NP-Complete *Problems*

- Problems
  - **Abstract Problems**
    - Decision Problem, Optimal value, Optimal solution
  - **Encodings**
    - //Data Structure
  - **Concrete Problem**
    - //Language
- Class of Problems
  - **P**
  - **NP**
  - **NP-Complete**
    - NP-Completeness Proofs
- Solving hard problems
  - **Approximation Algorithms**

TECH
Computer Science

Abstract Problems

- **a formal notion of what a "problem" is**
- **high-level description of a problem**
- We define an *abstract problem* Q to be
  - **a binary relation on**
  - **a set I of problem *instances*, and**
  - **a set S of problem *solutions*.**
  - **$Q \in I \times S$**
- Three Kinds of Problems
  - **Decision Problem**
    - e.g. Is there a solution better than some given bound?
  - **Optimal Value**
    - e.g. What is the value of a best possible solution?
  - **Optimal Solution**
    - e.g. Find a solution that achieves the optimal value.

Encodings

- **// Data Structure**
- **describing abstract problems (for solving by computers)**
- **in terms of data structure or binary strings**
- An *encoding* of a set S of abstract objects is
  - **a mapping e from S to the set of binary strings.**
- Encoding for Decision problems
  - **Problem instances, $e : I \rightarrow \{0, 1\}^*$**
  - **Solution, $e : S \rightarrow \{0, 1\}$**
- "Standard" encoding
  - **computing time may be a function of encoding**
    - // The size of the input (the number of bit to represent one input)
  - **polynomially related encodings**
  - **assume encoding in a reasonable concise fashion**

Concrete Problem

- **problem instances and solutions are represented in data structure or binary strings**
- **// Language (in formal-language framework)**
- We call a problem whose instance set (and solution set) is the set of binary strings a *concrete problem*.
- Computer algorithm solves concrete problems!
  - **solves a concrete problem in time O(T(n))**
  - **if provided a problem instance i of length n = |i|,**
  - **the algorithm can produce the solution**
  - **in a most O(T(n)) *time.***
- A concrete problem is *polynomial-time solvable*
  - **if there exists an algorithm to solve it in time $O(n^k)$**
  - **for some constant k. (also called polynomially bounded)**

Class of Problems

- **// What makes a problem hard?**
- **// Make simple: classify decision problems**
- Definition: The class *P*
  - ***P* is the class of decision problems that are polynomially bounded.**
    - // there exist a deterministic algorithm
- Definition: The class *NP*
  - ***NP* is the class of decision problems for which there is a polynomially bounded non-deterministic algorithm.**
    - The name *NP* comes from "Non-deterministic Polynomially bounded."
    - // there exist a non-deterministic algorithm
- Theorem: $P \subseteq NP$

The Class NP

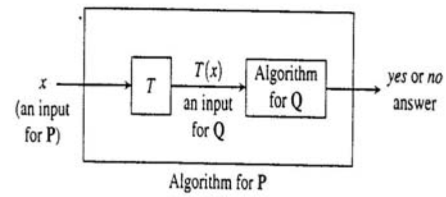
- NP is a class of decision problems for which
  - **a given proposed solution (called certificate) for**
  - **a given input**
  - **can be checked quickly (in polynomial time)**
  - **to see if it really is a solution.**
- A non-deterministic algorithm
  - **The non-deterministic "guessing" phase.**
    - Some completely arbitrary string s, "proposed solution"
    - each time the algorithm is run the string may differ
  - **The deterministic "verifying" phase.**
    - a deterministic algorithm takes the input of the problem and the proposed solution s, and
    - return value true or false
  - **The output step.**
    - If the verifying phase returned true, the algorithm outputs yes. Otherwise, there is no output.

The Class NP-Complete

- A problem Q is *NP-complete*
  - **if it is in NP and**
  - **it is NP-hard.**
- A problem Q is *NP-hard*
  - **if *every* problem in NP**
  - **is reducible to Q.**
- A problem P is *reducible* to a problem Q if
  - **there exists a polynomial reduction function T such that**
    - For every string x,
    - if x is a yes input for P, then T(x) is a yes input for Q
    - if x is a no input for P, then T(x) is a no input for Q.
    - T can be computed in polynomially bounded time.

Polynomial Reductions

- Problem P is reducible to Q
  - **P ≤p Q**
  - **Transforming inputs of P**
    - to inputs of Q
- Reducibility relation is transitive.



