Volume I Number 2 July, 1969

An Orthomodular Poset With a Full Set of States Not Embeddable in Hilbert Space

by

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1. Introduction

It is the purpose of this paper to disprove the following conjecture: Every orthomodular poset which admits a full set of states may be embedded in the lattice of all closed subspaces of a complex Hilbert space.

The conjecture derives its importance from the fact that orthomodular posets appear to be the proper setting for quantum logic [2]. Moreover, Arlan Ramsay's (unpublished) result that the horizontal sum of 2^3 and 2^2 may be embedded in Hilbert space indicates that even "pathological" orthomodular posets may be so embedded. However, the existence of orthomodular posets which admit no states [1] and the fact that any suborthomodular poset of Hilbert space admits a full set of states shows that there exist orthomodular posets which are not embeddable in Hilbert space. At the same time these two facts yield the above conjecture.

We present a counterexample to the conjecture by exhibiting an orthomodular poset with a full set of states such that the states are not strongly order determining [3]. An immediate corollary of the existence of such a poset is that the two orderings most frequently utilized in empirical logics may be distinct. A state α may be interpreted as a probability measure; if "fullness" determines the order, then $x \leq y$ is interpreted to mean that the probability that the event x occurs is less than or equal to the probability that the event y occurs; if the states are strongly order determining then the interpretation of $x \leq y$ is: if x occurs with certainty, then y occurs with certainty.

2. Definitions

By an orthomodular poset we mean a partially ordered set P with 0 and 1 together with an orthocomplementation $':P \to P$ such that $x'' = (x')' = x$, $x \leq y$ implies $y' \leq x'$, $x \vee x'$ exists and equals 1, $x \wedge x'$ exists and equals 0, $x \leq y'$ implies $x \vee y$ exists, and finally

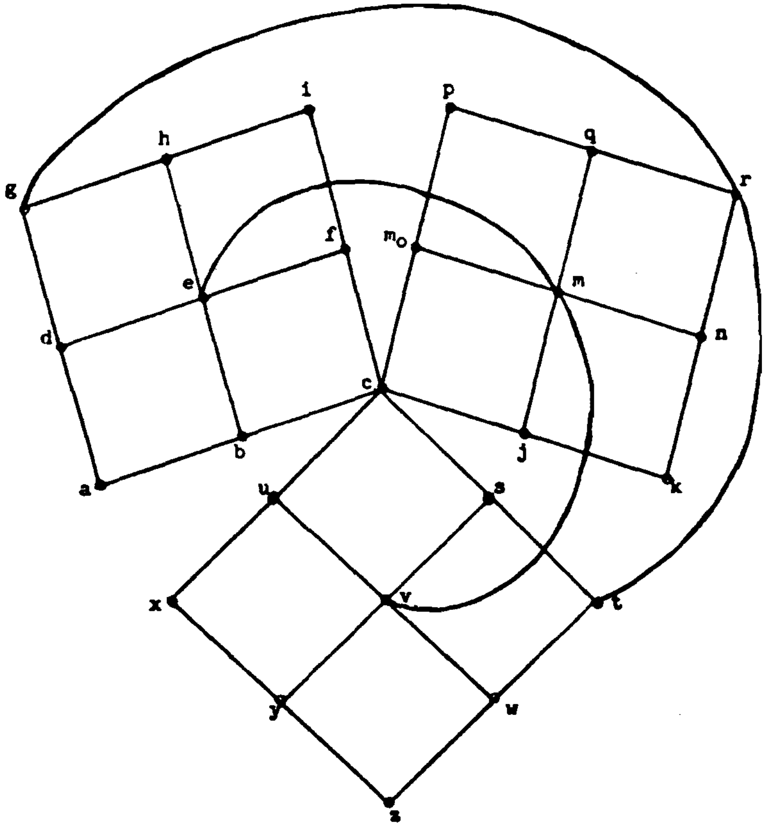
$$x \leq y \quad \text{implies} \quad y = x \vee (y' \vee x)'.$$

Given two orthomodular posets P and Q, we say that P is embeddable in Q if there exists an injection $\phi:P \to Q$ which preserves the orthocomplementation, and all existing joins (and therefore the ordering and all existing meets) such that ϕ^{-1} preserves the ordering.

