\ 0 & 0 & -14 & 7 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ 0 & 3 & 24 & 9 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \rightarrow \text{ z = -1/2 } \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \rightarrow \begin{gathered} y+8(-1/2)=3 \\ y-4=3 \\ y=7 \end{gathered}$$

and finally, $$\begin{gathered} x-2(7)-10(-1/2)=-6 \\ x-14+5=-6 \\ x=3 \end{gathered}$$

Therefore, x = 3; y = 7 and z = -1/2

Our basic approach is:

1. Get the top left term to be one by multiplying the top row by the reciprocal of the current top corner term.
2. Get zeroes for every term beneath this “one” by multiplying the top row by the opposite of the non-zero term and adding the rows.
3. Repeat “down diagonal” until you reach the bottom right term.

Example:

$$\begin{gathered} 3x+2y=6 \\ 2x-3y=-9 \end{gathered}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 6 \\ 2 & -3 & -9 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2/3 & 2 \\ 2 & -3 & -9 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2/3 & 2 \\ 0 & -13/3 & -13 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2/3 & 2 \\ 0 & 1 & 3 \end{array}\right] \rightarrow \text{y = 3}$$

$$\begin{gathered} x+(2/3)(3)=2 \\ x+2=2 \\ x=0 \end{gathered}$$

Work the following examples:

a) $$\begin{gathered} 4x-y-3z=1 \\ 8x+y-z=5 \\ 2x+y+2z=5 \end{gathered}$$

b) $$\begin{gathered} -2x+6y=4 \\ 3x-9y=-6 \end{gathered}$$ (special case)

c) $$\begin{gathered} x-y-z=10 \\ 3x+4y+2z=0 \\ 2x+3y+3z=-10 \end{gathered}$$ (do Gauss-Jordan to show)

d) $$\begin{gathered} 3x+2y+2z=3 \\ x+2y-z=5 \\ 2x-4y+z=0 \end{gathered}$$ (show row swap to easily get a one in $a_{1,1}$ position)

Day Two: Basic Matrix Operations

Last time we introduced the idea of a matrix and one use (Gaussian Elimination). In general, we say that

$$\text{matrix A} = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & & \\ \vdots & \cdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

A is called an (m x n) or m-row by n-column matrix. Two special shapes of matrices are:

1. Square where m = n
2. Vector where m or n = 1 (ie just a row or a column)

Intuitively, a matrix an be thought of as a short-hand holding place for “numbers”. (nothing special about numbers, but for us that is what we will put in the matrices). As a matrix is a representation of “numbers” we can perform some operations on them. We will look at the basic four (addition, subtraction, multiplication and division).

1. Addition/subtraction. (as subtraction is just addition of the opposite, we will consider them together)
   IF matrices A and B have equal dimension,
   THEN $A \pm B = \left[a_{ij} \pm b_{ij}\right]$

   EXAMPLE:

   $$A = \begin{bmatrix} -5 & 0 \\ 4 & 1/2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$$

   $$A + B = \begin{bmatrix} -5+6 & 0-3 \\ 4+2 & 1/2+3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 6 & 7/2 \end{bmatrix}$$

   do $A - B$ and $B - A$

2. Multiplication/division. (Division does not exist... BUT we will have a work around later)
   Two types of multiplication:
   1. Scaler (multiplying or “scaling” the matrix by a single number)
      $kA = \left[ka_{ij}\right]$

      EXAMPLE:

      $$2\text{A} = 2\begin{bmatrix} -5 & 0 \\ 4 & 1/2 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 8 & 2 \end{bmatrix}$$

      do $(-1/2)B$ and $3\text{A} \pm 2\text{B}$

   2. Matrix Multiplication If A is a (m x p) matrix and B is a (p x n) matrix, then AB exists and will result in a (m x n) matrix. We define AB=C in a (2 x 2) case as:

      $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11})+(a_{12}b_{21}) & (a_{11}b_{12})+(a_{12}b_{22}) \\ (a_{21}b_{11})+(a_{22}b_{12}) & (a_{21}b_{12})+(a_{22}b_{22}) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

