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Tight bounds on maximal and maximum matchings

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Abstract

In this paper, we study lower bounds on the size of maximal and maximum matchings in 3-connected planar graphs and graphs with bounded maximum degree. For each class, we give a lower bound on the size of matchings, and show that the bound is tight for some graph within the class.

Keywords: Matching; Bounds; Planar graphs; Bounded maximum degree

1. Introduction

The problem of finding a maximum matching in an undirected graph (all graphs in this paper are undirected) has a long and distinguished history beginning with the early work of Petersen [10] König [7], Hall [5], and Tutte [12]. The fastest algorithms to find a maximum matching in an n-vertex m-edge graph take $O(\sqrt{n}m)$ time, for bipartite graphs [6] as well as for general graphs [8].

One intensely studied topic is whether a graph has a perfect matching, i.e., a matching of size $\frac{n}{2}$. Perfect matchings exist in all 3-regular biconnected graphs [10] and all k-regular bipartite graphs [7], and a perfect matching can be found efficiently for these graphs [2,3,11]. Tutte [12] characterized when a graph has a perfect matching, but even if we know that a perfect matching exists, we do not know how to compute one faster than computing a maximum matching.

Not as much is known about lower bounds on the cardinality of matchings for graphs that do not have a perfect matching. Every 4-connected triangulated planar graph has a matching of size $\lfloor\frac{n}{2}\rfloor$, because it has a Hamiltonian cycle [13]. Nishizeki and Baybars [9] gave bounds for planar graphs as a function of the minimum degree and the connectivity. In particular, they showed that any 3-connected planar graph has a matching of size at least $\frac{1}{3}(n+4)$ for $n \geqslant 22$.

In this paper, we show that every 3-connected planar graphs has a matching of size $\lceil\min\{\frac{n-1}{2}, \frac{2n+4-\ell_4}{4}\}\rceil$, where ℓ_4 is the number of leaves of the 4-block tree, i.e., the tree of 4-connected components. We also show that this bound is no worse than $\frac{n+4}{3}$ for $n \geqslant 10$, so this result is a generalization of the bound by Nishizeki and Baybars. Moreover, if the graph is 4-connected, then $\ell_4 = 1$ and our bound yields a matching of size $\lceil\frac{n-1}{2}\rceil = \lfloor\frac{n}{2}\rfloor$, without using the Hamiltonicity

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Table 1
Overview of the results in this paper

Graph	Matching type	Bound 1	Bound 2
3-connected	Maximal	$\frac{n+4}{6}$	$\frac{2n+4-\ell_4}{8}$
planar	Maximum	$\frac{n+4}{3}$	$\frac{2n+4-\ell_4}{4}$
Max-deg-3	Maximum	$\frac{n-1}{3}$	$\frac{3n-n_2-2\ell_2}{6}$
Max-deg k	Maximal	$\frac{m}{2k-1}$	$\frac{m}{2k-1}$
3-regular	Maximum	$\frac{4n-1}{9}$	$\frac{3n-2\ell_2}{3}$

Here ℓ_4 denotes the number of leaves in the 4-block tree, ℓ_2 denotes the number of leaves in the 2-block tree, and n_2 denotes the number of vertices of degree 2 (see Section 2 for precise definitions). All bounds in the table are tight.

of such graphs. We also study bounds for maximal matchings, which are of interest because they can easily be computed in linear time.

We next study matchings in graphs with small maximum degree; the graphs do not have to be planar. It is known that every 3-regular biconnected graph has a perfect matching [10], but no other results appear to be known. (For bounds relative to the minimum degree in planar graphs, see [9].) We give bounds on matchings in graphs with maximum degree 3 that depend on the number of vertices of degree 2 and the number of leaves of the 2-block tree, i.e., the tree of maximal 2-connected components. The proof for maximal matchings generalizes even further to graphs of maximum degree k.

An overview of our results is given in Table 1. All entries are lower bounds on the size of the matching, and they are tight for some graph within this class. We typically give two bounds: one bound that depends only on n or m, and one bound that also includes other structural parameters of the graph.[3]

This paper is organized as follows. In Section 2 we define the basic graph-theoretical concepts used in this paper. We also review the theorem by Tutte and Berge which will be used extensively for the lower bound proofs for the size of maximum matchings. In Section 3 we study maximal and maximum matchings in 3-connected planar graphs. All our lower bounds on the size of maximal matchings follow directly from the corresponding bounds for maximum matchings (we just have to divide by 2, see Lemma 1). In Section 4 we study maximal matchings in max-deg-k graphs, and maximum matchings in max-deg-3 and 3-regular graphs. We end in Section 5 with a list of open problems.

