A Hybrid FE-FD Scheme for Solving Parabolic Two-Step Micro Heat Transport Equations in an Irregularly Shaped Three Dimensional Double-Layered Thin Film

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Abstract

Heat transport at the microscale is important in microtechnology applications. The heat transport equations are parabolic two-step equations, which differ from the traditional heat diffusion equation. In this study, a hybrid finite element - finite difference (FE-FD) method for solving the parabolic two-step heat transport equations in a three dimensional irregular geometry double-layered thin film exposed to ultrashort-pulsed lasers is developed. It is shown that the scheme is unconditionally stable with respect to the heat source. The method is illustrated by three numerical examples in which the temperature rise in a gold layer on a chromium padding layer is investigated.

NOMENCLATURE

$C_{(e,l)}$	volumetric heat capacity of electron gas (e) and metal lattice (l)
C_p	volumetric heat capacity
G	electron-phonon coupling factor
J	laser influence
κ	thermal conductivity
$\mathbf{K}, \mathbf{M}$	conductance and capacitance matrices, respectively
N	number of elemental nodal points
$\vec{n}$	unit outward normal vector
Q	heat flux
R	reflectivity
r_z	mesh ratio, $\frac{\Delta t}{\Delta z^2}$
S	source term
T, T_0	temperature function
t	time
t_p	laser pulse duration
w	weight function
δ	laser penetration depth
δ_z^2	second order finite difference operator
∇, ∇^2	gradient and Laplace operators, respectively
$\Delta t, \Delta z$	time increment and grid size, respectively
α_p	generalized coordinates
ϕ_p	eigenvector
γ	specific heat coefficient
φ	basis and weighting function designation
λ_p	eigenvalue

1 INTRODUCTION

Multi-layer thin films are important components in many micro-electronic devices. These films are often used when a single film layer is insufficient to meet device specifications. The continued reduction in component size has the side effect of increasing the thermal stress on these films and consequently the devices they comprise. Thus the transportation of thermal energy through thin films is of vital importance in micro-technological applications.

Understanding the transfer of heat-energy at the micro-scale is important for thermal processing using a pulse-laser [1]. Often, micro-voids may be found in processed devices. This is due to thermal expansion. Such defects may cause an amplification of neighboring defects resulting in severe damage and consequently the failure of the desired device. Thus a complete understanding of thermal dissipation and defects is necessary to avoid damage and to increase the efficiency of thermal processing.

Micro-scale heat transfer differs from macro-scale heat transfer in some important ways. On the micro-scale, energy transport is governed by lattice-electron interaction in metallic films. Macroscopic energy transport relies upon a heat diffusion model based on Fourier's law. This loses accuracy on a micro-scale because of its emphasis on averaged behavior over many grains. Research has resulted in an energy equation which captures both the classical heat equation and thermal waves in the same framework [1, 2]. The energy equations describing the continuous energy flow from hot electrons to lattices can be expressed as a parabolic two-step model [3, 4]

$$C_e \frac{\partial T_e}{\partial t} = \kappa \nabla^2 T_e - G(T_e - T_l) + S \tag{1}$$

$$C_l \frac{\partial T_l}{\partial t} = G(T_e - T_l) \tag{2}$$

where T_e is electron temperature and T_l is lattice temperature. Also, κ is the thermal conductivity, C_e and C_l are the volumetric electron heat capacity and the volumetric lattice heat capacity respectively. G is the electron-lattice coupling factor and S is the strength of the laser heating source. The standard notation ∇^2 is the Laplace operator.

In classical (macro scale) heat transfer, the electron and lattice temperatures are assumed to be equal. Thus this system reduces to the classical model when such an assumption is made. However, for sub-picosecond pulses and sub-microscale, the laser energy is absorbed primarily by free electrons confined within the thin material layer. This energy is then transferred to the lattice resulting in a lag between the excitement of the electrons and the transfer of energy to the lattice. As the duration of the laser pulse is short, the source of heat is turned off before thermal equilibrium between electrons and lattice is reached. This necessitates a two step model for describing energy transfer in such a situation. Eqs. (1) and (2) and its consideration over classical energy transfer models has been discussed in [2].

Analytical and numerical solutions for solving Eqs. (1) and (2) have been widely studied [2-53]. Among these, Qiu and Tien [1, 3-6] studied the heat transfer mechanism during short-pulse laser heating of metals using both numerical and experimental methods. Joshi and Majumdar [7] obtained numerical solutions using the explicit upstream difference method. Tzou and colleagues [2, 8-14] modified Eqs. (1) and (2) to a dual lag phase heat transport equation and studied the lagging behavior by using the Laplace transform method and the Reimann-sum approximation for the inverse. Ho et al. [15-16] studied heat transfer in a multilayered structure using the lattice Boltzmann method. Chen and colleagues [17-21] employed the corrective smoothed particle methods to solve a dual-phase-lag diffusion model. Al-Nimr et al. [22-27] investigated the phase lag effect on non-equilibrium entropy production and studied the effects of radiative and convective thermal losses during short-pulse laser heating of metals. They [28] also investigated the validity of the two-step model under harmonic boundary heating. Lin et al. [29] obtained an analytic solution of the dual-phase-lag model using Fourier series. Antaki and others [30-32] presented a solution of dual phase lag heat conduction in a semi-infinite slab and investigated the effect of dual phase lag heat conduction on ignition of a solid. Tang and Araki [33] derived an analytic solution in finite rigid slabs by using Green's function and a finite integral transform technique. Smith et al. [34] presented an analytical and numerical analysis for nonequilibrium heating in metal films. Lor and Chu [35] studied the propagation of thermal waves in a composite medium with interface thermal boundary resistance. Dai and Nassar [36-43] developed several finite difference schemes for solving a dual-phase-lag heat transport equation. Prakash et al. [44] investigated the non-Fourier microscale effect using a numerical method based on the finite element method and Runge-Kutta method. Wang et al. [45-48] analyzed the well-possedness and solution structure of 1D, 2D, and 3D dual-phase-lag heat transport equations. Kim and Daniel [49] studied an inverse heat conduct problem for nanoscale structure using sequential method. Lee et al. [50] investigated the transfer characteristics of a silicon microstructure irradiated by ultrashort pulsed lasers. Tsai and Hung [51] studied thermal wave propagation in a bi-layered composite sphere using the dual-phase-lag heat transport equation. Srinivasan et al. [52] proposed a parallel computation for microscale heat transfer in multilayered thin films. Recently, Dai and Nassar [53] have developed a three level in time finite difference scheme for solving the parabolic two-step heat transport equations in a 3D double-layered rectangular thin film exposed to ultrashort-pulsed lasers. In this article, we extend our study to the case that the double-layered thin film is irregular in the planar direction and develop an unconditionally stable FE-FD hybrid scheme for solving the parabolic two-step model. We employ the hybrid FE-FD method because of the nature of the geometry, where the thin film is irregular in the planar direction and regular in the z-direction. It should be pointed out that the hybrid FE-FD method has been widely employed in many research areas [37,41,54-63].