EXAMPLE:

$$A=\begin{bmatrix}-5 & 0\\ 4 & 1/2\end{bmatrix} \text{ and } B=\begin{bmatrix}6 & -3\\ 2 & 3\end{bmatrix}$$

$$AB=\begin{bmatrix}-5 & 0\\ 4 & 1/2\end{bmatrix}\begin{bmatrix}6 & -3\\ 2 & 3\end{bmatrix}=\begin{bmatrix}(-30+0) & (15+0)\\ (24+1) & (-12+3/2)\end{bmatrix}=\begin{bmatrix}-30 & 15\\ 25 & -21/2\end{bmatrix}$$

do BA

EXAMPLES:

$$A=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}; \quad B=\begin{bmatrix}1 & 6\\ 3 & -5\\ -2 & 4\end{bmatrix}; \quad C=\begin{bmatrix}4 & -6\\ 1 & 2\end{bmatrix}$$

a) AB b) BA c) BC d) AC

NOTE two matrices are equal iff all terms are equal. This enables us to solve situations like:

$$\begin{bmatrix}3 & 2x\\ y & -8\end{bmatrix}=\begin{bmatrix}3 & -2\\ 1 & -8\end{bmatrix}$$

$$2x=-2 \qquad y=1$$
$$x=-1$$

EXAMPLE:

$$\begin{bmatrix}x-1 & 4\\ y+3 & -7\end{bmatrix}=\begin{bmatrix}0 & 4\\ -2 & -7\end{bmatrix}$$

NOW ON TO DIVISION:

If we say $3x=4$ we can solve for x by “dividing” both sides of the equation by 3. But we are actually doing something much more subtle. We are actually multiplying by the inverse of three ($3^{(-1)}$)

$$3x=4$$
$$(3^{-1})3x=(3^{-1})4$$
$$(1)x=(3^{-1})4$$
$$x=\frac{4}{3}$$

NOTE: For matrices, we do not have a “1” BUT we do have something that is functionally a “1” In the real number system, “1” is the number that if we multiply it with any other number, we get the other number back. (1*3 = 3 for example). In the matrix world, the special square matrix “I” functions in this way. In a (3 x 3) matrix, we can see the pattern for I

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ Note the “ones” down the diagonal and zeroes everywhere else (or “off-diagonal”)

Therefore, to solve the system Au=b (where A, u and b are appropriately dimensioned matrices/vectors) we have:

$$\begin{aligned} Au &= b \\ A^{-1}Au &= A^{-1}b \\ Iu &= A^{-1}b \\ u &= A^{-1}b \end{aligned}$$

This means that to solve a system of linear equations, all we need to do is find the inverse, $\left(A^{-1}\right)$ of the matrix formed from the coefficients to our unknowns in our system. We will get to that shortly. But first:

<u>Note:</u> if $A^{-1}A=I$ we can check if two matrices are inverses of each other by multiplying them together. If we get the identity matrix for their dimensions, then we can claim that they are inverses of each other. Note also that we are living squarely in the realm of square matrices here!!

EXAMPLE:

$$A=\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}; \quad B=\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

$$AB=\begin{bmatrix} (-8+9) & (6-6) \\ (-12+12) & (9-8) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$$

EXAMPLE: Are the following matrices inverses of each other?

a) $A=\begin{bmatrix} 1 & -3 \\ -1 & 7 \end{bmatrix}; \quad B=\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ (yes)

b) $A=\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}: \quad B=\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$ (no)

So, how DO we get the inverse? Gauss-Jordan to the rescue!!

We will augment our original matrix with the matching identity matrix and perform Gauss-Jordan elimination to change the left hand side matrix into the identity matrix. The resulting right hand side matrix will be our inverse. We can always check our work by multiplying the two matrices together to be sure we get the identity matrix as our product.

