



Additional Function Topics

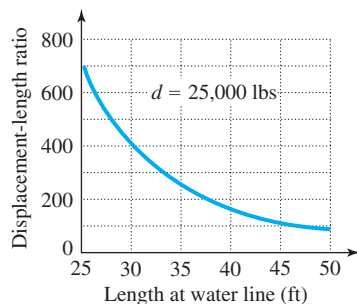
Working in a world of numbers, designers of racing boats blend art with science to design attractive boats that are also fast and safe. If the sail area is increased, the boat will go faster but will be less stable in open seas. If the displacement is increased, the boat will be more stable but slower. Increasing length increases speed but reduces stability. To make yacht racing both competitive and safe, racing boats must satisfy complex systems of rules, many of which involve mathematical formulas.

After the 1988 mismatch between Dennis Conner's catamaran and New Zealander Michael Fay's 133-foot monohull, an international group of yacht designers rewrote the America's Cup rules to ensure the fairness of the race. In addition to hundreds of pages of other rules, every yacht must satisfy the basic inequality

$$\frac{L + 1.25\sqrt{S} - 9.8\sqrt[3]{D}}{0.679} \leq 24,000,$$

which balances the length L , the sail area S , and the displacement D .

In the 1979 Fastnet Race 15 sailors lost their lives. After *Exide Challenger's* carbon-fiber keel snapped off, Tony Bullimore spent 4 days inside the overturned hull before being rescued by the Australian navy. Yacht racing is a dangerous sport. To determine the general performance and safety of a yacht, designers calculate the displacement-length ratio, the sail area-displacement ratio, the ballast-displacement ratio, and the capsize screening value. In Exercises 73 and 74 of Section 9.3 we will see how composition of functions is used to define the displacement-length ratio and the sail area-displacement ratio.



In This Section

- Linear and Constant Functions
- Absolute Value Functions
- Quadratic Functions
- Square-Root Functions
- Graphing Relations

9.1

GRAPHS OF FUNCTIONS AND RELATIONS

Functions were introduced in Section 3.5. In this section we will study the graphs of several types of functions. We graphed linear functions in Chapter 3 and quadratic functions in Chapter 8, but for completeness we will review them here.

Linear and Constant Functions

Linear functions get their name from the fact that their graphs are straight lines.

Linear Function

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$.

The graph of the linear function $f(x) = mx + b$ is exactly the same as the graph of the linear equation $y = mx + b$. If $m = 0$, then we get $f(x) = b$, which is called a **constant function**. If $m = 1$ and $b = 0$, then we get the function $f(x) = x$, which is called the **identity function**. When we graph a function given in function notation, we usually label the vertical axis as $f(x)$ rather than y .

EXAMPLE 1 Graphing a constant function

Graph $f(x) = 3$ and state the domain and range.

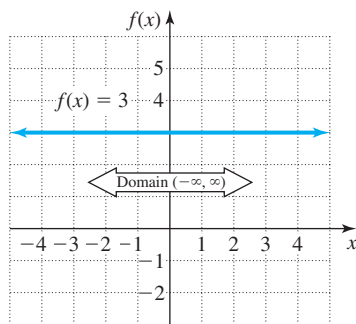


FIGURE 9.1

Solution

The graph of $f(x) = 3$ is the same as the graph of $y = 3$, which is the horizontal line in Fig. 9.1. Since any real number can be used for x in $f(x) = 3$ and since the line in Fig. 9.1 extends without bounds to the left and right, the domain is the set of all real numbers, $(-\infty, \infty)$. Since the only y -coordinate for $f(x) = 3$ is 3, the range is $\{3\}$. ■

The domain and range of a function can be determined from the formula or the graph. However, the graph is usually very helpful for understanding domain and range.

EXAMPLE 2 Graphing a linear function

Graph the function $f(x) = 3x - 4$ and state the domain and range.

Solution

The y -intercept is $(0, -4)$ and the slope of the line is 3. We can use the y -intercept and the slope to draw the graph in Fig. 9.2. Since any real number can be used for x in $f(x) = 3x - 4$, and since the line in Fig. 9.2 extends without bounds to the left and right, the domain is the set of all real numbers, $(-\infty, \infty)$. Since the graph extends without bounds upward and downward, the range is the set of all real numbers, $(-\infty, \infty)$.