2 FE-FD Scheme

Consider a double-layered thin film with thickness of order 0.1 m and with length and width of order 1 mm, which is subjected to a sub picosecond pulse irradiation, as shown in Figure 1. Based on Eqs. (1)-(2) the governing equations can be written

$$C_e^{(m)} \frac{\partial T_e^{(m)}}{\partial t} = \kappa^{(m)} \nabla^2 T_e^{(m)} - G^{(m)}(T_e^{(m)} - T_l^{(m)}) + S^{(m)} \tag{3}$$

$$C_l^{(m)} \frac{\partial T_l^{(m)}}{\partial t} = G^{(m)}(T_e^{(m)} - T_l^{(m)}) \tag{4}$$

where $m = 1, 2$. The initial condition is

$$T_e^{(m)}(x, y, z, 0) = T_l^{(m)}(x, y, z, 0) = T_0 \tag{5}$$

where T_0 is the ambient temperature. The boundary conditions are assumed to be Neumann boundary conditions

$$\frac{\partial T_e^{(m)}}{\partial \vec{n}} = \frac{\partial T_l^{(m)}}{\partial \vec{n}} = 0 \tag{6}$$

where $\vec{n}$ is outward unit normal vector. Such boundary conditions arise from the case where the film is subjected to a short-pulse laser irradiation. Hence, one may assume no heat loss from the film surfaces in the short time response [2].

At the interface, we assume perfect thermal contact

$$T_e^{(1)} = T_e^{(2)}, \quad T_l^{(1)} = T_l^{(2)} \tag{7}$$

$$\kappa^{(1)} \nabla T_e^{(1)} = \kappa^{(2)} \nabla T_e^{(2)} \tag{8}$$

The exact solution is difficult in general to obtain due to the irregular geometry. Hence we develop a hybrid finite element - finite difference scheme for solving the above initial and boundary value problem. To this end, we first employ the finite element method to the planar direction. That is, multiplying Eq. (3) by a testing function w and applying Green's formula and the boundary, Eq. (6), we obtain a weak form as

$$\begin{aligned} & C_e^{(m)} \iint\limits_D \frac{\partial T_e^{(m)}}{\partial t} w dxdy - \kappa^{(m)} \oint\limits_{\partial D} \frac{\partial T_e^{(m)}}{\partial \vec{n}} w ds + \kappa \iint\limits_D \left(\frac{\partial T_e^{(m)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial T_e^{(m)}}{\partial y}\frac{\partial w}{\partial y}\right) dxdy \\ & + \iint\limits_D G^{(m)}(T_e^{(m)} - T_l^{(m)}) w dxdy + \iint\limits_D S^{(m)} w dxdy \\ = \; & 0 \end{aligned} \tag{9}$$

We define the test functions as

$$T_e^{(m)}(x,y,z,t) = \sum_{p=1}^{N_p} (T_e^{(m)})_p(z,t)\varphi_p(x,y), \quad T_l^{(m)}(x,y,z,t) = \sum_{p=1}^{N_p} (T_l^{(m)})_p(z,t)\varphi_p(x,y) \tag{10}$$

where $\varphi_p(x,y)$ is a basis function, and N_p is the number of points in the xy cross section. Further, the source term is written as $S_h^{(m)} = \sum\limits_{p=1}^{N_p} S_p^{(m)} \varphi_p(x,y)$. Letting $w(x,y) = \varphi_q(x,y)$ and substituting T_e by $(T_e^{(m)})_p$ in Eq. (9), we obtain

$$\sum_{p=1}^{N_p} C_e^{(m)} \frac{\partial (T_e^{(m)})_p}{\partial t} \iint\limits_D \varphi_p \varphi_q dxdy + \sum_{p=1}^{N_p} \kappa^{(m)} (T_e^{(m)})_p \iint\limits_D \left(\frac{\partial \varphi_p}{\partial x}\frac{\partial \varphi_q}{\partial x} + \frac{\partial \varphi_p}{\partial y}\frac{\partial \varphi_q}{\partial y}\right) dxdy$$

$$
\begin{aligned}
& -\sum_{p=1}^{N_p} \kappa^{(m)} \frac{\partial^2 (T_e^{(m)})_p}{\partial z^2} \iint_D \varphi_p \varphi_q dxdy + \sum_{p=1}^{N_p} G^{(m)} [(T_e^{(m)})_p - (T_l^{(m)})_p] \iint_D \varphi_p \varphi_q dxdy \\
& -\sum_{p=1}^{N_p} S_p^{(m)} \iint_D \varphi_p \varphi_q dxdy \\
= \ & 0, \qquad (11)
\end{aligned}
$$

and

$$
\sum_{p=1}^{N_p} C_l^{(m)} \frac{\partial (T_l^{(m)})_p}{\partial t} \iint_D \varphi_p \varphi_q dxdy - \sum_{p=1}^{N_p} G^{(m)} [(T_e^{(m)})_p - (T_l^{(m)})_p] \iint_D \varphi_p \varphi_q dxdy = 0 \qquad (12)
$$

where $q = 1, ..., N_p$. Denoting $m_{qp} = \iint_D \varphi_p \varphi_q dxdy$ and $k_{qp} = \iint_D (\frac{\partial \varphi_p}{\partial x} \frac{\partial \varphi_q}{\partial x} + \frac{\partial \varphi_p}{\partial y} \frac{\partial \varphi_q}{\partial y}) dxdy$, we rewrite Eqs. (11) and (12) into matrix form

$$
C_e^{(m)} \mathbf{M} \frac{\partial \vec{T}_e^{(m)}}{\partial t} + \kappa^{(m)} \mathbf{K} \vec{T}_e^{(m)} - \kappa^{(m)} \mathbf{M} \frac{\partial^2 \vec{T}_e^{(m)}}{\partial z^2} + G^{(m)} \mathbf{M} (\vec{T}_e^{(m)} - \vec{T}_l^{(m)}) = \mathbf{M} \vec{S} \qquad (13)
$$

and

$$
C_l^{(m)} \mathbf{M} \frac{\partial \vec{T}_l^{(m)}}{\partial t} - G^{(m)} \mathbf{M} (\vec{T}_e^{(m)} - \vec{T}_l^{(m)}) = 0 \qquad (14)
$$

where $\mathbf{M}$ and $\mathbf{K}$ are the capacitance and conductance matrices, respectively.