2. Definitions

Let $G=(V,E)$ be an undirected graph with $|V(G)|=n$ vertices and $|E(G)|=m$ edges. We denote by n_i the number of vertices of degree i, i.e., with exactly i incident edges. G is 3-*regular* if every vertex has degree 3, and it is a *max-deg-k graph* if every vertex has degree at most k. G is *simple* if there are no loops and no multiple edges, and *connected* if for any pair of vertices there exists a path from one vertex to the other. Throughout this paper, we assume that graphs are undirected, simple and connected.

2.1. The 2-block tree

A connected graph G is *k-connected* if, for any set C of at most $k-1$ vertices, the graph that results from deleting the vertices in C is still connected. A 2-connected graph is also called *biconnected.* If a connected graph is not biconnected, then it must have a vertex v such that $G-v$ is not connected; such a vertex is called a *cutvertex*. If G has cutvertices, then its *biconnected components* are the maximal biconnected subgraphs of the graph. In the 2-*block tree* of G, each biconnected component and each cutvertex correspond to a node (see Fig. 1). If a cutvertex is contained in some biconnected component, we connect the two nodes in the 2-block tree corresponding to the cutvertex and the biconnected component. This defines a graph without cycles (otherwise, all the biconnected components on a cycle would actually belong to a single large biconnected component), justifying the name “2-block tree” for this graph. We denote the number of *leaves* in the 2-block tree by $\ell_2(G)$, or just ℓ_2 if the graph is clear from the context.

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[3] All lower bounds in this paper are given as fractions. Clearly, the size of the matching is an integer, and thus we could obtain a slightly better bound by rounding up these fractions. To avoid obscureness, we will not do this.

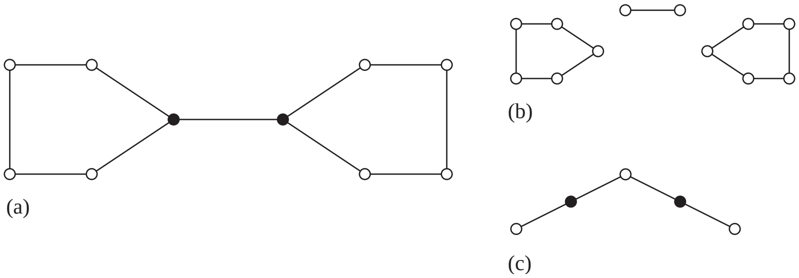


Fig. 1. (a) A graph with two cutvertices (black); (b) its maximal biconnected components; and (c) its 2-block tree.

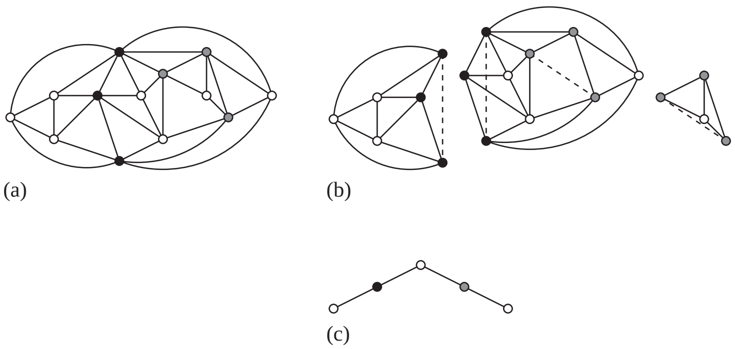


Fig. 2. (a) A graph with two separating triplets (black and grey); (b) its 4-connected components (added edges are dashed); and (c) its 4-block tree.

2.2. The 4-block tree

Similar to the 2-block tree, we can define a 4-block tree that captures the relationships among the 4-connected components of a graph (Fig. 2). Recall that a graph is 4-connected if removing any three arbitrary vertices leaves a connected graph. Assume that a graph is 3-connected, but not 4-connected. Then it contains three vertices $\{v,w,x\}$ such that removing them from the graph yields at least two connected components; we call $\{v,w,x\}$ a *separating triplet*. For each connected component C obtained from removing $\{v,w,x\}$, we create a new graph by adding to C the vertices v,w,x, as well as all their edges incident to another vertex in C, and the three edges $(v,w),(w,x)$ and (x,v) if they did not exist already.

