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HYPER-IRREDUCIBILITY IN AN ORTHOMODULAR LATTICE

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1. Summary

In (1) D. E. Catlin generalizes the concept of "atomic bisection property" to non-atomic orthomodular lattices. He replaces a condition on pairs of atoms by a condition on comparable non-distinguished ($\neq 0, 1$) elements of L. We replace his condition by a seemingly more restrictive condition on distinct non-distinguished elements of L which are not orthocomplements of one another. This latter condition (which turns out to be equivalent to Catlin's) simply states that the blocks (maximal Boolean suborthomodular lattices) of L "separate the elements of L". We give an example to show that the condition "The blocks separate the atoms of L" is not equivalent to separation of elements of L.

Undefined terms and symbols are defined in (2). Throughout this paper L denotes an orthomodular lattice; seemingly homeless elements belong to L.

2. Main Theorem

Definition. Following Catlin we define L to be *hyper-irreducible* (HI) in case $L \neq 2^2$ and $0 < e < f < 1$ implies there exists $g \in L$ such that $e \, C \, g, f \not C \, g$. We say that the set of blocks of L, $\mathcal{B}_L$, *separates the elements of L* in case $0 < e, f < 1$, $e \neq f$, $e \neq f'$ implies there exists $B \in \mathcal{B}_L$ such that $e \in B$ and $f \notin B$. If L is atomic, then we say that $\mathcal{B}_L$ *separates the atoms of L* in case, for any two distinct atoms a, b of L such that $a \neq b'$, there exists $B \in \mathcal{B}_L$ such that $a \in B$ and $b \notin B$.

Lemma. L is hyper-irreducible if and only if $L \neq 2^2$ and $0 < e < f < 1$ implies there exists $B \in \mathcal{B}_L$ such that $e \in B$ and $f \notin B$.

Proof. Let L be hyper-irreducible. If $0 < e < f < 1$, then there exists $g \in L$ such that $e \, C \, g$ and $f \not C \, g$. Extend $\{e, g\}$ to a maximal family B of mutually commuting elements of L. Then $B \in \mathcal{B}_L$, $e \in B$ and $f \notin B$. Conversely, let $0 < e < f < 1$, then there exists $B \in \mathcal{B}_L$, such that $e \in B$

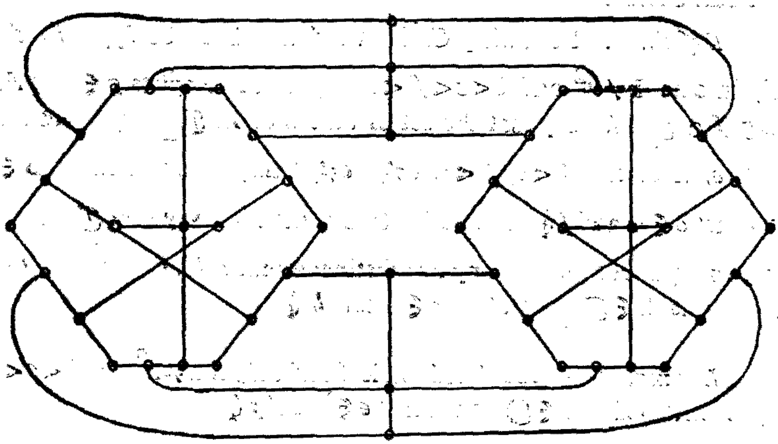
and $f \notin B$; by the maximality of B there exists $g \in B$ such that $f \not\subset g$. But $e \, C \, g$ since $e, g \in B$.

Theorem. If $L \neq 2^2$, then L is hyper-irreducible if and only if $\mathcal{B}_L$ separates the elements of L.

Proof. Necessity. Let $0 < e, f < 1$, $e \neq f$, and $e \neq f'$. In case $e \not\subset f$, let B be any block containing e; then $e \in B$ and $f \notin B$. In case $e \, C \, f$, by the above Lemma, we may assume that $e \not\leq f$. But $e = (e \wedge f) \vee (e \wedge f')$ so that $e \wedge f' > 0$ and $1 > (e \wedge f')' = e' \vee f > e' > 0$. Hence, by hyper-irreducibility, there exists $B \in \mathcal{B}_L$ such that $e' \in B$ and $e' \vee f \notin B$; consequently $e \in B$ and $f \notin B$. The sufficiency is clear.

3. Counterexample.

Since separating the elements of L ($\neq 2^2$) by blocks is equivalent to hyper-irreducibility and since, in the atomic case, the latter condition is equivalent to the atomic bisection property, it seems reasonable to conjecture that hyper-irreducibility in an atomic lattice L is equivalent to separating atoms of L by blocks. The following example proves this conjecture false. (We utilize the notation expounded in (3); Theorem 3 of that paper shows that our counterexample is in fact an orthomodular lattice.)



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