Eqs. (13) and (14) are then discretized using the finite difference method as follows

$$
\begin{aligned}
& C_e^{(m)} \mathbf{M} \frac{(\vec{T}_e^{(m)})_k^{n+1} - (\vec{T}_e^{(m)})_k^{n-1}}{2\Delta t} + \kappa^{(m)} \mathbf{K} \frac{(\vec{T}_e^{(m)})_k^{n+1} + 2(\vec{T}_e^{(m)})_k^{n} + (\vec{T}_e^{(m)})_k^{n-1}}{4} \\
& -\kappa^{(m)} \mathbf{M} \delta_z^2 \frac{(\vec{T}_e^{(m)})_k^{n+1} + 2(\vec{T}_e^{(m)})_k^{n} + (\vec{T}_e^{(m)})_k^{n-1}}{4} \\
& +G^{(m)} \mathbf{M} [\frac{(\vec{T}_e^{(m)})_k^{n+1} + 2(\vec{T}_e^{(m)})_k^{n} + (\vec{T}_e^{(m)})_k^{n-1}}{4} - \frac{(\vec{T}_l^{(m)})_k^{n+1} + 2(\vec{T}_l^{(m)})_k^{n} + (\vec{T}_l^{(m)})_k^{n-1}}{4}] \\
= \ & \mathbf{M} (\vec{S}^{(m)})^n \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
& C_l^{(m)} \mathbf{M} \frac{(\vec{T}_l^{(m)})_k^{n+1} - (\vec{T}_l^{(m)})_k^{n-1}}{2\Delta t} \\
& -G^{(m)} \mathbf{M} [\frac{(\vec{T}_e^{(m)})_k^{n+1} + 2(\vec{T}_e^{(m)})_k^{n} + (\vec{T}_e^{(m)})_k^{n-1}}{4} - \frac{(\vec{T}_l^{(m)})_k^{n+1} + 2(\vec{T}_l^{(m)})_k^{n} + (\vec{T}_l^{(m)})_k^{n-1}}{4}] \\
= \ & 0 \qquad (16)
\end{aligned}
$$

where $(\vec{T}_e^{(m)})_k^n$ is the approximation of $\vec{T}_e^{(m)}(k\Delta z, n\Delta t)$, $k = 0, 1, \cdots, N_z + 1$, and δ_z^2 is the second-order central difference operator. The interfacial equations are discretized as follows

$$\kappa^{(1)} \frac{(\vec{T}_e^{(1)})_{N_z+1}^n - (\vec{T}_e^{(1)})_{N_z}^n}{\Delta z} = \kappa^{(2)} \frac{(\vec{T}_e^{(2)})_1^n - (\vec{T}_e^{(2)})_0^n}{\Delta z} \tag{17}$$

$$(\vec{T}_e^{(1)})_{N_z+1}^n = (\vec{T}_e^{(2)})_0^n \tag{18}$$

Since Eqs. (15) and (16) are three-level in time, we assume

$$(\vec{T}_e^{(m)})_k^0 = (\vec{T}_l^{(m)})_k^0 = (\vec{T}_e^{(m)})_k^1 = (\vec{T}_l^{(m)})_k^1 = T_0 \tag{19}$$

The boundary conditions are assumed to be

$$(\vec{T}_e^{(1)})_0^n = (\vec{T}_e^{(1)})_1^n,\ (\vec{T}_l^{(1)})_0^n = (\vec{T}_l^{(1)})_1^n,\ (\vec{T}_e^{(2)})_{N_z}^n = (\vec{T}_e^{(2)})_{N_z+1}^n,\ (\vec{T}_l^{(2)})_{N_z}^n = (\vec{T}_l^{(2)})_{N_z+1}^n \tag{20}$$

Based on the FEM, Eqs. (11) and (12) are the first-order approximations of Eqs. (7) and (8) while the truncation errors of Eqs. (15) and (16) are $O(\Delta t^2 + \Delta z^2)$.

3 NUMERICAL METHOD

We now analyze the stability of the above scheme, Eqs. (15)-(20), with respect to the source term. To this end, we consider the following eigenvalue problem

$$\mathbf{K}\phi_p - \lambda_p \mathbf{M}\phi_p = 0 \tag{21}$$

where λ_p is the eigenvalue corresponding to the eigenvector ϕ_p. Since $\mathbf{K}$ and $\mathbf{M}$ are symmetric positive definite, we assume that the eigenvectors are orthonormalized with respect to the capacitance matrix $\mathbf{M}$ such that

$$\phi_j^T \mathbf{M} \phi_p = \delta_{pj} \tag{22}$$

where δ_{pj} is 1 if $p = j$ and 0 if $p \neq j$. Then multiplying Eq. (21) by ϕ_j^T and using Eq. (22) yields

$$\phi_j^T \mathbf{K} \phi_p - \lambda_p \delta_{pj} = 0 \tag{23}$$

implying that the eigenvectors are also orthogonal with respect to the matrix $\mathbf{K}$. Further, $\lambda_p > 0$ since $\mathbf{K}$ is positive definite. As the eigenvectors form a basis for the semi discrete system in our system, the solution $\vec{T}^{(m)}$ may be represented as a linear combination of the eigenvectors,

$$\vec{T}_e^{(m)}(x,y,z,t) = \sum_{p=1}^{N_p}(\alpha_e^{(m)}(z,t))_p\varphi_p(x,y), \quad \vec{T}_l^{(m)}(x,y,z,t) = \sum_{p=1}^{N_p}(\alpha_l^{(m)}(z,t))_p\varphi_p(x,y) \tag{24a}$$

$$\vec{S}_h^{(m)} = \sum_{p=1}^{N_p} S_p^{(m)}(z,t)_p\varphi_p(x,y) \tag{24b}$$

where $(\alpha_e^{(m)})_p$ and $(\alpha_l^{(m)})_p$ are the generalized electron and lattice coordinates respectively. Substituting Eq. (24) into Eqs. (15) and (16), pre-multiplying by ϕ_p^T, and using the orthogonal properties in Eqs. (22) and (23) leads to the result

