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On the Structure of Orthomodular Lattices Satisfying the Chain Condition

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ABSTRACT

Beginning with the external point of view we show how orthomodular lattices may be "pasted" together to yield a new orthomodular lattice. Changing to the internal point of view we show that any two *blocks* (maximal Boolean suborthomodular lattices) of an orthomodular lattice satisfying the chain condition can be "connected" by blocks which intersect in a specific fashion. Returning to our initial point of view we obtain a method of constructing a given orthomodular lattice from Boolean lattices.

1. INTRODUCTION

In this paper[1] we give a method for constructing "new" orthomodular lattices from "old" by "pasting" together isomorphic parts of the "old" ones. Our method generalizes that of MacLaren's construction of the horizontal sum of two orthomodular lattices [6], it extends the results of [3] which in turn generalizes the procedure given in [2], and it allows us to exhibit many examples of pristine orthomodular lattices (we shall do this in a later paper). In the case in which the two "old" lattices are complete Boolean lattices we obtain conditions which yield the idea behind a structure theorem for orthomodular lattices satisfying the chain condition.

Roughly speaking, the theorem reads as follows: Let L be an orthomodular lattice which satisfies the chain condition. Then corresponding to any two maximal Boolean suborthomodular lattices B_1 and B_2 (henceforth called blocks) of L is a finite sequence of blocks beginning at B_1 and ending at B_2 such that the intersection of any two consecutive blocks

[1] Section 3 of this paper essentially appears in the author's 1966 Ph. D. dissertation submitted to the University of Florida and written under the guidance of Professor D. J. Foulis.

is the union of a principal ideal and its dual ideal intersected with the union of the two consecutive blocks.

An immediate corollary of this theorem is a construction procedure, whereby any orthomodular lattice satisfying the chain condition may be built out of Boolean lattices by "pasting" together isomorphic ideals and dual ideals.

We wish to express our indebtedness to Professor D. J. Foulis for his interest and encouragement in the course of our research.

2. Definitions

Recall that an *orthomodular lattice* is a lattice with zero and unit elements such that there exists a mapping $' : L \to L$ having the following properties: for all $x, y \in L$

(1) $x'' = x$,

(2) if $x \leqslant y$, then $y' \leqslant x'$,

(3) $x \vee x' = 1$ and $x \wedge x' = 0$,

(4) if $x \leqslant y$, then $y = x \vee (y \wedge x')$.

If L satisfies all the above properties except possibly (4), then L is called an *orthocomplemented lattice*.

A sublattice M of an orthomodular lattice L is said to be a *suborthomodular lattice* of L in case the restriction of the orthocomplementation on L makes M an orthomodular lattice. A suborthomodular lattice M of an orthomodular lattice L is called *subcomplete* in case $N \subset M$ and sup N exists as computed in L implies sup N is in M.

Let L be a lattice, and let $N \subset L$, and let $a, b \in L$; then by $N[a, b]$ we mean $\{z \in N : a \leqslant z \leqslant b\}$ and by $U(N)$ we mean $\{z \in L : x \leqslant z$ for all $x \in N\}$. By an *order ideal* (resp., *order filter* or *dual order ideal*) of a lattice L we mean a non-empty subset I (resp., F) of L such that if $y \in I$ and $x \leqslant y$ then $x \in I$ (resp., $x \in F$ and $x \leqslant y$ then $y \in F$). An order ideal I is *principal* in case $I = L[0, x]$ for some $x \in L$. A *principal order filter* is defined dually.

A subset S of an orthocomplemented lattice L is said to be a *section* of L in case $S = I \cup F$ where I is an order ideal of L, and $F = \{x \in L : x' \in I\}$. (Note that F is an order filter of L.) If $S = L[x, 1] \cup L[0, x']$, then S is called a *principal section* of L and S is denoted by S_x. (Unless otherwise mentioned, the subscripts "1" and "2" in "S_1" and "S_2" are only indices and do not connote principality.)