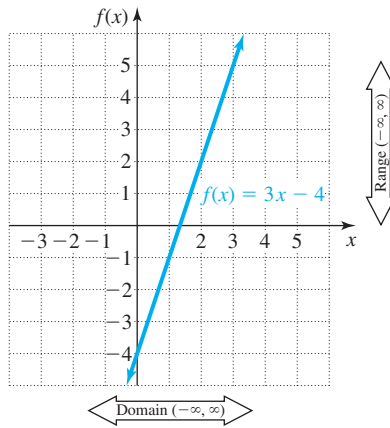


FIGURE 9.2

Absolute Value Functions

The equation $y = |x|$ defines a function because every value of x determines a unique value of y . We call this function the absolute value function.

Absolute Value Function

The **absolute value function** is the function defined by

$$f(x) = |x|.$$

To graph the absolute value function, we simply plot enough ordered pairs of the function to see what the graph looks like.

EXAMPLE 3 The absolute value function

Graph $f(x) = |x|$ and state the domain and range.

Solution

To graph this function, we find points that satisfy the equation $f(x) = |x|$.

x	-2	-1	0	1	2
$f(x) = x $	2	1	0	1	2

Plotting these points, we see that they lie along the V-shaped graph shown in Fig. 9.3. Since any real number can be used for x in $f(x) = |x|$ and since the graph extends without bounds to the left and right, the domain is $(-\infty, \infty)$. Because the graph does not go below the x -axis and because $|x|$ is never negative, the range is the set of nonnegative real numbers, $[0, \infty)$.

Helpful Hint

The most important feature of an absolute value function is its V-shape. If we had plotted only points in the first quadrant, we would not have seen the V-shape. So for an absolute value function we always plot enough points to see the V-shape.

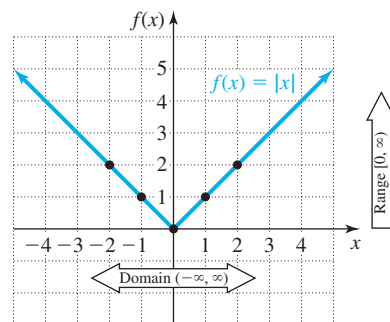


FIGURE 9.3

Many functions involving absolute value have graphs that are V-shaped, as in Fig. 9.3. To graph functions involving absolute value, we must choose points that determine the correct shape and location of the V-shaped graph.

EXAMPLE 4 Other functions involving absolute value

Graph each function and state the domain and range.

a) $f(x) = |x| - 2$

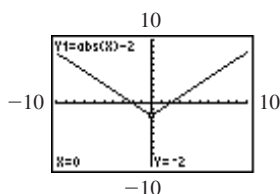
b) $g(x) = |2x - 6|$

Calculator Close-Up

To check Example 4(a) set

$$y_1 = \text{abs}(x) - 2$$

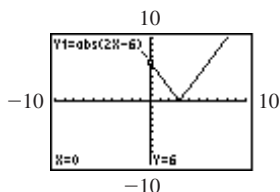
and then press GRAPH.



To check Example 4(b) set

$$y_2 = \text{abs}(2x - 6)$$

and then press GRAPH.



Solution

a) Choose values for x and find $f(x)$.

x	-2	-1	0	1	2
$f(x) = x - 2$	0	-1	-2	-1	0

Plot these points and draw a V-shaped graph through them as shown in Fig. 9.4. The domain is $(-\infty, \infty)$, and the range is $[-2, \infty)$.

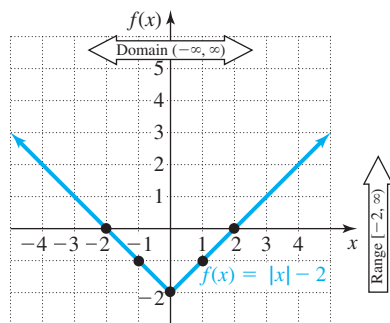


FIGURE 9.4

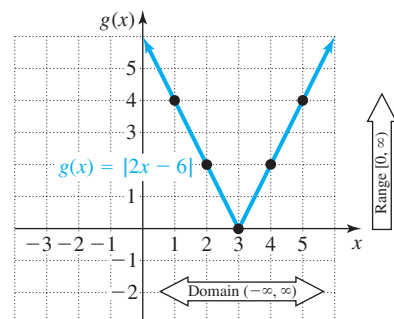


FIGURE 9.5

b) Make a table of values for x and $g(x)$.

x	1	2	3	4	5
$g(x) = 2x - 6 $	4	2	0	2	4

Draw the graph as shown in Fig. 9.5. The domain is $(-\infty, \infty)$, and the range is $[0, \infty)$.