We iterate this process until all resulting graphs are 4-connected; these are the 4-*connected components* of the graph. The 4-block tree is then defined as follows. We create one node for every 4-connected component, and one node for every separating triplet, and add an edge if and only if the separating triplet was contained in the 4-connected component. The resulting graph is again a tree. We denote its number of leaves by $\ell_4(G)$, or just ℓ_4 if the graph is clear from the context.

Note that each leaf of the 4-block tree corresponds to some subgraph of G that would be 4-connected if we added all edges between the vertices of the separating triplet that defined it.

2.3. Planar graphs

A *planar graph* is a graph that can be drawn in the plane without a crossing. Such a planar drawing divides the plane into connected areas, called *faces*. The *degree* of a face is the number of edges on its boundary. In a simple planar graph with at least three vertices, every face has degree at least 3. A planar graph is *triangulated* if all faces have degree 3. Such faces are also called *triangles*, and their boundary is a 3-*cycle*. A triangulated planar graph is 3-connected and has exactly $3n-6$ edges and $2n-4$ faces, each face is a triangle, and each edge borders exactly two triangles.

2.4. Matchings

For a graph G, a *matching* M is a subset of the edges of G such that any vertex has at most one incident edge in M. We denote by V_M the set of *matched vertices*, i.e., the vertices with an incident edge in M, and by $V_{\neg M}$ the set of *unmatched vertices*, i.e., $V_{\neg M} = V - V_M$. A matching is *maximal* if there is no edge between two unmatched vertices, i.e., we cannot add one more edge to the matching. A matching is *maximum* if it has the maximum possible cardinality among all matchings. A *perfect matching* is a matching that leaves no unmatched vertices. Only graphs of even order can have a perfect matching, and the size of such a matching is $\frac{n}{2}$. The following well-known lemma relates the size of maximal and maximum matchings.

Lemma 1. *If a graph G has a maximum matching of size k, then any maximal matching has at least size $\frac{k}{2}$.*

Proof. Let M be a maximum matching in G of size k, and let M' be a maximal matching in G. For each edge e in M, at least one of the two endpoints of e must be matched in M' (otherwise we could add e to M'). Thus, at least k vertices of G are matched in M'. We conclude that M' must have size at least $\frac{k}{2}$. □

Observe that a maximal (maximum) matching in a disconnected graph consists of the union of maximal (maximum) matchings in each of its components. Hence, in the following we only consider connected graphs. Also, loops or multiple edges cannot change the size of a maximal or maximum matching. Therefore, we will assume that the graphs are simple.

2.5. Tutte's theorem and Berge's generalization

Let T be an arbitrary subset of the vertices of a graph G. Removing T will split G into a number of connected components. Some of those may have an even number of vertices, and some may have an odd number of vertices. The latter are the *odd components* of $G - T$, and we denote their number by $\mathrm{odd}(T)$.

In 1947, Tutte [12] characterized graphs that have a perfect matching as exactly those graphs that have at most $|T|$ odd components, for any subset T of the vertices. In 1957, Berge [1] extended this theorem to bound the size of maximum matchings.

Theorem 2 (Berge [1]). *Let G be a graph. For any set $T \subset V$ and any matching M, the number of unmatched vertices in M is at least* $\mathrm{odd}(T) - |T|$. *Moreover, there exists a set $T \subset V$ such that any maximum matching of G contains exactly* $\mathrm{odd}(T) - |T|$ *unmatched vertices.*

3. 3-Connected planar graphs

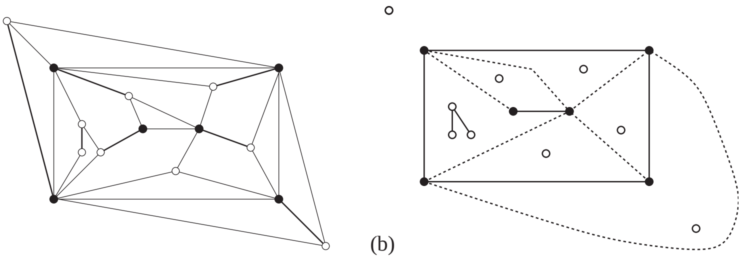
Nishizeki and Baybars showed that every 3-connected planar graph has a matching of size $\frac{n+4}{3}$ [9]. In this section, we strengthen this result by including the number of leaves of the 4-block tree in the bound; in particular we obtain a bound that resolves to $\lfloor \frac{n}{2} \rfloor$ if the graph is 4-connected.