By a state on an orthomodular poset P we mean a mapping $\alpha:P \to [0,1]$ such that $0\,\alpha = 0$, $1\,\alpha = 1$ and $x \leq y'$ implies $(x \vee y)\,\alpha = x\,\alpha + y\,\alpha$. The set S_P of all states on P is said to be full in case $x \leq y$ if and only if $x\,\alpha \leq y\,\alpha$ for all $\alpha \in S_P$. The set S_P is strongly order determining in case $x \leq y$ if and only if for all $\alpha \in S_P$, $x\,\alpha = 1$ implies $\alpha = 1$.

Note that if S_P is strongly order determining then S_P is full. Moreover, every suborthomodular poset of the closed subspaces of a Hilbert space has a strongly order determining set of states [3]. The poset G_{52}, given in Figure 1, satisfies the weaker condition but not the stronger and therefore provides the desired counterexample.

Figure 1. G_{52}.



The notation is that presented in [1]. By Theorem 1 of [1], G_{52} is an orthomodular poset (all the blocks of G_{52} are isomorphic to 2^3).

Lemma. $S_{G_{52}}$ is not strongly order determining.

Proof. Noting that $c \nleq w$, we prove that $c\,\alpha = 1$ implies $w\,\alpha = 1$. If $c\,\alpha = 1$, then $a\,\alpha = b\,\alpha = f\,\alpha = i\,\alpha = m_o\alpha = p\,\alpha = j\,\alpha = k\,\alpha = s\,\alpha$

$t\alpha = u\alpha = x\alpha = 0$. Let $e\alpha = \epsilon$, $0 \le \epsilon \le 1$. Then $h\alpha = 1 - \epsilon$ so that $g\alpha = \epsilon$. Since $t\alpha = 0$ it follows that $r\alpha = 1 - \epsilon$, $q\alpha = \epsilon$ and therefore $m\alpha = 1 - \epsilon$. Since $e\alpha = \epsilon$ and $m\alpha = 1 - \epsilon$, we have $v\alpha = 0$; but $u\alpha = 0$ so $w\alpha = 1$.

Lemma. G_{52} admits a full set of states.

Proof. We need only exhibit enough states in order to satisfy the following condition:

(I) $x \nleq y$ implies there exists $\alpha \in S_{G_{52}}$ such that $x\alpha > y\alpha$.

Table 1 lists all of the dispersion free states, i.e., the states whose range is $\{0,1\}$. Table 2 lists selected states needed to fill in Tables 3 and 4. In Tables 3 and 4 the symbol "<" in the i^{th} row, j^{th} column indicates that there is a dispersion free state α such that $i\alpha < j\alpha$; a "____" indicates that $j \le i$ so that we cannot expect to find such an α; the symbol β_k in the i^{th} row, j^{th} column indicates that there is no dispersion free state such that $i\alpha < j\alpha$ but that $i\beta_k < j\beta_k$, where β_k is the k^{th} state in Table 2.