Circuit-satisfiability problem is NP-Complete

- Circuit-satisfiability problem
  - **belongs to the class NP, and**
  - **is NP-hard, i.e.**
    - every problem in NP is reducible to circuit-satisfiability problem!
- Circuit-satisfiablity problem
  - **we say that a one-output Boolean combinational circuit is satisfiable**
    - if it has a satisfying assignment,
    - a truth assignment (a set of Boolean input values) that
    - causes the output of the circuit to be 1
- Proof…

NP-Completeness Proofs

- **Once we proved a NP-complete problem**
- To show that the problem Q is NP-complete,
  - **choose a know NP-complete problem P**
  - **reduce P to Q**
- The logic is as follows:
  - **since P is NP-complete,**
    - all problems R in NP are reducible to P, R ≤p P.
  - **show P ≤p Q**
  - **then all problem R in NP satisfy R ≤p Q,**
    - by transitivity of reductions
  - **therefore Q is NP-complete**

Solving hard problems: Approximation Algorithms

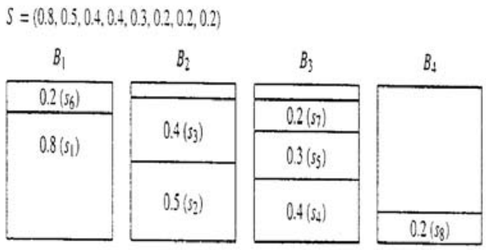
- **an algorithm that returns near-optimal solutions**
- **may use heuristic methods**
  - e.g. greedy heuristics
- Definition:Approximation algorithm
  - An approximation algorithm for a problem is
  - a polynomial-time algorithm that,
  - when given input I, outputs an element of FS(I).
- Definition: Feasible solution set
  - A feasible solution is
  - an object of the right type but not necessarily an optimal one.
  - FS(I) is the set of feasible solutions for I.

Approximation Algorithm e.g. Bin Packing

- **How to pack or store objects of various sizes and shapes**
- **with a minimum of wasted space**
- Bin Packing
  - **Let S = (s_1, …, s_n)**
    - where $0 < s_i <= 1$ for $1 <= i <= n$
  - **pack s_1, …, s_n into as few bin as possible**
    - where each bin has capacity one
- Optimal solution for Bin Packing
  - **considering all ways to**
  - **partition S into n or fewer subsets**
  - **there are more than**
  - **$(n/2)^{n/2}$ possible partitions**

Bin Packing: First fit decreasing strategy

- places an object in the first bin in which it fits
- W(n) in $\theta(n^2)$



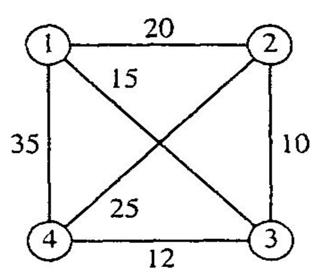
Algorithm: Bin Packing (first fit decreasing)

- Input: A sequence $S=(s_1, \ldots, s_n)$ of type float, where $0<s_i<1$ for $1<=i<=n$. S represents the sizes of objects {1,...,n} to be placed in bins of capacity 1.0 each.
- Output: An array bin where for 1<=i<=n, bin[i] is the number of the bin into which object i is placed.For simplicity,objects are indexed after being sorted in the algorithm.The array is passed in and the algorithm fills it.

```
binpackFFd(S,n,bin)
 float[] used=new float[n+1];
 //used[j] is the amount of space in bin j already used up.
 int i,j;
 Initialize all used entries to 0.0
 Sort S into descending(nonincreasing)order,giving the sequence s1>=S2>=…>=Sn.
 for(i=1;i<=n;i++)
   //Look for a bin in which s[i] fits.
   for(j=1;j<=n;j++)
    if(used[j]+si<+1.0)
      bin[i]=j;
      used[j] += si;
      break;  //exit for(j)
   //continue for(i).
```

The Traveling Salesperson Problem

- given a complete, weighted graph
- find a tour (a cycle through all the vertices) of
- minimum weight
- e.g.



Approximation algorithm for TSP

- The Nearest-Neighbor Strategy
  - **as in Prim's algorithm …**
- NearestTSP(V, E, W)
  - **Select an arbitrary vertex s to start the cycle C.**
  - **v = s;**
  - **While there are vertices not yet in C:**
    - Select an edge vw of minimum weight, where w is not in C.
    - Add edge vw to C;
    - v = w;
  - **Add the edge vs to C.**
  - **return C;**
- W(n) in $O(n^2)$
  - where n is the number of vertices