$$\begin{aligned}
& C_e^{(m)}\frac{(\alpha_e^{(m)})_{pk}^{n+1} - (\alpha_e^{(m)})_{pk}^{n-1}}{2\Delta t} + \lambda_p\kappa^{(m)}\frac{(\alpha_e^{(m)})_{pk}^{n+1} + 2(\alpha_e^{(m)})_{pk}^{n} + (\alpha_e^{(m)})_{pk}^{n-1}}{4} \\
& -\frac{\kappa^{(m)}}{4\Delta z^2}\delta_z^2[(\alpha_e^{(m)})_{pk}^{n+1} + 2(\alpha_e^{(m)})_{pk}^{n} + (\alpha_e^{(m)})_{pk}^{n-1}] \\
& +\frac{G^{(m)}}{4}[(\alpha_e^{(m)})_{pk}^{n+1} + 2(\alpha_e^{(m)})_{pk}^{n} + (\alpha_e^{(m)})_{pk}^{n-1} - (\alpha_l^{(m)})_{pk}^{n+1} - 2(\alpha_l^{(m)})_{pk}^{n} - (\alpha_l^{(m)})_{pk}^{n-1}] \\
= \; & (S^{(m)})_{pk}^{n}
\end{aligned} \tag{25}$$

$$\begin{aligned}
& C_l^{(m)}\frac{(\alpha_l^{(m)})_{pk}^{n+1} - (\alpha_l^{(m)})_{pk}^{n-1}}{2\Delta t} \\
& -\frac{G^{(m)}}{4}[(\alpha_e^{(m)})_{pk}^{n+1} + 2(\alpha_e^{(m)})_{pk}^{n} + (\alpha_e^{(m)})_{pk}^{n-1} - (\alpha_l^{(m)})_{pk}^{n+1} - 2(\alpha_l^{(m)})_{pk}^{n} - (\alpha_l^{(m)})_{pk}^{n-1}] \\
= \; & 0
\end{aligned} \tag{26}$$

$$\kappa^{(1)}\frac{(\alpha_e^{(1)})_{pN_z+1}^n - (\alpha_e^{(1)})_{pN_z}^n}{\Delta z} = \kappa^{(2)}\frac{(\alpha_e^{(2)})_{p1}^n - (\alpha_e^{(2)})_{p0}^n}{\Delta z} \tag{27a}$$

$$(\alpha_e^{(1)})_{pN_z+1}^n = (\alpha_e^{(2)})_{p0}^n \tag{27b}$$

and

$$(\alpha_e^{(m)})_{pk}^0 = (\alpha_l^{(m)})_{pk}^0 = (\alpha_e^{(m)})_{pk}^1 = (\alpha_l^{(m)})_{pk}^1 = [(\alpha_e^{(m)})_0]_{pk} \tag{28}$$

$$(\alpha_e^{(1)})_{p0}^n = (\alpha_e^{(1)})_{p1}^n, \quad (\alpha_e^{(2)})_{pN_z+1}^n = (\alpha_e^{(2)})_{pN_z}^n \tag{29a}$$

$$(\alpha_l^{(1)})_{p0}^n = (\alpha_l^{(1)})_{p1}^n, \quad (\alpha_l^{(2)})_{pN_z+1}^n = (\alpha_l^{(2)})_{pN_z}^n \tag{29b}$$

for any time level n. Hence, the analysis of the stability of the scheme, Eqs. (16)-(21), can be switched to analyze the stability of the scheme Eqs. (25)-(29).

We now employ the discrete energy method [64] to analyze the stability of the above scheme with respect to the source term. Let S_h be a set of $\{u^n = \{u^n_{pk}\}$, with $u^n_{p0} = u^n_{p1}$ and $u^n_{pN_z} = u^n_{pN_z+1}\}$. For any $u^n, v^n \in S_h$, the inner products and norms are defined as

$$\begin{aligned}
(u^n, v^n) &= \Delta z \sum_{p=1}^{N_p} \sum_{k=1}^{N_z} u^n_{pk} v^n_{pk}, \|u^n\|^2 = (u^n, u^n) \\
\|\nabla_{\bar{z}} u^n\|_1^2 &= (\nabla_{\bar{z}} u^n, \nabla_{\bar{z}} v^n) = \Delta z \sum_{p=1}^{N_p} \sum_{k=1}^{N_z+1} \nabla_{\bar{z}} u^n_{pk} \nabla_{\bar{z}} v^n_{pk}
\end{aligned}$$

We also define $\nabla_z U_k = U_{k+1} - U_k$ and $\nabla_{\bar{z}} U_k = U_k - U_{k-1}$, the forward difference and backward difference operators respectively. It can also been seen that the central difference operator satisfies $\delta_z^2 = \nabla_z \nabla_{\bar{z}} U_k$.

Lemma 1. For any $u^n \in S_h$,

$$[u^{n+1}_{pk} + 2u^n_{pk} + u^{n-1}_{pk}][u^{n+1}_{pk} - u^{n-1}_{pk}] = [u^{n+1}_{pk} + u^n_{pk}]^2 - [u^n_{pk} + u^{n-1}_{pk}]^2 \tag{30}$$

Lemma 2. If $(\alpha_e^{(m)})^n_{pk}$, $m = 1, 2$, are the solutions of Eqs. (25)-(29), then

$$\begin{aligned}
&\kappa^{(1)} \Delta z \sum_{k=1}^{N_z} \delta_z^2 [(\alpha_e^{(1)})^{n+1}_{pk} + 2(\alpha_e^{(1)})^n_{pk} + (\alpha_e^{(1)})^{n-1}_{pk}] \cdot [(\alpha_e^{(1)})^{n+1}_{pk} + 2(\alpha_e^{(1)})^n_{pk} + (\alpha_e^{(1)})^{n-1}_{pk}] \\
&+\kappa^{(2)} \Delta z \sum_{k=1}^{N_z} \delta_z^2 [(\alpha_e^{(2)})^{n+1}_{pk} + 2(\alpha_e^{(2)})^n_{pk} + (\alpha_e^{(2)})^{n-1}_{pk}] \cdot [(\alpha_e^{(2)})^{n+1}_{pk} + 2(\alpha_e^{(2)})^n_{pk} + (\alpha_e^{(2)})^{n-1}_{pk}] \\
= &-\kappa^{(1)} \Delta z \sum_{k=1}^{N_z+1} (\nabla_{\bar{z}} [(\alpha_e^{(1)})^{n+1}_{pk} + 2(\alpha_e^{(1)})^n_{pk} + (\alpha_e^{(1)})^{n-1}_{pk}])^2 \\
&-\kappa^{(2)} \Delta z \sum_{k=1}^{N_z+1} (\nabla_{\bar{z}} [(\alpha_e^{(2)})^{n+1}_{pk} + 2(\alpha_e^{(2)})^n_{pk} + (\alpha_e^{(2)})^{n-1}_{pk}])^2
\end{aligned} \tag{31}$$