Quadratic Functions

In Chapter 8 we learned that the graph of any quadratic function is a parabola, which opens upward or downward. The vertex of a parabola is the lowest point on a parabola that opens upward or the highest point on a parabola that opens downward. Parabolas will be discussed again when we study conic sections later in this text.

Quadratic Function

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a , b , and c are real numbers, with $a \neq 0$.

EXAMPLE 5 A quadratic function

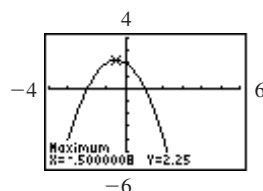
Graph the function $g(x) = 4 - x^2$ and state the domain and range.

Calculator Close-Up

You can find the vertex of a parabola with a calculator. For example, graph

$$y = -x^2 - x + 2.$$

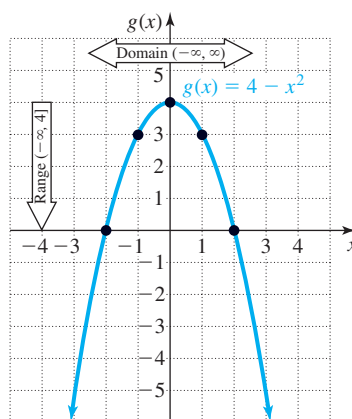
Then use the maximum feature, which is found in the CALC menu. For the left bound pick a point to the left of the vertex; for the right bound pick a point to the right of the vertex; and for the guess pick a point near the vertex.

**Solution**

We plot enough points to get the correct shape of the graph.

x	-2	-1	0	1	2
$g(x) = 4 - x^2$	0	3	4	3	0

See Fig. 9.6 for the graph. The domain is $(-\infty, \infty)$. From the graph we see that the largest y -coordinate is 4. So the range is $(-\infty, 4]$.

**FIGURE 9.6****Square-Root Functions**

Functions involving square roots typically have graphs that look like half a parabola.

Square-Root Function

The **square-root function** is the function defined by

$$f(x) = \sqrt{x}.$$

EXAMPLE 6 Square-root functions

Graph each equation and state the domain and range.

a) $y = \sqrt{x}$

b) $y = \sqrt{x + 3}$

Solution

a) The graph of the equation $y = \sqrt{x}$ and the graph of the function $f(x) = \sqrt{x}$ are the same. Because \sqrt{x} is a real number only if $x \geq 0$, the domain of this function is the set of nonnegative real numbers. The following ordered pairs are on the graph:

x	0	1	4	9
$y = \sqrt{x}$	0	1	2	3

The graph goes through these ordered pairs as shown in Fig. 9.7. Note that x is chosen from the nonnegative numbers. The domain is $[0, \infty)$ and the range is $[0, \infty)$.

- b) Note that $\sqrt{x+3}$ is a real number only if $x+3 \geq 0$, or $x \geq -3$. So we make a table of ordered pairs in which $x \geq -3$:

x	-3	-2	1	6
$y = \sqrt{x+3}$	0	1	2	3

The graph goes through these ordered pairs as shown in Fig. 9.8. The domain is $[-3, \infty)$ and the range is $[0, \infty)$.

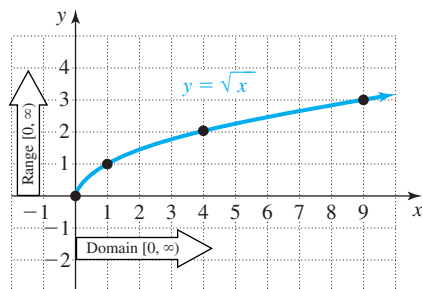


FIGURE 9.7

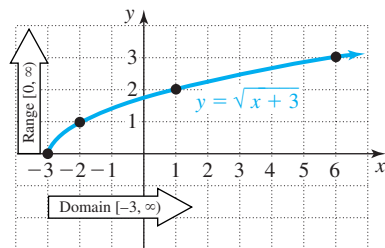


FIGURE 9.8

Graphing Relations

A function is a set of ordered pairs in which no two have the same first coordinate and different second coordinates. A relation is any set of ordered pairs. The domain of a relation is the set of x -coordinates of the ordered pairs and the range of a relation is the set of y -coordinates of the ordered pairs. In Example 7 we graph the relation $x = y^2$. Note that this relation is not a function because ordered pairs such as $(4, 2)$ and $(4, -2)$ satisfy $x = y^2$.