Table 1. Dispersion Free States

	a	b	c	d	e	f	g	h	i	j	k	m_o	m	n	p	q	r	s	t	u	v	w	x	y	z
α_1	1	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
α_2	1	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	1	1	0	0	0	1	1	0	0
α_3	1	0	0	0	1	0	0	0	1	0	1	1	0	0	0	1	0	0	1	1	0	0	0	1	0
α_4	1	0	0	0	1	0	0	0	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0	1	0
α_5	1	0	0	0	0	1	0	1	0	0	1	1	0	0	0	1	0	0	1	0	1	0	1	0	0
α_6	1	0	0	0	0	1	0	1	0	1	0	0	0	1	1	0	0	0	1	0	1	0	1	0	0
α_7	1	0	0	0	0	1	0	1	0	0	1	0	1	0	1	0	0	0	1	1	0	0	0	1	0
α_8	0	1	0	1	0	0	0	0	1	0	1	1	0	0	0	1	0	0	1	0	1	0	1	0	0
α_9	0	1	0	1	0	0	0	0	1	1	0	0	0	1	1	0	0	0	1	0	1	0	1	0	0
α_{10}	0	1	0	1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	1	1	0	0	0	1	0
α_{11}	0	1	0	0	0	1	1	0	0	0	1	0	1	0	1	0	0	1	0	1	0	0	0	0	1
α_{12}	0	1	0	0	0	1	1	0	0	0	1	0	1	0	1	0	0	1	0	0	0	1	1	0	0
α_{13}	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	1	0
α_{14}	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0

Table 2. Some other states

	a	b	c	d	e	f	g	h	i	j	k	m_o	m	n	p	q	r	s	t	u	v	w	x	y	z
β_1	1/9	7/9	1/9	4/9	1/9	4/9	4/9	1/9	4/9	3/9	5/9	3/9	5/9	1/9	5/9	1/9	3/9	6/9	2/9	4/9	3/9	2/9	4/9	0	5/9
β_2	5/9	3/9	1/9	1/9	5/9	3/9	3/9	1/9	5/9	7/9	1/9	4/9	1/9	4/9	4/9	1/9	4/9	6/9	2/9	4/9	3/9	2/9	4/9	0	5/9
β_3	2/9	4/9	3/9	3/9	5/9	1/9	4/9	0	5/9	5/9	1/9	5/9	1/9	3/9	1/9	3/9	5/9	6/9	0	4/9	3/9	2/9	2/9	0	7/9
β_4	1/9	5/9	3/9	3/9	1/9	5/9	5/9	3/9	1/9	4/9	2/9	1/9	5/9	3/9	5/9	0	4/9	6/9	0	4/9	3/9	2/9	2/9	0	7/9
β_5	2/9	4/9	3/9	4/9	1/9	4/9	3/9	4/9	2/9	4/9	2/9	3/9	1/9	5/9	3/9	4/9	2/9	2/9	4/9	0	7/9	2/9	6/9	0	3/9
β_6	2/9	4/9	3/9	5/9	1/9	3/9	2/9	4/9	3/9	4/9	2/9	4/9	1/9	4/9	2/9	4/9	3/9	2/9	4/9	0	7/9	2/9	6/9	0	3/9
β_7	6/9	3/9	0	0	2/9	7/9	3/9	4/9	2/9	4/9	5/9	4/9	5/9	0	5/9	0	4/9	7/9	2/9	0	2/9	7/9	1	0	0
β_8	5/9	4/9	0	0	5/9	4/9	4/9	0	5/9	3/9	6/9	7/9	2/9	0	2/9	4/9	3/9	7/9	2/9	0	2/9	7/9	1	0	0
β_9	5/9	3/9	1/9	4/9	3/9	2/9	0	3/9	6/9	6/9	2/9	6/9	3/9	0	2/9	0	7/9	6/9	2/9	0	3/9	6/9	8/9	0	1/9
β_{10}	2/9	6/9	1/9	0	3/9	6/9	7/9	0	2/9	3/9	5/9	2/9	3/9	4/9	6/9	3/9	0	6/9	2/9	0	3/9	6/9	8/9	0	1/9
β_{11}	1/9	1/9	7/9	1/9	6/9	2/9	7/9	2/9	0	2/9	0	2/9	0	7/9	0	7/9	2/9	2/9	0	2/9	3/9	4/9	0	4/9	5/9
β_{12}	0	4/9	5/9	3/9	5/9	1/9	6/9	0	3/9	0	4/9	4/9	3/9	[illegible]/9	0	6/9	3/9	4/9	0	4/9	1/9	4/9	0	4/9	5/9
β_{13}	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3