Proof. We let $u_{pk} = (\alpha_e^{(1)})^{n+1}_p + 2(\alpha_e^{(1)})^n_p + (\alpha_e^{(1)})^{n-1}_p$and $v_{pk} = (\alpha_e^{(2)})^{n+1}_p + 2(\alpha_e^{(2)})^n_p + (\alpha_e^{(2)})^{n-1}_p$. Based on Eqs. (27) and (29), the left hand side (LHS) of the equation can be expressed

$$\begin{aligned}
LHS &= \kappa^{(1)} \Delta z \sum_{k=1}^{N_z} [(u_{pk+1} - u_{pk}) - (u_{pk} - u_{pk-1})] u_{pk} \\
&\quad +\kappa^{(2)} \Delta z \sum_{k=1}^{N_z} [(v_{pk+1} - v_{pk}) - (v_{pk} - v_{pk-1})] v_{pk} \\
&= \kappa^{(1)} \Delta z \sum_{k=2}^{N_z+1} [(u_{pk} - u_{pk-1})] \cdot u_{pk-1} - \kappa^{(1)} \Delta z \sum_{k=1}^{N_z} [(u_{pk} - u_{pk-1})] \cdot u_{pk}
\end{aligned}$$

$$
\begin{aligned}
& +\kappa^{(2)}\Delta z \sum_{k=2}^{N_z+1} [(v_{pk}-v_{pk-1})]\cdot v_{pk-1} - \kappa^{(2)}\Delta z \sum_{k=1}^{N_z} [(v_{pk}-v_{pk-1})]\cdot v_{pk} \\
= \; & \kappa^{(1)}\Delta z \sum_{k=1}^{N_z+1} [(u_{pk}-u_{pk-1})]\cdot u_{pk-1} - \kappa^{(1)}\Delta z \sum_{k=1}^{N_z+1} [(u_{pk}-u_{pk-1})]\cdot u_{pk} \\
& +\kappa^{(2)}\Delta z \sum_{k=1}^{N_z+1} [(v_{pk}-v_{pk-1})]\cdot v_{pk-1} - \kappa^{(2)}\Delta z \sum_{k=1}^{N_z+1} [(v_{pk}-v_{pk-1})]\cdot v_{pk} \\
& +\kappa^{(1)}\Delta z(u_{pN_{z+1}}-u_{pN_z})u_{pN_z+1} - \kappa^{(2)}\Delta z(v_{p1}-v_{p0})v_{p0} \\
& -\kappa^{(1)}\Delta z(u_{p1}-u_{p0})u_{p0} + \kappa^{(2)}\Delta z(v_{pN_z+1}-v_{pN_z})v_{pN_z+1} \\
= \; & -\kappa^{(1)}\Delta z \sum_{k=1}^{N_z+1} \nabla_{\bar z}u_{pk}\cdot\nabla_{\bar z}u_{pk} - \kappa^{(2)}\Delta z \sum_{k=1}^{N_z+1} \nabla_{\bar z}v_{pk}\cdot\nabla_{\bar z}v_{pk}
\end{aligned}
$$

which is the RHS of Eq. (31).

Theorem. Suppose that $(U_e^{(m)})_{pk}^n$ and $(U_l^{(m)})_{pk}^n$, $(V_e^{(m)})_{pk}^n$ and $(V_l^{(m)})_{pk}^n$ are solutions of the proposed scheme in Eqs. (25)-(29) with the same initial and boundary conditions but with different source terms. We represent $(\varepsilon_e^{(m)})_{pk}^n = (U_e^{(m)})_{pk}^n - (V_e^{(m)})_{pk}^n$ and $(\varepsilon_l^{(m)})_{pk}^n = (U_l^{(m)})_{pk}^n - (V_l^{(m)})_{pk}^n$, $m=1,2$. Then for any n in $0 \le n\Delta t \le t_0$, $(\varepsilon_e^{(m)})_{pk}^n$ and $(\varepsilon_l^{(m)})_{pk}^n$, $m=1,2$,

$$
F(n) \le e^{6t_0}F(0) + ce^{6t_0}\max_{1\le\xi\le n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2] \tag{32}
$$

where

$$
\begin{aligned}
F(n) \; = \; & 2C_e^{(1)}\left\|(\varepsilon_e^{(1)})^{n+1} + (\varepsilon_e^{(1)})^n\right\|^2 + 2C_l^{(1)}\left\|(\varepsilon_l^{(1)})^{n+1} + (\varepsilon_l^{(1)})^n\right\|^2 \\
& +2C_e^{(2)}\left\|(\varepsilon_e^{(2)})^{n+1} + (\varepsilon_e^{(2)})^n\right\|^2 + 2C_l^{(2)}\left\|(\varepsilon_l^{(2)})^{n+1} + (\varepsilon_l^{(2)})^n\right\|^2
\end{aligned} \tag{33}
$$

$c = \max\{\frac{1}{C_e^{(1)}}, \frac{1}{C_e^{(2)}}\}$ and $e_m(\xi), m=1,2$, are the difference of corresponding source terms in layers 1 and 2 respectively. Hence, this scheme is unconditionally stable with respect to the source term.

Proof. It can be seen from Eqs. (25) and (26) that $(\varepsilon_e^{(m)})_{pk}^n$ and $(\varepsilon_l^{(m)})_{pk}^n$, $m=1,2$, satisfy

$$
\begin{aligned}
& 2C_e^{(m)}[(\varepsilon_e^{(m)})_{pk}^{n+1} - (\varepsilon_e^{(m)})_{pk}^{n-1}] + \lambda_p\kappa^{(m)}\Delta t[(\varepsilon_e^{(m)})_{pk}^{n+1} + 2(\varepsilon_e^{(m)})_{pk}^n + (\varepsilon_e^{(m)})_{pk}^{n-1}] \\
= \; & \frac{\kappa^{(m)}\Delta t}{\Delta z^2}\delta_z^2[(\varepsilon_e^{(m)})_{pk}^{n+1} + 2(\varepsilon_e^{(m)})_{pk}^n + (\varepsilon_e^{(m)})_{pk}^{n-1}] - G^{(m)}\Delta t\{[(\varepsilon_e^{(m)})_{pk}^{n+1} + 2(\varepsilon_e^{(m)})_{pk}^n + (\varepsilon_e^{(m)})_{pk}^{n-1}] \\
& -[(\varepsilon_l^{(m)})_{pk}^{n+1} + 2(\varepsilon_l^{(m)})_{pk}^n + (\varepsilon_l^{(m)})_{pk}^{n-1}]\} + 4\Delta t e_m(n)_{pk}
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
2C_l^{(m)}[(\varepsilon_l^{(m)})_{pk}^{n+1}-(\varepsilon_l^{(m)})_{pk}^{n-1}] &= G^{(m)}\Delta t\{[(\varepsilon_e^{(m)})_{pk}^{n+1}+2(\varepsilon_e^{(m)})_{pk}^{n}+(\varepsilon_e^{(m)})_{pk}^{n-1}] \\
&\quad -[(\varepsilon_l^{(m)})_{pk}^{n+1}+2(\varepsilon_l^{(m)})_{pk}^{n}+(\varepsilon_l^{(m)})_{pk}^{n-1}]\} \qquad (35)
\end{aligned}
$$