EXAMPLE 7

Graphing relations that are not functions

Graph each relation and state the domain and range.

- a) $x = y^2$ b) $x = |y - 3|$

Solution

- a) Because the equation $x = y^2$ expresses x in terms of y , it is easier to choose the y -coordinate first and then find the x -coordinate:

$x = y^2$	4	1	0	1	4
y	-2	-1	0	1	2

Figure 9.9 shows the graph. The domain is $[0, \infty)$ and the range is $(-\infty, \infty)$.

- b) Again we select values for y first and find the corresponding x -coordinates:

$x = y - 3 $	2	1	0	1	2
y	1	2	3	4	5

Plot these points as shown in Fig. 9.10. The domain is $[0, \infty)$ and the range is $(-\infty, \infty)$.

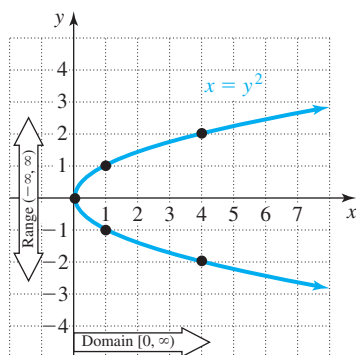


FIGURE 9.9

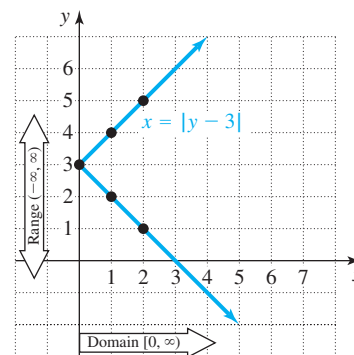


FIGURE 9.10

WARM - UPS

True or false? Explain your answer.

1. The graph of a function is a picture of all ordered pairs of the function.
2. The graph of every linear function is a straight line.
3. The absolute value function has a V-shaped graph.
4. The domain of $f(x) = 3$ is $(-\infty, \infty)$.
5. The graph of a quadratic function is a parabola.
6. The range of any quadratic function is $(-\infty, \infty)$.
7. The y-axis and the $f(x)$ -axis are the same.
8. The domain of $x = y^2$ is $[0, \infty)$.
9. The domain of $f(x) = \sqrt{x - 1}$ is $(1, \infty)$.
10. The domain of any quadratic function is $(-\infty, \infty)$.

9.1 EXERCISES

Reading and Writing After reading this section, write out the answers to these questions. Use complete sentences.

1. What is a linear function?

2. What is a constant function?

3. What is the graph of a constant function?

4. What shape is the graph of an absolute value function?

5. What is the graph of quadratic function called?

6. What is the identity function?

Graph each function and state its domain and range. See Examples 1 and 2.

7. $h(x) = -2$

8. $f(x) = 4$

9. $f(x) = 2x - 1$

10. $g(x) = x + 2$

19. $h(x) = |x + 1|$

20. $f(x) = |x - 2|$

11. $g(x) = \frac{1}{2}x + 2$

12. $h(x) = \frac{2}{3}x - 4$

21. $g(x) = |3x|$

22. $h(x) = |-2x|$

13. $y = -\frac{2}{3}x + 3$

14. $y = -\frac{3}{4}x + 4$

23. $f(x) = |2x - 1|$

24. $y = |2x - 3|$

15. $y = -0.3x + 6.5$

16. $y = 0.25x - 0.5$

25. $f(x) = |x - 2| + 1$

26. $y = |x - 1| + 2$

Graph each absolute value function and state its domain and range. See Examples 3 and 4.

17. $f(x) = |x| + 1$

18. $g(x) = |x| - 3$

Graph each quadratic function and state its domain and range. See Example 5.

27. $g(x) = x^2 + 2$

28. $f(x) = x^2 - 4$

29. $f(x) = 2x^2$

30. $h(x) = -3x^2$

39. $y = \sqrt{x} + 2$

40. $y = 2\sqrt{x} + 1$

31. $y = 6 - x^2$

32. $y = -2x^2 + 3$

Graph each relation and state its domain and range. See Example 7.

41. $x = |y|$

42. $x = -|y|$

Graph each square-root function and state its domain and range. See Example 6.