EXAMPLE:

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & -1/3 & -5/3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & 5 & -3 \end{array}\right] \rightarrow$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{array}\right]$$

So, our inverse is, as we saw earlier, $A^{-1}=\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}=B$

One application of the inverse is solving systems of linear equations. For example:

$$3x+2y=4$$
$$5x+3y=7$$

In this case, $A=\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix};\quad u=\begin{bmatrix}x\\y\end{bmatrix};\quad b=\begin{bmatrix}4\\7\end{bmatrix}$

so, Au = b becomes $u=A^{-1}b$ as $A^{-1}=\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$, $u=\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}\begin{bmatrix}4\\7\end{bmatrix}=\begin{bmatrix}(-12+14)\\(20-21)\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

thus as $u=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$ then x = 2 and y = -1

EXAMPLES:

Find the inverse of the following matrices:

a) $\begin{bmatrix}2 & 3 & 2\\3 & 3 & 4\\-1 & -1 & -1\end{bmatrix}$ b) $\begin{bmatrix}1 & 2 & -1\\-2 & 0 & 1\\1 & -1 & 0\end{bmatrix}$

Solve using the inverse:

$$11x+3y=-4$$
$$7x+2y=5$$

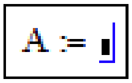
Matrix Operations Using Mathcad (Supplemental)

These notes describe how to use Mathcad to perform matrix operations. As an example you'll be able to solve a series of simultaneous linear equations using Mathcad's capabilities.

Create Matrices

1. Open Mathcad. Move the cross shaped cursor a little to the right and below its initial position. You will begin by entering a definition of the matrix "A". Press the "A" and ":" keys. Your screen should look like figure 1. The black box represents information they you must enter. In this case you will define a matrix.

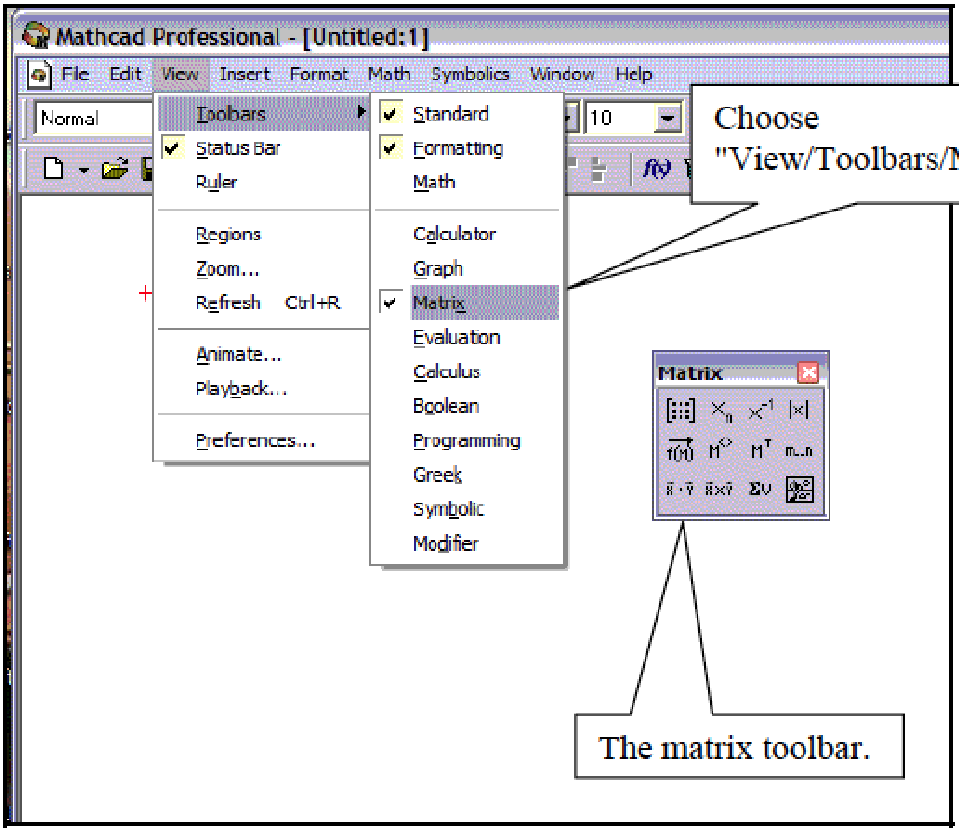
Figure 1
Defining Matrix A



A := ▮

2. Open the matrix toolbar by choosing "View/Toolbars/Matrix" from the menu bar. The menu in the matrix toolbar are shown in Figure 2.