and also Eqs. (27)-(29). Multiplying Eq. (34) with $m=1$ by $\Delta z[(\varepsilon_e^{(1)})_{pk}^{n+1}+2(\varepsilon_e^{(1)})_{pk}^{n}+(\varepsilon_e^{(1)})_{pk}^{n-1}]$ and Eq. (35) with $m=2$ by $\Delta z[(\varepsilon_e^{(2)})_{pk}^{n+1}+2(\varepsilon_e^{(2)})_{pk}^{n}+(\varepsilon_e^{(2)})_{pk}^{n-1}]$, summing over p,k from $p=1,\cdots,N_p$ and $k=1,\cdots,N_z$, respectively, combining them together and using lemmas 1 and 2 gives

$$
\begin{aligned}
& 2C_e^{(1)}\left\|(\varepsilon_e^{(1)})^{n+1}+(\varepsilon_e^{(1)})^{n}\right\|^2-2C_e^{(1)}\left\|(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1}\right\|^2+2C_e^{(2)}\left\|(\varepsilon_e^{(2)})^{n+1}+(\varepsilon_e^{(2)})^{n}\right\|^2 \\
& -2C_e^{(2)}\left\|(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1}\right\|^2+r_z\kappa^{(1)}\left\|\nabla_{\bar z}[(\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1}]\right\|^2 \\
& +\lambda_p\kappa^{(1)}\Delta t\left\|(\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1}\right\|^2+r_z\kappa^{(2)}\left\|\nabla_{\bar z}[(\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1}]\right\|^2 \\
& +\lambda_p\kappa^{(2)}\Delta t\left\|(\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1}\right\|^2+\Delta tG^{(1)}\left\|(\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1}\right\|^2 \\
& +\Delta tG^{(2)}\left\|(\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1}\right\|^2 \\
= {} & \Delta tG^{(1)}((\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1},(\varepsilon_l^{(1)})^{n+1}+2(\varepsilon_l^{(1)})^{n}+(\varepsilon_l^{(1)})^{n-1}) \\
& +\Delta tG^{(2)}((\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1},(\varepsilon_l^{(2)})^{n+1}+2(\varepsilon_l^{(2)})^{n}+(\varepsilon_l^{(2)})^{n-1}) \\
& +4\Delta t(e_1(n),(\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1}) \\
& +4\Delta t(e_2(n),(\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1}) \qquad (36)
\end{aligned}
$$

where $r_z=\frac{\Delta t}{\Delta z^2}$. Similarly, we obtain from Eq. (35)

$$
\begin{aligned}
& 2C_l^{(1)}\left\|(\varepsilon_l^{(1)})^{n+1}+(\varepsilon_l^{(1)})^{n}\right\|^2+2C_l^{(2)}\left\|(\varepsilon_l^{(2)})^{n+1}+(\varepsilon_l^{(2)})^{n}\right\|^2-[2C_l^{(1)}\left\|(\varepsilon_l^{(1)})^{n}+(\varepsilon_l^{(1)})^{n-1}\right\|^2 \\
& +2C_l^{(2)}\left\|(\varepsilon_l^{(2)})^{n}+(\varepsilon_l^{(2)})^{n-1}\right\|^2]+\Delta tG^{(1)}\left\|(\varepsilon_l^{(1)})^{n+1}+2(\varepsilon_l^{(1)})^{n}+(\varepsilon_l^{(1)})^{n-1}\right\|^2 \\
& +\Delta tG^{(2)}\left\|(\varepsilon_l^{(2)})^{n+1}+2(\varepsilon_l^{(2)})^{n}+(\varepsilon_l^{(2)})^{n-1}\right\|^2 \\
= {} & \Delta tG^{(1)}((\varepsilon_e^{(1)})^{n+1}+2(\varepsilon_e^{(1)})^{n}+(\varepsilon_e^{(1)})^{n-1},(\varepsilon_l^{(1)})^{n+1}+2(\varepsilon_l^{(1)})^{n}+(\varepsilon_l^{(1)})^{n-1}) \\
& +\Delta tG^{(2)}((\varepsilon_e^{(2)})^{n+1}+2(\varepsilon_e^{(2)})^{n}+(\varepsilon_e^{(2)})^{n-1},(\varepsilon_l^{(2)})^{n+1}+2(\varepsilon_l^{(2)})^{n}+(\varepsilon_l^{(2)})^{n-1}) \qquad (37)
\end{aligned}
$$

Adding both Eqs. (36) and (37), and using the fact

$$
\left\|(\varepsilon_e^{(m)})^{n+1}+2(\varepsilon_e^{(m)})^{n}+(\varepsilon_e^{(m)})^{n-1}\right\|^2+\left\|(\varepsilon_l^{(m)})^{n+1}+2(\varepsilon_l^{(m)})^{n}+(\varepsilon_l^{(m)})^{n-1}\right\|^2
$$

$$
\begin{aligned}
& -2((\varepsilon_e^{(m)})^{n+1} + 2(\varepsilon_e^{(m)})^n + (\varepsilon_e^{(m)})^{n-1}, (\varepsilon_l^{(m)})^{n+1} + 2(\varepsilon_l^{(m)})^n + (\varepsilon_l^{(m)})^{n-1}) \\
\geq\ & 0
\end{aligned} \tag{38}
$$