Theorem 3. *Any* 3*-connected planar graph G of order n has a matching of size* $\min\{\frac{n-1}{2}, \frac{2n+4-\ell_4}{4}\}$, *where ℓ_4 is the number of leaves of the* 4*-block tree of G.*

Proof. Let G be a 3-connected planar graph of order n, and let M be a maximum matching in G. By Theorem 2, there exists a vertex set T in G such that there are exactly $|V_{\neg M}| = \mathrm{odd}(T) - |T|$ unmatched vertices in M. If $|T| \leqslant 2$, then $G - T$ is still connected, i.e., $|V_{\neg M}| \leqslant \mathrm{odd}(T) \leqslant 1$. But then clearly $|M| \geqslant \frac{n-1}{2}$.

If $|T| = 3$, then there can be at most two odd components in $G - T$. If there were three or more components, they would all have to be incident to all vertices of T by 3-connectivity, and the graph would contain $K_{3,3}$ as a minor. But G is planar, so this is impossible. Since we assumed that there are $\mathrm{odd}(T) - |T| < 0$ unmatched vertices, this case is actually impossible.

If $|T| \geqslant 4$, then we greedily add edges between any two non-adjacent vertices of T that lie on the same face of G, without destroying the planarity of the graph. Let G_T denote the subgraph of this augmented graph induced by the vertices of T (see Fig. 3). Note that no two components of $G - T$ can be within the same face of G_T, because then we would have introduced an edge to split the face between them. Therefore, for every odd component there must be a unique face in G_T. This immediately proves $\mathrm{odd}(T) \leqslant 2|T| - 4$, but in fact, we can do better and show $2\mathrm{odd}(T) \leqslant 2|T| - 4 + \ell_4$.



(a) (b)

Fig. 3. (a) A set T (in black) in a 3-connected planar graph G. Note that $|T| = 6$, $\mathrm{odd}(T) = 7$, and G has a maximum matching leaving only one node unmatched (the thick edges). (b) The corresponding graph G_T with the connected components of $G - T$; added edges are dashed.

More precisely, let f_3 and $f_{\geqslant 4}$ be the number of faces of G_T of degree 3 and degree at least 4, respectively. An easy counting argument shows that $f_3 + 2f_{\geqslant 4} \leqslant 2|T| - 4$. Let C be an odd component, and let f_C be the face of G_T containing C. If f_C has degree 3, then C has only three neighbors in T, and these three neighbors form a separating triplet of G (separating C from the rest of T, remember that $|T| \geqslant 4$). This separating triplet is the ancestor of at least one leaf of the 4-block tree of G. So C can be associated with one face of G_T that has degree 3 and one leaf of the 4-block tree. If f_C has a higher degree, then C can be associated with one face of G_T that counts towards $f_{\geqslant 4}$. So $2\mathrm{odd}(T) \leqslant f_3 + \ell_4 + 2f_4 \leqslant 2|T| - 4 + \ell_4$.

But then $|V_{\neg M}| \leqslant \frac{2|T|-4+\ell_4}{2} - |T| = \frac{\ell_4-4}{2}$, which implies $|V_M| \geqslant n - |V_{\neg M}| \geqslant \frac{2n+4-\ell_4}{2}$. □

To obtain a bound that only depends on n, we need a bound on ℓ_4, the number of leaves in the 4-block tree.

Lemma 4. *The 4-block tree of any planar 3-connected graph of order $n \geqslant 4$ has at most $\frac{2n-4}{3}$ leaves.*

Proof. Let G be a planar 3-connected graph, and let G' be the graph obtained from G by adding all edges between two vertices that belong to a separating triplet, if they did not appear already. One can show that G' is still planar. Now any leaf of the 4-block tree of G corresponds to a 4-connected subgraph of G' that has at least 4 faces. For any two leaves, the interior faces of these subgraphs are disjoint. Thus for every leaf of the 4-block tree of G, there must be at least 3 faces of G'.