Figure 2
The Matrix Toolbar

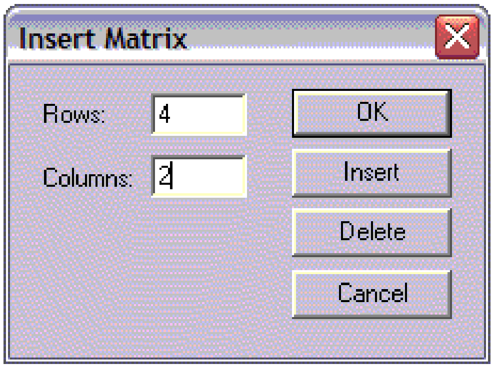


Choose "View/Toolbars/M

The matrix toolbar.

2. Choose the button that looks like a matrix ([:::]) from the matrix toolbar. The "Insert Matrix" dialog box should appear. Enter "4" for the number of rows and "2" for the number of columns. The dialog box should she appear as shown in figure 2.

Figure 2
The Insert Matrix Dialog Box

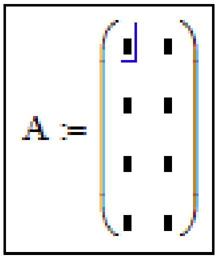


Rows: 4

Columns: 2

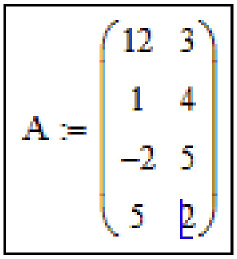
3. Click "OK". The dialog box will disappear and the cross cursor will be replaced on an empty 4 x 2 matrix. The black box placeholder shows the location of each and element in the matrix. Your screen should appear like Figure 3A. The blue inverted L cursor will appear around the upper left element. As you type numbers they will appear in the element surrounded by the inverted L cursor. Use the arrow buttons to move the inverted L cursor from one element to another. Enter all the elements so that your matrix looks like Figure 3B.

Figure 3A
The Empty "A" Matrix



$$A := \begin{pmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{pmatrix}$$

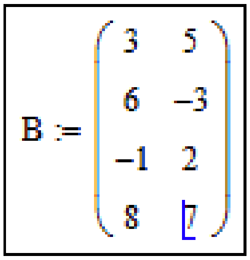
Figure 3B
The Completed Matrix



$$A := \begin{pmatrix} 12 & 3 \\ 1 & 4 \\ -2 & 5 \\ 5 & 2 \end{pmatrix}$$

4. Move the mouse cursor to a place just below the matrix you have created and click the left mouse button. The red cross cursor will appear. Create a second 4x2 matrix called "B" and defined as shown in figure 4.

Figure 4
The "B" Matrix

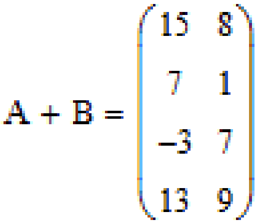


$$B := \begin{pmatrix} 3 & 5 \\ 6 & -3 \\ -1 & 2 \\ 8 & 7 \end{pmatrix}$$

Simple Matrix Operations

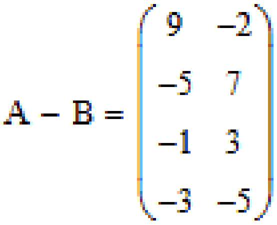
5. Matrix addition and subtraction can be performed by using the “+” and “-“ operators. As an example, add the vector “A” to the vector “B”. Move the mouse cursor to somewhere below the definition of vector “B” and click the left mouse button. Type “A+B=” (use the equals key to create the “=” symbol). Immediately after you press the equals key, Mathcad performs the operation and displays the result to the right of the =. The result is shown in figure 5A. Move your cursor below this result and type A-B=”. Again the result is shown (figure 5B). Multiplication of a matrix by a scalar is done in the same way. Type the keys “2*A=” and the result should appear like figure 5C.