where $m = 1, 2$, we obtain

$$
\begin{aligned}
& 2C_e^{(1)} \left\|(\varepsilon_e^{(1)})^{n+1} + (\varepsilon_e^{(1)})^n\right\|^2 + 2C_l^{(1)} \left\|(\varepsilon_l^{(1)})^{n+1} + (\varepsilon_l^{(1)})^n\right\|^2 \\
& +2C_e^{(2)} \left\|(\varepsilon_e^{(2)})^{n+1} + (\varepsilon_e^{(2)})^n\right\|^2 + 2C_l^{(2)} \left\|(\varepsilon_l^{(2)})^{n+1} + (\varepsilon_l^{(2)})^n\right\|^2 \\
& -[2C_e^{(1)} \left\|(\varepsilon_e^{(1)})^n + (\varepsilon_e^{(1)})^{n-1}\right\|^2 + 2C_l^{(1)} \left\|(\varepsilon_l^{(1)})^n + (\varepsilon_l^{(1)})^{n-1}\right\|^2] \\
& -[2C_e^{(2)} \left\|(\varepsilon_e^{(2)})^n + (\varepsilon_e^{(2)})^{n-1}\right\|^2 + 2C_l^{(2)} \left\|(\varepsilon_l^{(2)})^n + (\varepsilon_l^{(2)})^{n-1}\right\|^2] \\
\leq\ & 4\Delta t(e_1(n), (\varepsilon_e^{(1)})^{n+1} + 2(\varepsilon_e^{(1)})^n + (\varepsilon_e^{(1)})^{n-1}) \\
& +4\Delta t(e_2(n), (\varepsilon_e^{(2)})^{n+1} + 2(\varepsilon_e^{(2)})^n + (\varepsilon_e^{(2)})^{n-1})
\end{aligned} \tag{39}
$$

By the generalized Cauchy-Schwarz's inequality, we have

$$
\begin{aligned}
& 2(e_m(n), (\varepsilon_e^{(m)})^{n+1} + 2(\varepsilon_e^{(m)})^n + (\varepsilon_e^{(m)})^{n-1}) \\
\leq\ & 2C_e^{(m)} \left\|(\varepsilon_e^{(m)})^{n+1} + (\varepsilon_e^{(m)})^n\right\|^2 + 2C_e^{(m)} \left\|(\varepsilon_e^{(m)})^n + (\varepsilon_e^{(m)})^{n-1}\right\|^2 + \frac{1}{C_e^{(m)}} \left\|e_m(n)\right\|^2
\end{aligned} \tag{40}
$$

where $m = 1, 2$. Substituting the above inequality into Eq. (39), and using the notation $F(n)$ as defined in Eq. (33), Eq. (39) can be simplified as

$$
(1 - 2\Delta t)F(n) \leq (1 + 2\Delta t)F(n-1) + 2c\Delta t(\|e_1(n)\|^2 + \|e_2(n)\|^2) \tag{41}
$$

where $c = \max\{\frac{1}{C_e^{(1)}}, \frac{1}{C_e^{(2)}}\}$. Hence, when $1 - 2\Delta t > \frac{\Delta t}{2}$, we have

$$
\begin{aligned}
F(n) \ \leq\ & (\frac{1 + 2\Delta t}{1 - 2\Delta t})F(n-1) + \frac{2c\Delta t}{1 - 2\Delta t}(\|e_1(n)\|^2 + \|e_2(n)\|^2) \\
& \vdots \\
\leq\ & (\frac{1 + 2\Delta t}{1 - 2\Delta t})^n F(0) + \frac{2c\Delta t}{1 - 2\Delta t}[1 + \frac{1 + 2\Delta t}{1 - 2\Delta t} + \cdots + (\frac{1 + 2\Delta t}{1 - 2\Delta t})^{n-1}] \\
& \cdot \max_{1 \leq \xi \leq n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2] \\
\leq\ & (\frac{1 + 2\Delta t}{1 - 2\Delta t})^n F(0) + \frac{c}{2}[(\frac{1 + 2\Delta t}{1 - 2\Delta t})^n - 1] \max_{1 \leq \xi \leq n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2] \\
\leq\ & (\frac{1 + 2\Delta t}{1 - 2\Delta t})^n \{F(0) + c \max_{1 \leq \xi \leq n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2]\}
\end{aligned} \tag{42}
$$

Using the inequalities, $(1+x)^n \leq e^{nx}$ when $x > 0$, and $(1-x)^{-1} \leq e^{2x}$ when $x < \frac{1}{2}$ and letting $x = 2\Delta t$ and Δt be sufficiently small, we see $(\frac{1+2\Delta t}{1-2\Delta t})^n \leq e^{2n\Delta t}e^{4n\Delta t} \leq e^{6n\Delta t}$ and hence

$$\begin{aligned} F(n) &\leq e^{6n\Delta t}F(0) + ce^{6n\Delta t}\max_{1\leq\xi\leq n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2] \\ &\leq e^{6t_0}\{F(0) + c\max_{1\leq\xi\leq n}[\|e_1(\xi)\|^2 + \|e_2(\xi)\|^2]\} \end{aligned} \quad (43)$$

which completes the proof.

4 NUMERICAL EXAMPLE

To demonstrate the applicability of the scheme, Eqs. (15) and (16), with initial and boundary conditions Eqs. (17) - (20), without loss of generality we consider a simple cylindrical double-layered thin film with thickness of 0.1 m and radius of 0.5 mm as shown in Fig. 2a. The chosen thermal properties are listed in Table 1 [2, 3, 65]. The source term is chosen to be

$$S_p = 0.94J(\frac{1-R}{t_p\delta})e^{-\frac{z}{\delta}-2.77(\frac{t-2t_p}{t_p})^2} \quad (44)$$

where R is given as 0.93, δ is given as 15.3 nm and J is given as 13.4 J/m^2. J represents the total energy carried by the laser pulse divided by the laser spot cross section. The figure $t_p = 100$ fs and represents the full-width-at-half-maximum (FWHM) duration of the laser pulse. Time, $t = 0$, is when the laser arrives at the metal surface [1].

The information to formulate the xy planar mesh is used to create our matrix coefficient system. Figure 2b indicates the triangular mesh created for the matrix coefficient system. Linear triangle elements are used in this computation.

Three different xy planar meshes were chosen - these being planar meshes with thirty-three nodes, one with sixty-five nodes and a final mesh with one hundred twenty-nine nodes. A z-directional grid size, Δz, of 10^{-6} m was chosen. A time step of 0.001 ps was also chosen. Figure 3 gives normalized temperature profiles at the surface directly beneath the heat source. Figure 3a depicts the electron temperature on the surface over time. The maximum temperature of T_e on the surface was about 1100 K. This plot shows good agreement with the one obtained by Tzou [2]. The results also show good agreement with Dai and Nassar [53]. Figure 3b shows the normalized lattice temperature profiles at the surface directly beneath the heat source. The maximum temperature rise of T_l was about 10.99 K which is again in good agreement with other numerical studies [53].