Since G' has at most $2n - 4$ faces, the number ℓ_4 of leaves of the 4-block tree of G satisfies $3\ell_4 \leqslant 2n - 4$, or $\ell_4 \leqslant \frac{2n-4}{3}$. □

We can use this lemma to obtain the following general bound on the size of maximum matchings in 3-connected planar graphs.

Theorem 5. *Any 3-connected planar graph of order $n \geqslant 10$ has a matching of size $\frac{n+4}{3}$.*

Proof. By Lemma 4, we have

$$\frac{2n+4-\ell_4}{4} \geqslant \frac{3(2n+4)-(2n-4)}{12} = \frac{4n+16}{12} = \frac{n+4}{3}.$$

Combining this with Theorem 3 and noting that $\lceil \frac{n+4}{3} \rceil \leqslant \lceil \frac{n-1}{2} \rceil$ for $n \geqslant 10$ yields the result. □

By Lemma 1, we also get a lower bound for the size of maximal matchings.

Theorem 6. *Any maximal matching in a planar 3-connected graph G of order $n \geqslant 4$ has size at least $\frac{2n+4-\ell_4}{8}$, where ℓ_4 is the number of leaves in the 4-block tree of G. If $n \geqslant 10$, then the size of the maximal matching is at least $\frac{n+4}{6}$.*

The bounds in Theorems 5 and 6 are tight. For even $n \geqslant 4$, there exists a triangulated (hence 3-connected) planar graph G' of order $n' = 3n - 4$ with a maximal matching of size $\frac{n'+4}{6}$. This graph is shown in Fig. 4(a). It consists of a set T of n vertices (black vertices in the figure), where $n - 2$ of the vertices are connected by a cycle and the other two vertices are each connected to every vertex on the cycle (solid edges). This graph G has a perfect matching M (bold edges).

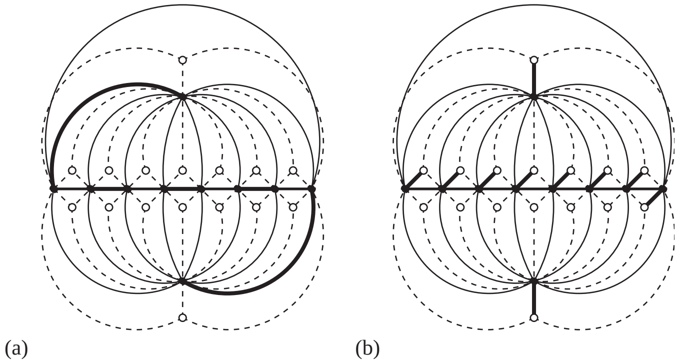


Fig. 4. (a) The black vertices and solid edges are a triangulated planar graph G of order $n = 10$ with a perfect matching M of size $\frac{n}{2}$ (bold edges). In the graph G' of order $n' = 3n - 4$ which also includes the white vertices and dashed edges, M is a maximal matching of size $\frac{n'+4}{6}$. (b) G' has a maximum matching of size $\frac{n'+4}{3}$ (bold edges).

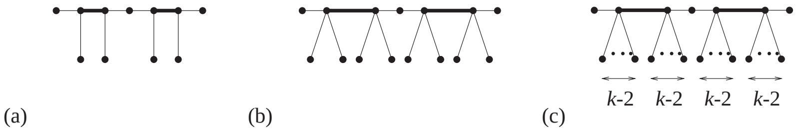


Fig. 5. A max-deg-k graph with a maximal matching of size $\frac{m}{2k-1}$ for (a) $k = 3$, (b) $k = 4$, (c) arbitrary k.

We now construct G' by adding another vertex (white vertices) into each of the $2n - 4$ faces of G and triangulating the resulting graph (dashed edges). In G', M is a maximal matching of size $\frac{n'+4}{6}$.

G' has a matching of size $\frac{n'+4}{3}$, see Fig. 4(b). To see that this is a maximum matching, observe that there are no edges between the white vertices. Thus, each white vertex forms an odd component of $G - T$, i.e., $\mathrm{odd}(T) - |T| = (2n-4) - n = n - 4$. Hence, in any matching in G' at least $n - 4$ vertices are unmatched and at most $2n$ vertices are matched, so no matching can have size larger than $n = \frac{n'+4}{3}$.