33. $g(x) = 2\sqrt{x}$

34. $g(x) = \sqrt{x} - 1$

43. $x = -y^2$

44. $x = 1 - y^2$

35. $f(x) = \sqrt{x - 1}$

36. $f(x) = \sqrt{x + 1}$

45. $x = 5$

46. $x = -3$

37. $h(x) = -\sqrt{x}$

38. $h(x) = -\sqrt{x - 1}$

47. $x + 9 = y^2$

48. $x + 3 = |y|$

49. $x = \sqrt{y}$

50. $x = -\sqrt{y}$

59. $y = \sqrt{x} - 3$

60. $y = 2|x|$

51. $x = (y - 1)^2$

52. $x = (y + 2)^2$

61. $y = 3x - 5$

62. $g(x) = (x + 2)^2$

Graph each function and state the domain and range.

53. $f(x) = 1 - |x|$

54. $h(x) = \sqrt{x - 3}$

63. $y = -x^2 + 4x - 4$

64. $y = -2|x - 1| + 4$

55. $y = (x - 3)^2 - 1$

56. $y = x^2 - 2x - 3$

57. $y = |x + 3| + 1$

58. $f(x) = -2x + 4$



GRAPHING CALCULATOR EXERCISES

65. Graph the function $f(x) = \sqrt{x^2}$ and explain what this graph illustrates.
66. Graph the function $f(x) = \frac{1}{x}$ and state the domain and range.
67. Graph $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$ on the same coordinate system. What can you say about the graph of $y = kx^2$?
68. Graph $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 3$ on the same screen. What can you say about the position of $y = x^2 + k$ relative to $y = x^2$.

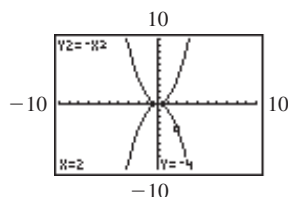
69. Graph $y = x^2$, $y = (x + 5)^2$, and $y = (x - 2)^2$ on the same screen. What can you say about the position of $y = (x - k)^2$ relative to $y = x^2$.
70. You can graph the relation $x = y^2$ by graphing the two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$. Try it and explain why this works.
71. Graph $y = (x - 3)^2$, $y = |x - 3|$, and $y = \sqrt{x - 3}$ on the same coordinate system. How does the graph of $y = f(x - k)$ compare to the graph of $y = f(x)$?

In This Section

- Reflecting
- Translating
- Stretching and Shrinking
- Multiple Transformations

Calculator Close-Up

With a graphing calculator, you can quickly see the result of modifying the formula for a function. If you have a graphing calculator, use it to graph the functions in the examples. Experimenting with it will help you to understand the ideas in this section.



9.2 TRANSFORMATIONS OF GRAPHS

We can discover what the graph of almost any function looks like if we plot enough points. However, it is helpful to know something about a graph so that we do not have to plot very many points. In this section we will learn how one graph can be transformed into another by modifying the formula that defines the function.

Reflecting

Consider the graphs of $f(x) = x^2$ and $g(x) = -x^2$ shown in Fig. 9.11. Notice that the graph of g is a mirror image of the graph of f . For any value of x we compute the y -coordinate of an ordered pair of f by squaring x . For an ordered pair of g we square first and then find the opposite because of the order of operations. This gives a correspondence between the ordered pairs of f and the ordered pairs of g . For every ordered pair on the graph of f there is a corresponding ordered pair directly below it on the graph of g , and these ordered pairs are the same distance from the x -axis. We say that the graph of g is obtained by reflecting the graph of f in the x -axis or that g is a reflection of the graph of f .

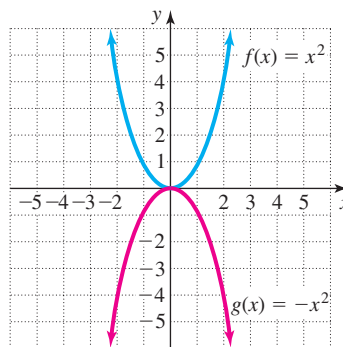


FIGURE 9.11

Reflection

The graph of $y = -f(x)$ is a **reflection** in the x -axis of the graph of $y = f(x)$.

EXAMPLE 1

Reflection

Sketch the graphs of each pair of functions on the same coordinate system.

a) $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$

b) $f(x) = |x|$, $g(x) = -|x|$