3. The Paste Job

CONVENTION 3.1. In what follows we assume that $(L_1, \leqslant_1, {}^{\#})$ and $(L_2, \leqslant_2, {}^{+})$ are two disjoint orthomodular lattices, that S_i is a proper suborthomodular lattice of $L_i (i = 1, 2)$, and that there exists an order orthoisomorphism $\theta : S_1 \to S_2$.

DEFINITION 3.2.

(1) Let $L_0 = L_1 \cup L_2$.
(2) Let $P_1 = \{(x, y) \in L_0 \times L_0 : y = x\theta\}$.
(3) Let $\Delta = \{(x, x) : x \in L_0\}$.
(4) Let P be the equivalence relation defined by $P = \Delta \cup P_1 \cup P_1^{-1}$ where $P_1^{-1} = \{(y, x) : (x, y) \in P_1\}$.
(5) Let $L = L_0/P$.
(6) For $i = 1, 2$, let $R_i = \{([x], [y]) \in L \times L$: there exist $x_i \in [x]$ and $y_i \in [y]$ such that $x_i <_i y_i\}$.
(7) Let $\leqslant$ be the relation $(R_1 \cup R_2)^2$.
(8) Define $[0]$ to be $[0_1]$ and $[1]$ to be $[1_1]$ where 0_1 and 1_1 are the zero and unit elements of L_1.
(9) Define $' : L \to L$ by the following prescription:

$$\text{for} \quad [x] \in L, [x]' = \begin{cases} [x_1{}^{\#}], & \text{if there exists } x_1 \in L_1 \text{ such that } x_1 \in [x], \\ [x_2{}^{+}], & \text{if there exists } x_2 \in L_2 \text{ such that } x_2 \in [x]. \end{cases}$$

(10) Two sections S_1 and S_2 are said to be *corresponding* sections of L_1 and L_2 in case there exists $M_i \subset S_i (i = 1, 2)$ such that $M_1\theta = M_2$ and $S_1 = \cup \{S_m{}^{\#} : m \in M_1\}$ and $S_2 = \cup \{S_m{}^{+} : m \in M_2\}$

NOTATION. (1) If $[x] \leqslant [y]$, then, by the definition of $\leqslant$, there exists $[z] \in L$ such that $([x], [z]) \in R_i$ and $([z], [y]) \in R_j$ where $i, j \in \{1, 2\}$. For each such $[z]$ we write $[z] : [x] \leqslant [y]$ and say that $[z]$ implements the comparability $[x] \leqslant [y]$.

(2) Let S denote the set of all elements of L having a representative in both L_1 and L_2.

(3) If $[x] \in S$, then we write $[x] = \{x_1, x_2\}$, meaning that $x_1 \in L_1$, $x_2 \in L_2$, $x_1\theta = x_2$, and $[x] = [x_1] = [x_2]$. If $[x]$ is a singleton, then $[x] = \{x_1\}$ or $[x] = \{x_2\}$ simply denotes "the address" of $[x]$'s unique representative.

LEMMA 3.3. *Let S_1 and S_2 be corresponding sections of L_1 and L_2. If $[x], [y] \in L$ are such that $[x] = \{x_1\}$ and $[y] = \{y_2\}$ then $[x] \nleqslant [y]$.*

PROOF: Suppose that the statement is false; then there exists $[z]$ such that $[z] : [x] < [y]$, $[x] = \{x_1\}$, and $[y] = \{y_2\}$. It follows that $[z] = \{z_1, z_2\}$, $x_1 <_1 z_1$, $z_2 <_2 y_2$, $z_1 \in S_1$, and $z_2 \in S_2$. Since S_1 is a section of L_1, $L_1[0, z_1] \subset S_1$ or $L_1[z_1, 1] \subset S_1$. If $L_1[0, z_1] \subset S_1$, then $x_1 \in S_1$, contradicting the fact that $[x] = \{x_1\}$. If $L_1[z_1, 1] \subset S_1$, then $L_2[z_2, 1] \subset S_2$ (since S_1 and S_2 are corresponding sections), contradicting the fact that $[y] = \{y_2\}$. Therefore no such $[z]$ exists and $[x] \not\leqslant [y]$.