Note that any of these white vertices together with its neighbors forms a leaf in the 4-block tree, so this graph has a 4-block tree with $\frac{2n'-4}{3}$ leaves, and Lemma 4 is tight as well.

4. Graphs with maximum degree k

In this section we study another class of graphs, namely graphs with maximum degree k, for fixed $k \geqslant 3$. These graphs are called *max-deg-k* graphs. 3-regular biconnected graphs always have a perfect matching. We obtain bounds on matchings for graphs with higher maximum degree.

4.1. Maximal matchings in max-deg-k graphs

Theorem 7. *Any maximal matching in a max-deg-k graph with m edges has size at least* $\frac{m}{2k-1}$.

Proof. Let G be a max-deg-k graph, and let M be a maximal matching in G. An edge $e = (u, v)$ of M can be adjacent to at most $2k - 2$ other edges (adjacent to u or v). If we greedily remove edges of M together with their adjacent edges, then we partition the set of edges of G into subsets of size at most $2k - 1$, where each subset contains at least one edge of M. Since M is maximal, all edges of G must appear in one of the subsets. But then $|M| \geqslant \frac{m}{2k-1}$. □

This bound is tight, as illustrated in Fig. 5. The bold edges indicate a maximal matching of size $\frac{m}{2k-1}$.

Unfortunately, we do not have a stronger bound for the size of maximum matchings in max-deg-k graphs, except for the special case of $k = 3$.

4.2. Maximum matchings in max-deg-3 graphs

Theorem 8. *Any max-deg-3 graph G of order n has a matching of size $\frac{3n-n_2-2\ell_2}{6}$, where ℓ_2 is the number of leaves in the 2-block tree of G and n_2 is the number of vertices of degree 2.*

Proof. Let G be a max-deg-3 graph of order n, and let M be a maximum matching in G. By Theorem 2, there exists a vertex set T in G such that there are exactly $|V_{\neg M}| = \mathrm{odd}(T) - |T|$ unmatched vertices in M. We define the following three quantities: $\mathrm{odd}_1(T)$, $\mathrm{odd}_2(T)$, and $\mathrm{odd}_{\geqslant 3}(T)$ are the number of odd components of $G - T$ joined to T by one edge, two edges, and at least three edges, respectively. Every odd component joined to T by exactly one edge contains a leaf of the 2-block tree, so $\mathrm{odd}_1(T) \leqslant \ell_2$. Every odd component joined to T by exactly two edges must contain at least one vertex of degree 2 (otherwise there would be an odd number of vertices of odd degree), so $\mathrm{odd}_2(T) \leqslant n_2$. The number of edges incident to T is at least $\mathrm{odd}_1(T) + 2\mathrm{odd}_2(T) + 3\mathrm{odd}_{\geqslant 3}(T)$, but also at most $3 \cdot |T|$ because G has maximum degree 3. Therefore,

$$\begin{aligned} |V_{\neg M}| &= \mathrm{odd}(T) - |T| \\ &= \frac{\mathrm{odd}_1(T) + 2\mathrm{odd}_2(T) + 3\mathrm{odd}_{\geqslant 3}(T)}{3} + \frac{2}{3}\mathrm{odd}_1(T) + \frac{1}{3}\mathrm{odd}_2(T) - |T| \\ &\leqslant \frac{3 \cdot |T| + 2\ell_2 + n_2}{3} - |T| \\ &= \frac{2\ell_2 + n_2}{3} \end{aligned}$$

or

$$|V_M| \geqslant \frac{3n - n_2 - 2\ell_2}{3}. \qquad \square$$

To obtain a bound that only depends on n, we need to bound ℓ_2 and n_2.

Lemma 9. *Every connected max-deg-3 graph of order n has $2\ell_2 + n_2 \leqslant n + 2$, where ℓ_2 is the number of leaves in the 2-block tree and n_2 is the number of vertices of degree 2.*

Proof. Let G be a connected max-deg-3 graph of order n. Nothing is to show if G is 2-connected or $\ell_2 = 1$, so assume that G has cutvertices. Every vertex of degree 1 corresponds to a leaf in the 2-block tree of G, so we have $n_1 \leqslant \ell_2$. We distinguish two cases.