Figure 5A
A+B



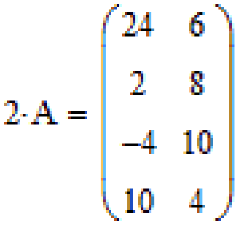
$$A + B = \begin{pmatrix} 15 & 8 \\ 7 & 1 \\ -3 & 7 \\ 13 & 9 \end{pmatrix}$$

Figure 5B
A-B



$$A - B = \begin{pmatrix} 9 & -2 \\ -5 & 7 \\ -1 & 3 \\ -3 & -5 \end{pmatrix}$$

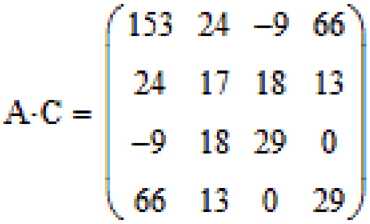
Figure 5C
2A



$$2 \cdot A = \begin{pmatrix} 24 & 6 \\ 2 & 8 \\ -4 & 10 \\ 10 & 4 \end{pmatrix}$$

6. Matrix multiplication can be performed using the ordinary multiplication symbol. Move the mouse cursor to a space below the definition of some matrix C and click the left mouse button. Press the keys “A”, “*”, “C” and, “=”. After the equals key is pressed the product of A and C will appear to the right of the =. If you defined matrix A with the values given earlier, your answer should be the same as shown in figure 6

Figure 6
An Example Of Matrix Multiplication

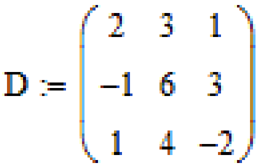


$$A \cdot C = \begin{pmatrix} 153 & 24 & -9 & 66 \\ 24 & 17 & 18 & 13 \\ -9 & 18 & 29 & 0 \\ 66 & 13 & 0 & 29 \end{pmatrix}$$

Inverse of a matrix

7. You will now find the inverse of a square matrix. First define a square 3x3 matrix D using the approach you used in step 1 and 2 except that you will enter “3” for the number of rows and columns in the Insert Matrix Dialog Box. Create a matrix with the values shown in Figure 7.

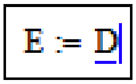
Figure 7
The Matrix D



$$D := \begin{pmatrix} 2 & 3 & 1 \\ -1 & 6 & 3 \\ 1 & 4 & -2 \end{pmatrix}$$

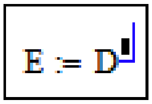
8. Define a second matrix, E, that is the inverse of matrix D. Begin by pressing the “E”, “=” and “D” keys. The inverted L cursor appears around the D as shown in Figure 8A. You can use the “^” symbol to create an exponent (remember that X^Y represents XY in Excel, Basic and many other software applications). After you press the “^” key, a superscript block is creates as shown in Figure 8B. Press the “-“ and “1” keys in that order. Your formula should look like Figure 8C

Figure 8A



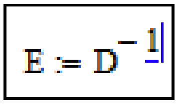
$E := D$

Figure 8B



$E := D^{\blacksquare}$

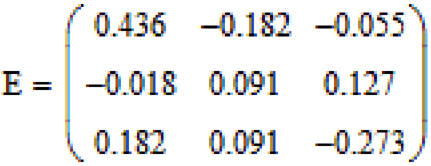
Figure 8C



$E := D^{-1}$

9. Display the contents of matrix E by typing "E" and "=" somewhere below the definition of E. If you entered the values shown in Figure 7, your matrix should look like figure 9.

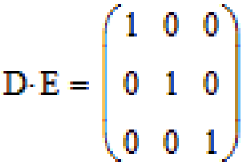
Figure 9
Matrix E Is The Inverse Of D



$$E = \begin{pmatrix} 0.436 & -0.182 & -0.055 \\ -0.018 & 0.091 & 0.127 \\ 0.182 & 0.091 & -0.273 \end{pmatrix}$$

10. You can prove that E is the inverse of D by multiplying D and E. Move the cursor below your work on the Mathcad worksheet. Press the "D". "*" , "E" and "=" keys. As soon as you press the "=" key, the result should appear as shown in Figure 10.

Figure 10
The Product Of D Times E Is The Identity Matrix



$$D \cdot E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

11. Now you can use these procedures to set up a solution to the system Au = b by entering $u := A^{-1} b$ after defining the coefficient matrix A and constant vector b. Finally simply type $u =$ and Mathcad will return the answer in vector form!