THEOREM 3.4. *Let S_1 and S_2 be corresponding sections of L_1 and L_2. Let L_i be complete and let S_i be subcomplete $(i = 1, 2)$. Then L is a complete orthomodular lattice.*

PROOF: It is clear that L is an orthocomplemented poset with zero and unit elements $[0]$ and $[1]$, respectively. To show that L is a lattice, we need only show that the join of any two elements of L exists. Let $[e], [f] \in L$. If, for a fixed $i \in \{1, 2\}$, there exists $e_i \in [e]$ and $f_i \in [f]$, then the fact that θ is a lattice isomorphism between the suborthomodular lattices S_i of L_i yields the result that $[e] \vee [f]$ exists and equals $[e_i \vee_i f_i]$. By symmetry the only case we need consider is the one in which $[e] = \{e_1\}$ and $[f] = \{f_2\}$. In this case, as an immediate consequence of Lemma 3.3, we have $[1] \in U([x], [y]) \subset S$. Let $M_1 = \{z_1 :$ there exists $[z] \in U([x], [y])$ such that $[z] = \{z_1, z_2\}\}$. Now $\inf_1 M_1$ exists since L_1 is complete. But since S_1 is subcomplete, $\inf M_1$, as computed in S_1, exists and equals $\inf_1 M_1$. Let $z^{(1)} = \inf_1 M_1$; let $z^{(2)} = z^{(1)}\theta$; let $[z]_0 = [z^{(1)}] = [z^{(2)}]$, and let $M_2 = \{z_2 :$ there exists $[z] \in U([x], [y])$ such that $[z] = \{z_1, z_2\}\}$. Then $z^{(2)} = \inf_2 \{z_1\theta : z_1 \in M_1\} = \inf_2 M_2 = \inf M_2$, as computed in S_2 since S_2 is subcomplete. It follows that $U([z]_0) = U([x], [y])$. Consequently L is an orthomodular lattice.

To show that L is complete, let M be any subset of L. Let $N = \{[x] \in M:$ there exists $x_1 \in [x]\}$, let $n = \sup_1 \{x : x \in L_1$ and $[x] \in N\}$ (the supremum exists since L_1 is complete), let $P = M - N$, let $p = \sup_2 \{x : x \in L_2$ and $[x] \in P\}$ (the supremum exists since L_2 is complete). It is easy to see that $\sup N$ exists and equals $[n]$, that $\sup P$ exists and equals $[p]$, and that $\sup M$ exists and equals $[n] \vee [p]$. Hence L is complete.

We conclude the proof by showing that L satisfies the orthomodular identity. Let $[e], [f] \in L$ be such that $[e] < [f]$. By Lemma 3.3 and by symmetry we need only check the case in which both $[e]$ and $[f]$ have a representative in, say, L_1. In this case

$$\begin{aligned}[e] \vee ([e] \vee [f]')' &= [e_1] \vee ([e_1] \vee [f_1^{\#}])' = [e_1] \vee [e_1 \vee_1 f_1^{\#}]' \\ &= [e_1] \vee [e_1^{\#} \wedge_1 f_1] = [e_1 \vee_1 (e_1^{\#} \wedge_1 f_1)] = [f_1] = [f].\end{aligned}$$

If S_i is not a section of L_i, then L need not be a lattice. For example, let L_1 and L_2 be disjoint "copies" of 2^4, the Boolean lattice of all subsets of a four-element set X partially ordered by set-theoretic inclusion. Let S_i be the sublattice of L_i corresponding to $\{\varphi, X, M, X - M\}$ where M is a two-element set. Then S_i is not a section of L_i ($i = 1, 2$). There exist elements $a_1 \in L_1$, $b_2 \in L_2$ such that a_1 corresponds to a singleton subset of M, and b_2 corresponds to a singleton of $X - M$, and $[a_1] \vee [b_2]$ does not exist. Hence L is not a lattice.