In the first case every leaf of the 2-block tree is a single vertex of degree 1, so $\ell_2 = n_1$. Since G is connected, $m \geqslant n - 1$, but also $n_1 + 2n_2 + 3n_3 = 2m \geqslant 2n - 2 = 2n_1 + 2n_2 + 2n_3 - 2$. Hence $n_1 \leqslant n_3 + 2$, and we have $2\ell_2 + n_2 = 2n_1 + n_2 \leqslant n_1 + n_2 + n_3 + 2 = n + 2$.

In the second case some leaves of the 2-block tree are not a single vertex of degree 1. We then obtain from G a new graph G' by deleting from these leaves all vertices except the cutvertex. Note that the cutvertex must have had two neighbors in the leaves before (and in particular must have had degree 3), and hence now becomes a vertex of degree 1 (and again a leaf of the 2-block tree). So $\ell_2(G') = \ell_2(G)$ and $n_1(G') + n_3(G') \leqslant n_1(G) + n_3(G)$. Since the claim holds for G', we have

$$\begin{aligned} 2\ell_2(G) + n_2(G) &= 2\ell_2(G') + n_2(G') + (n_2(G) - n_2(G')) \\ &\leqslant n(G') + 2 + (n_2(G) - n_2(G')) \\ &= n_1(G') + n_3(G') + 2 + n_2(G) \\ &\leqslant n(G) + 2. \qquad \square \end{aligned}$$

Theorem 10. *Any max-deg-3 graph of order n has a matching of size $\frac{n-1}{3}$.*

Proof. By Theorem 8 and Lemma 9, the number of unmatched vertices is at most $\frac{2\ell_2+n_2}{3} \leqslant \frac{(n+2-n_2)+n_2}{3} \leqslant \frac{n+2}{3}$, hence the maximum matching has size at least $\frac{n-1}{3}$. $\square$

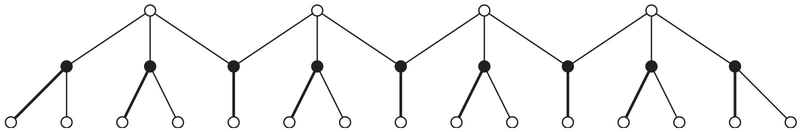


Fig. 6. A max-deg-3 graph of order $n = 28$ which has a maximum matching of size $\frac{n-1}{3}$ (bold edges).

This bound is tight, as can be seen from the graph in Fig. 6, for which the maximum matching has size $\frac{n-1}{3}$.

This graph does not have any vertices of degree 2, and hence one might conjecture whether the term "$-\frac{n_2}{6}$" in the bound of Theorem 8 is really needed. Consider the following graph: Let G be an arbitrary 3-regular graph with n vertices and $m = \frac{3}{2}n$ edges. Split every edge into two edges by introducing a new degree-2 vertex in the middle of the edge. This gives a new graph G' with $n' = n + m = \frac{5}{2}n$ vertices and $n_2' = \frac{3}{2}n$ vertices of degree 2. With T the set of the n vertices of G, Theorem 2 tells us that no matching in G' can have less than $n_2' - n = \frac{n}{2}$ unmatched vertices. Thus, any maximum matching in G' has size at most $\frac{n' - \frac{n}{2}}{2} = \frac{\frac{15}{2}n - \frac{3}{2}n}{6} = \frac{3n' - n_2'}{6}$. Thus, the term "$-\frac{n_2}{6}$" in Theorem 8 cannot be avoided in general.

On the other hand, the size of the maximum matching in this graph is $\frac{2}{5}n' > \frac{n'-1}{3}$. It remains open whether there exists a better bound on the size of the maximum matching if the graph has vertices of degree 2. In the example, a bound of $\frac{3n'+n_2'}{9}$ holds for the size of a maximum matching, but is this bound true for all max-deg-3 graphs?

4.3. *Maximum matchings in 3-regular graphs*

For 3-regular graphs we can improve the bounds of Theorem 10.

Theorem 11. *Any* 3-*regular graph* G *of order* n *has a matching of size* $\frac{3n-2\ell_2}{6}$, *where* ℓ_2 *is the number of leaves in the* 2-*block tree of* G.