Figure 4 gives temperature rises for both electron and lattice along the z axis for time (a) $t = 0.2$ ps, (b) $t = 0.25$ ps, and (c) $t = 0.5$ ps. Figures 5 and 6 give contour plots at (a) $t = 0.2$ ps, (b) $t = 0.25$ ps and (c) $t = 0.5$ ps for the electron and lattice temperatures respectively along the xz plane.

Figure 7 shows the surface contours for both the electron and lattice temperatures at time $t = 0.25$ ps, when the peak electron temperature appears. The lag in coupling electron and lattice energy exchange is evident. This contour comparison reveals a high electron temperature and only a slight variation in the lattice/bulk temperature.

To further examine the numerical method, two different cases are explored. First is a repetitive-pulse heating case. The heat source for this case is chosen to be

$$S_p = 0.94J(\frac{1-R}{t_p\delta})e^{-\frac{z}{\delta}}[e^{-2.77(\frac{t-2t_p}{t_p})^2} + e^{-2.77(\frac{t-4t_p}{t_p})^2}] \tag{45}$$

where $J = 13.4$ J/m2, R is given to be 0.93, $t_p = 100$ fs, and $\delta = 15.3$ nm. The dense mesh of the finite element representation was chosen for highest accuracy and the same time and z-directional (depth) values as in case one were chosen. Figure 8a shows the normalized change in electron temperature $(\Delta T/(\Delta T)_{max})$ on the surface of the gold/chromium disk. It can be seen from Figure 8a that there are two peaks in electron temperature due to the two laser pulses. Figure 8b gives the normalized temperature change for the lattice. A slight bend is evident at time $t = 0.5$ ps, as the second electron temperature peak begins to transfer energy to the lattice. Figure 9a demonstrates the temperature profiles through the disk along the z-direction through the film. The same three times as in case one were chosen ($t = 0.2$ ps, $t = 0.25$ ps and $t = 0.5$ ps). This figure shows a peak temperature at time $t = 0.5$ ps of nearly 1400 K. Figure 9b demonstrates the lattice temperature profiles through the film in the z-direction. Again, the temperature profile is similar to the ones in case one, except that the peak is higher due to the second pulse of the laser.

A third case demonstrates a moving source. The laser is pulsed five times at even time intervals about the center of the disk. The first pulse is focused on the center of the disk. The second pulse moves along the x-axis in the positive direction. The third pulse is located the same distance from the center along the positive direction of the y-axis. The fourth and fifth pulses are located likewise along the negative directions of the x- and y-axis respectively. Figure10 gives a graphical representation.

Figure 11a demonstrates the normalized electron temperature change for the central point of the top of the thin film's surface. The distinct laser pulses are evident as the electron temperature rises sharply with each pulse. Figure 11b demonstrates the normalized lattice temperature change for the same point. Again, a steady climb is evident with changes caused by each pulse as the energy is transferred from the electron cloud to the lattice. Contours for the electron temperature distribution through the thin films are shown in Figure 12. The first contour is after $t = 0.25$ ps. This is at the peak electron temperature profile for the first laser pulse. The second contour demonstrates the temperature distribution through the thin films after $t = 0.75$ ps. This represents the peak electron temperature rise for the second pulse. It is evident that the temperature distribution moves slightly toward the positive x-direction. The final contour demonstrates the electron temperature distribution through the thin films after $t = 1.75$ ps. This represents the peak electron temperature rise for the fourth laser pulse. This pulse is located along the negative x-axis. A shift in temperature toward that point is evident in this contour. Figures 13-15 show contours of the temperature distributions on the surface (xy-plane) at time $t = 0.25$ ps, $t = 0.75$ ps and $t = 1.75$ ps, respectively. The heat propagation from the moving source is evident in these figures.

5 CONCLUSION

A hybrid finite element-finite difference method has been developed for solving parabolic two-step micro heat transport equations in a three dimensional double-layered thin film exposed to ultrashort-pulsed lasers. It is shown that the scheme is unconditionally stable with respect to the heat source. Numerical results for thermal analysis of a gold layer on a chromium padding layer are obtained. The method can be readily applied to multiple layers and irregularly shaped geometries. Further research will focus on the temperature-dependant thermal property case and thermal deformation induced by ultrashort-pulsed lasers.

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Table 1.Thermal properties for Gold and Chromium [2, 3, 65].

Parameters	Gold	Chromium
T_0 (K)	300	300
C_e^0 (J/m^{3}K)	2.1×10^4	5.8×10^4
C_l (J/m^{3}K)	2.5×10^6	3.3×10^4
G (W/m^{3}K)	2.6×10^{16}	42×10^{16}
κ (W/mK)	315	94
γ (J/m^{3}K^2)	70	193.33

FIGURE CAPTIONS

Figure 1. Three-dimensional configuration of a double-layered thin film.

Figure 2. A double-layered thin film (a) and a triangular mesh for the xy plane (b).

Figure 3. Electron temperature change (a) and lattice temperature change (b) on the surface.

Figure 4. Electron temperature profiles (a) and lattice temperature profiles (b) along the z-direction.

Figure 5. Contours of electron temperature distributions in the xz plane at (a) $t = 0.2$ ps, (b) $t = 0.25$ ps, and (c) $t = 0.5$ ps.

Figure 6. Contours of lattice temperature distributions in the xz plane at (a) $t = 0.2$ ps, (b) $t = 0.25$ ps, and (c) $t = 0.5$ ps.

Figure 7. Electron temperature distribution (a) and lattice temperature distribution (b) in the xy plane at $t = 0.25$ ps.

Figure 8. Normalized electron temperature change (a) and normalized lattice temperature change (b) on surface for the repetitive-pulse case.

Figure 9. Electron temperature profiles (a) and lattice temperature profiles (b) for the repetitive-pulse case.

Figure 10. Graphical representation of pulsed laser on thin film surface.

Figure 11. Normalized electron temperature change (a) and normalized lattice temperature change (b) on center surface of disk for the moving source case.

Figure 12. Contours of electron temperature distributions at (a) $t = 0.25$ ps, (b) $t = 0.75$ ps, and (c) $t = 1.75$ ps for the moving source case.

Figure 13. Contours of electron temperature distribution (a) and lattice temperature distribution (b) in the xy plane at $t = 0.25$ ps.

Figure 14. Contours of electron temperature distribution (a) and lattice temperature distribution (b) in the xy plane at $t = 0.75$ ps.

Figure 15. Contours of electron temperature distribution (a) and lattice temperature distribution (b) in the xy plane at $t = 1.75$ ps.