The following example illustrates the fact that, if the suborthomodular lattice S_i is not a subcomplete suborthomodular lattice of the complete orthomodular lattice L_i ($i = 1, 2$), then L need not be a lattice. Let L_1 be the power set of an infinite set M, and let S_1 be the suborthomodular lattice of L_1 consisting of all finite or cofinite subsets of L_1. (Recall that a cofinite subset of M is a subset of M whose complement in M is finite.) Let L_2 be a disjoint "copy" of L_1. Then there is a natural orthoisomorphism $\varphi : L_1 \to L_2$. Let $\varphi = \varphi|_{S_1}$ and let $S_2 = S_1\theta$. Then S_i, L_i, and L have the required properties.

4. Structure Theorem

Definition 4.1. Let L be an orthomodular lattice. As in [1] we define a relation C on L by eCf in case $(e \vee f') \wedge f = e \wedge f$. If eCf, then we say that *e commutes with f*. For $M \subset L$, we define $C(M) = \{e \in L : eCf$ for all $f \in M\}$. By $CC(M)$ we mean $C(C(M))$. The set $C(L)$ is called the *center* of L. Note that $\{0, 1\} \subset C(L) \subset L$ always holds. If $\{0, 1\} = C(L)$, then L is said to be irreducible; if $\{0, 1\} \subsetneq C(L)$, then L is said to be reducible; if $C(L) = L$, then L is said to be a *Boolean lattice* (*Boolean algebra*).

By a *block* of an orthomodular lattice L we mean a maximal Boolean suborthomodular lattice of L. The set of all blocks of L is denoted by $\mathscr{B}_L$.

Lemma 4.2. *Let L be an orthomodular lattice and let $M, N \subset L$. Then:*

(1) *$C(M)$ is a subcomplete suborthomodular lattice of L.*

(2) *If $M \subset N$, then $C(N) \subset C(M)$.*

(3) *$M \subset CC(M)$*

(4) *$C(M) = C(CC(M))$.*

(5) *$M = CC(M)$ if an only if there exists $N \subset L$ such that $M = C(N)$*

(6) *Let M be a suborthomodular lattice of L. Then T.A.E.*

(a) *M is a Boolean lattice,*
(b) $M \subset C(M)$,
(c) $CC(M) \subset C(M)$.

(7) *If $M \subset C(M)$, then $CC(M)$ is a subcomplete Boolean subortho-modular lattice of L containing M.*

PROOF: (1) is essentially proved in [1, Lemma 3, p. 67]. (2) and (3) follow immediately from Definition 4.1. (4), (5), and (6) are consequences of (2) and (3). (7) follows from (1), (4), and (6).

Since any commuting family of elements of an orthomodular lattice may be extended to a maximal commuting family, it follows that $\cup \mathscr{B}_L = L$. Hence a theorem which shows how the blocks intersect could be interpreted as a structure theorem. The clue as to how the blocks intersect is provided by the following Remark, which is included here for the purpose of motivating Theorem 4.6 and whose proof is omitted because of its length.

REMARK. If L is a complete orthomodular lattice and if $L = B_1 \cup B_2$ where $B_1, B_2 \in \mathscr{B}_L$, then $\mathscr{B}_L = \{B_1, B_2\}$ and $B_1 \cap B_2 = S_e$ for some $e \in L$.

This remark suggests the naive conjecture that any pair of blocks B_1, B_2 of (even) a finite orthomodular lattice have an intersection of the form $S_e \cap (B_1 \cup B_2)$ for some $e \in B_1 \cup B_2$. The conjecture is false. (An example may be constructed by several applications of Theorem 3.4.) In this section we prove that for any pair of blocks of an orthomodular lattice satisfying the chain condition there is a finite sequence of blocks which "connect" the two and which intersect in the suggested fashion.

LEMMA 4.3. *Let B_1 and B_2 be any two Boolean suborthomodular lattices of the orthomodular lattice L. If $e \in B_1 \cap B_2 - \{0, 1\}$, then a necessary and sufficient condition for $B_1 \cap B_2 = S_e \cap (B_1 \cup B_2)$ is that e be an atom of $B_1 \cap B_2$ and $B_1[0, e'] = B_2[0, e']$.*

PROOF: The simple proof is omitted.