Day Three: Determinant and Cramer's Rule

Today we look at one more aspect to matrices: The Determinant.

Of a (2x2) matrix, the determinant is:

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } detA=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=(a_{11}a_{22})-(a_{21}a_{12})$$

For example:

$$\text{if } A=\begin{bmatrix} \sqrt{2} & -3 \\ -4 & -\sqrt{2} \end{bmatrix}, \quad detA=\begin{vmatrix} \sqrt{2} & -3 \\ -4 & -\sqrt{2} \end{vmatrix}=((\sqrt{2})(-\sqrt{2}))-((-3)(-4))=(-2)-(12)=-14$$

Find the determinant of the following matrices:

a) $\begin{bmatrix} -8 & 6 \\ -1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

We use the determinant of a matrix in many applications. One in particular is called "Cramer's Rule" It states:

In a (2x2) system, if

$$\begin{array}{l} ax+by=k_1 \\ cx+dy=k_2 \end{array} \quad \text{then} \quad \begin{array}{l} x=\dfrac{D_x}{D} \\ y=\dfrac{D_y}{D} \end{array}$$; D being the determinant and not equal to zero.

Also note the new terms D_x and D_y These are obtained by first replacing the proper column in our matrix with the constant column vector and then determining the determinant. In other words:

$$D_x=\begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix} \text{ and } D_y=\begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix}$$

EXAMPLE:

Use Cramer's Rule to solve:

$$\begin{array}{l} 2x+5y=7 \\ 5x-2y=-3 \end{array}$$

$$D=\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}=-29$$

$$D_x=\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}=1$$

$$D_y=\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}=-41$$

Therefore, $x=\dfrac{D_x}{D}=\dfrac{1}{-29}$ and $y=\dfrac{D_y}{D}=\dfrac{41}{29}$

What do we do with larger systems? Well, for a (3 x 3) system, we would have:

$$x=\frac{D_x}{D};\ \text{and}\ y=\frac{D_y}{D};\ \text{and}\ z=\frac{D_z}{D}$$

The trick is, to get the proper determinants. Before we get to that, we must introduce some definitions.

<u>DEF</u>: For a square matrix $A=\left[a_{ij}\right]$, the <u>minor</u> M_{ij} of an entry a_{ij} is the determinant formed by deleting the i^{th} row and the j^{th} column of the matrix A. For example:

$$A=\begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}, \text{ find } M_{11}$$

The determinant matrix formed by deleting the first row and first column is:

$$M_{11}=\begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix}=((-6)(5))-((-3)(7))=(-30)-(-21)=-9$$

find M_{23}

<u>DEF</u>: For a square matrix $A=\left[a_{ij}\right]$ the <u>cofactor</u> A_{ij} of an entry a_{ij} is given by $A_{ij}=(-1)^{i+j}M_{ij}$

This effectively alternates the sign depending on the column/row we are looking at.

Therefore, to find the determinant of a square matrix we choose any row or column. Multiply each element in that row or column by its cofactor and add the results.

EXAMPLE: Find the determinant of the following matrix:

$$A=\begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

Using the first row, we get: $(1)(-8)(-30+21)-(0)(20+7)+(6)(-12-6)=-36$

Find the determinant of $A=\begin{bmatrix} 3 & 2 & 2 \\ 5 & -1 & -6 \\ 2 & 3 & 3 \end{bmatrix}$ (answer = 25)

Solve by Cramer's Rule:

a) $\begin{array}{l} 3x+2y+2z=1 \\ 5x-y-6z=3 \\ 2x+3y+3z=4 \end{array}$ (answer = (-1,4,-2))

b) $\begin{array}{l} x-2y-3z=4 \\ x \qquad -2z=8 \\ 2x+y+4z=13 \end{array}$

EXTRA PROBLEMS:

Solve:

a) $\begin{vmatrix} x & 5 \\ -4 & x \end{vmatrix}=24$ (±2)

b) $\begin{vmatrix} y & -5 \\ -2 & y \end{vmatrix}<0$ (answer = $(-\sqrt{10},\sqrt{10})$)