Proof. This follows immediately from Theorem 8 because 3-regular graphs do not have vertices of degree 2. □

By bounding ℓ_2 we can obtain a stronger general bound for 3-regular graphs than Theorem 10.

Lemma 12. *The* 2-*block tree of any* 3-*regular graph of order* n *has at most* $\frac{n+2}{6}$ *leaves.*

Proof. Let G be a 3-regular graph of order n, let C be a biconnected component that is a leaf in the 2-block tree of G, and let v be the unique cutvertex of C. We claim that C has at least five vertices. Let w be one of the neighbors of v in C. w is not a cutvertex, therefore all three neighbors of w must be in C. So C has at least four vertices. Any graph has an even number of vertices of odd degree, so C must have at least four odd-degree vertices. v has even degree within C, so C must have at least five vertices.

Let G_L be the graph that results from G by deleting all vertices that are part of a leaf of the 2-block tree and not a cutvertex. For every leaf we delete at least four vertices, so $n(G_L) \leqslant n - 4\ell_2$. G_L is connected, hence $m(G_L) \geqslant n(G_L) - 1$. Also, every cutvertex that belonged to a leaf of G has degree 1 in G_L, whereas all other vertices have degree 3, so $2m(G_L) = \ell_2 + 3(n(G_L) - \ell_2)$. Thus, $\ell_2 + 3(n(G_L) - \ell_2) = 2m(G_L) \geqslant 2n(G_L) - 2$, which implies $2\ell_2 \leqslant n(G_L) + 2 \leqslant n - 4\ell_2 + 2$, or $\ell_2 \leqslant \frac{n+2}{6}$. □

Theorem 13. *Any* 3-*regular graph of order* n *has a matching of size* $\frac{4n-1}{9}$.

Proof. Let G be a 3-regular graph of order n. By Theorem 11 and Lemma 12, G has a matching of size

$$\frac{3n - 2\ell_2}{6} \geqslant \frac{3n - \frac{n+2}{3}}{6} = \frac{8n-2}{18} = \frac{4n-1}{9}. \qquad \square$$

The bound in Theorem 13 is tight, which can be seen by attaching the smallest possible 3-regular graph to every leaf of the graph in Fig. 6. The resulting graph, shown in Fig. 7, is defined for $n \equiv 16 \bmod 18$. There are $\frac{n-7}{9}$ black vertices, inducing $\frac{4n-10}{18}$ odd components. Hence, any matching has at least $\frac{4n-10}{18} - \frac{n-7}{9} = \frac{n+2}{9}$ unmatched vertices and therefore at most $\frac{8n-2}{9}$ matched vertices.

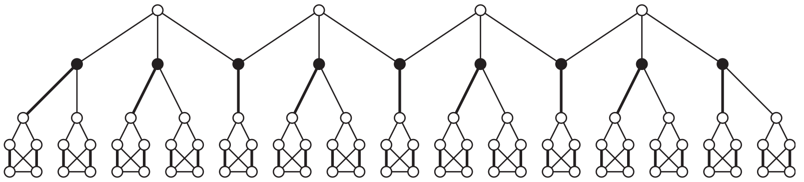


Fig. 7. A 3-regular graph of order $n = 88$ with a maximum matching of size $\frac{4n-1}{9} = 39$ (bold edges).

5. Conclusions

In this paper, we studied bounds on the size of maximal and maximum matchings in special graph classes, in particular 3-connected planar graphs, graphs with maximum degree k, graphs with maximum degree 3, and 3-regular graphs. We obtained lower bounds on the size of such matchings, and showed that the bounds are tight for some graph within the class. We leave a number of open problems:

- How quickly can we find matchings that are known to exist? A maximal matching can be found in linear time, but can we, say, find a matching of size $\frac{2n+4-\ell_4}{4}$ in a 3-connected planar graph in less than $O(m\sqrt{n}) = O(n^{1.5})$ time?
- What can be said about the size of maximum matchings in graphs with maximum degree k, for some fixed $k \geqslant 4$? Can we obtain a bound better than $\frac{m}{2k-1}$?
- Is there a graph with maximum degree 3 for which a maximum matching has size $\frac{3n-n_2-2\ell_2}{6}$ and which has a significant number of vertices of degree 2? Or if not, can we show a better bound?

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