LEMMA 4.4. *Let B be any subset of the orthomodular lattice L. Then:*

(1) *$B \in \mathscr{B}_L$ if and only if $B = C(B)$.*
(2) *if $B \in \mathscr{B}_L$ and a is an atom of B, then a is an atom of L.*

PROOF. AD 1: If $B \in \mathscr{B}_L$, then $B \subset C(B)$; moreover $x \in C(B)$ implies $x \in B$ since B is a maximal family of commuting elements of L. Hence $B = C(B)$. If $B = C(B)$ then by, Lemma 4.2, $B = C(B) = CC(B)$ is a subcomplete Boolean suborthomodular lattice of L. If $B \subset B'$ for some $B' \in \mathscr{B}_L$, then $B' = C(B') \subset C(B) = B$ and hence $B = B' \in \mathscr{B}_L$.

AD 2: If there is an atom a of B which is not an atom of L, then there exists an element b in L such that $0 < b < a$. But $x \in B$ implies $a < x$ or $a < x'$, so that $b < x$ or $b < x'$. Hence $b \in C(B) = B$, contradicting the fact that a is an atom of B.

LEMMA 4.5. *Let L be an orthomodular lattice, let $\{e_\alpha : \alpha \in I\}$ be a maximal orthogonal family of non-zero elements of L, let $\{B_\alpha : \alpha \in I\}$ be a collection of atomic blocks of L such that $e_\alpha \in B_\alpha$ for all $\alpha \in I$, let $M = \cup \{B_\alpha[0, e_\alpha] : \alpha \in I\}$, let $B = C(M)$, and let A be the set of all atoms in L. Then:*

(1) *$M \cap A$ is a maximal family of mutually orthogonal atoms of L.*

(2) *$B = C(M) = CC(M) = CC(M \cap A) = C(M \cap A)$,*

(3) *$B \in \mathscr{B}_L$,*

(4) *B is atomic,*

(5) *The atoms of B are atoms of L.*

PROOF. AD 1: $M \cap A$ is clearly a non-empty family of mutually orthogonal atoms of L. If it is not maximal, then there exists $b \in A - M$ such that $b < a'$ for all $a \in M \cap A$. Hence $b < e_\alpha'$ for all $\alpha \in I$ so that $b < \inf \{e_\alpha' : \alpha \in I\} = 0$, which is a contradiction.

AD 2: $C(M) \subset C(M \cap A)$ since $M \cap A \subset M$. Since every element in M is the join of elements in $M \cap A$, $C(M \cap A) \subset C(M)$. Hence $B = C(M) = C(M \cap A)$ and consequently $CC(M) = CC(M \cap A)$. We need only show that $C(M \cap A) = CC\ M \cap A)$. But $M \cap A \subset C(M \cap A)$ implies $CC(M \cap A) \subset C(M \cap A)$. To show the reverse inclusion, let $x, y \in C(M \cap A)$, let $K_1 = \{a \in M \cap A : a < x\}$, and let $K_2 = \{a \in M \cap A : a < x'\}$. Then $K_1 \cup K_2 = M \cap A$ and $K_1 \cap K_2 = \varphi$. Moreover $x = \sup K_1$ by the maximality of $M \cap A$. Hence xCy by [1, Lemma 2, p. 67]. Consequently $x, y \in C(M \cap A)$ implies xCy, i.e., $C(M \cap A) \subset CC(M \cap A)$.

AD 3: By Part 2, $B = C(B)$ so that $B \in \mathscr{B}_L$ by Lemma 4.4.

AD 4: If $x \in B = C(M \cap A)$, then, as in Part 2, x is the join of elements in $M \cap A$. Hence B is atomic.

AD 5: This follows from Part 2 of Lemma 4.4.

THEOREM 4.6. *Let L be an orthomodular lattice. Let B_1 and B_2 be atomic* *blocks of L such that* $^{\#}(B_1 \cap B_2) < \infty$, *let* $e_1, e_2, \ldots, e_n$ *be the distinct atoms of* $B_1 \cap B_2$, *and for* $1 \leqslant k \leqslant n+1$, *let*

$$B^k = C\left(B_2\left[0, \bigvee_{i=1}^{k-1} e_i\right] \cup B_1\left[0, \bigvee_{i=k}^{n} e_i\right]\right).$$

Then $B^1 = B_1$, $B^{n+1} = B_2$, *each* $B^k \in \mathscr{B}_L$, *and for* $1 \leqslant k \leqslant n$,

$$B^k \cap B^{k+1} = S_e \cap (B^k \cup B^{k+1}).$$

Proof: By Lemma 5.5, parts 2 and 3, $B^1 = B_1$ and $B^{n+1} = B_2$ and each $B^k \in \mathscr{B}_L$. For $1 \leqslant k \leqslant n$,

$$B^k[0, e_k] \cap B^{k+1}[0, e_k] = B_1[0, e_k] \cap B_2[0, e_k] = \{0, e_k\}.$$

Hence e_k is an atom of $B^k \cap B^{k+1}$. Moreover, since $B^k[0, e_j] = B^{k+1}[0, e_j]$ whenever $j \neq k$, it follows that

$$B^k\left[0, \bigvee_{j \neq k} e_j\right] = B^{k+1}\left[0, \bigvee_{j \neq k} e_j\right];$$

hence $B^k[0, e_k'] = B^{k+1}[0, e_k']$. The result follows by Lemma 4.3.

Definition 4.7. An orthomodular lattice L is said to satisfy the ascending *chain condition* in case every chain (i.e., linearly ordered set) in L is of finite cardinality.

Remark. If an orthomodular lattice L satisfies the chain condition, then each block of L is a finite Boolean lattice. Hence each block is atomic and the intersection of any two blocks is atomic.

Corollary 4.8. *If L is an orthomodular lattice which satisfies the chain condition (in particular, if* $^{\#}L < \infty$*), and if for any two blocks* $B_1, B_2 \in \mathscr{B}_L$, *we let*

$$B^k = C\left(B_2\left[0, \bigvee_{i=1}^{k-1} e_i\right] \cup B_1\left[0, \bigvee_{i=k}^{n} e_i\right]\right),$$

where $e_1, e_2, \ldots, e_n$ *are the distinct atoms of* $B_1 \cap B_2$, *then the conclusion of Theorem* 4.6 *obtains.*

We now change from the internal to the external point of view in order to describe a procedure whereby any orthomodular lattice satisfying the chain condition may be constructed from Boolean lattices by "pasting" together only corresponding principal sections.

REMARK 4.9. Let L satisfy the chain condition and let $\mathscr{C}$ be a disjointification of $\mathscr{B}_L$, i.e., a collection of disjoint Boolean lattices which is in one-to-one correspondence with $\mathscr{B}_L$ in such a way that corresponding lattices are isomorphic. Then L may be constructed by "pasting" together the lattices of $\mathscr{C}$ in the following way:

For each pair C_1, C_2 of lattices in $\mathscr{C}$, Theorem 5.4 singles out a finite sequence of elements of $\mathscr{B}_L$. The intersections of consecutive lattices of this distinguished sequence are principal sections in their respective unions. Now "paste" together the corresponding principal sections of the lattices in the corresponding sequence in $\mathscr{C}$. Do this for each pair C_1, C_2 of lattices in $\mathscr{C}$. (We assume that the identity of each $C \in \mathscr{C}$ is not destroyed by any previous "pasting" so that we can distinguish it at any future moment for necessary "pasting." It may be clearer to think not of "pasting" two elements together but rather of "attaching a string" to connect corresponding elements; then, after all the "strings have been attached", "paste" together all elements connected by a chain of strings.)

Then any two elements of $\cup\mathscr{C}$ which correspond to the same element in L will be pasted together (this follows from the fact that $B_1 \cap B_2 = \cap\{B^i : i = 1, 2, \ldots, n + 1\}$), all comparabilities in L will be realized (since each comparability in L appears in at least one block of L), and no superfluous comparabilities will be introduced (because only comparabilities found in blocks appear).

At each step only principal sections (in some union) are pasted together. If L is finite, then so is $\mathscr{B}_L$ and hence $\mathscr{C}$, so that the process takes only a finite number of steps.

Examples of lattices illustrating the block gestalt with which we have been concerned are given in [5] together with an expository account of the paste job described in Section 3.

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