Table 3.

	a	b	c	d	e	f	g	h	i	j	k	m_o	m	n	p	q	r	s	t	u	v	w	x	y	z
a	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
b	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
c	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
d	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
e	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
f	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
g	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
h	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
i	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
j	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
k	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<	<
m_o	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<	<
m	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<	<
n	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<	<
p	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<	<
q	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<	<
r	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	<	<
s	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	<	<	β_{12}
t	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	β_1	<	<	<	<
u	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<	β_1
v	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<	<
w	<	<	β_{12}	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<	<	<
x	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	β_{12}	<	·	<	<
y	<	<	β_1	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·	<
z	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	·

Table 4.

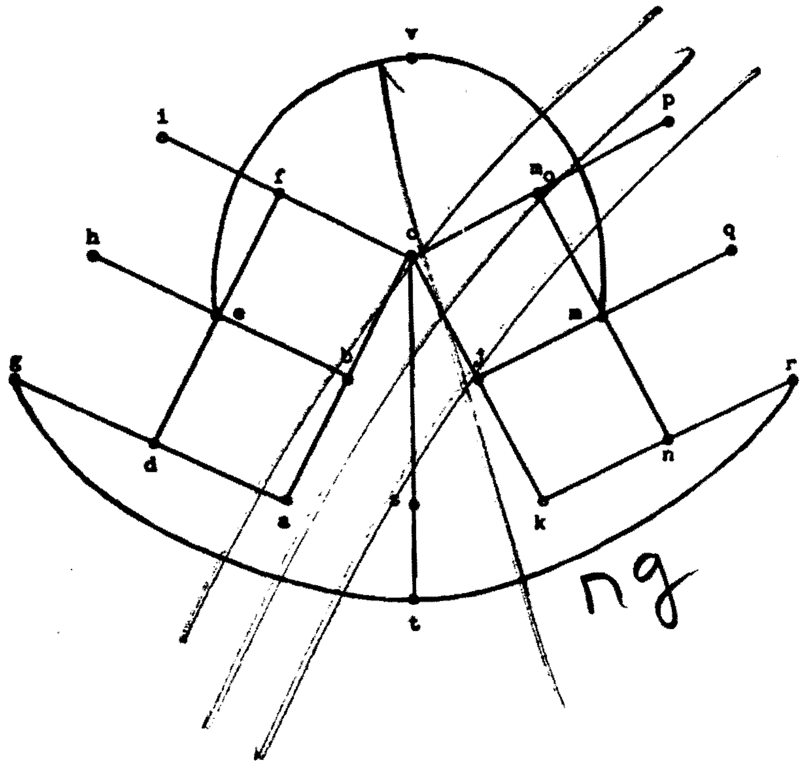
	a	b	c	d	e	f	g	h	i	j	k	m_0	m	n	p	q	r	s	t	u	v	w	x	y	z
a'	<	—	—	—	<	<	—	<	<	<	<	<	<	<	<	<	.	.	<	<	<	<	<	<	<
b'	—	<	—	<	—	<	<	—	<	<	<	<	<	<	<	<	β_1	<	<	<	<	<	<	<	<
c'	—	—	<	<	<	—	<	<	—	—	—	—	<	<	—	<	<	—	—	—	β_{11}	<	—	<	β_3
d'	—	<	<	<	—	—	—	<	<	<	<	<	<	<	<	<	<	β_1	<	<	<	<	<	<	β_3
e'	<	—	<	—	<	—	<	—	<	<	<	<	—	<	<	<	<	<	<	<	—	<	<	<	<
f'	<	<	—	—	—	<	<	<	—	<	<	<	<	<	<	<	β_7	<	<	<	<	<	<	<	<
g'	—	<	<	—	<	<	<	—	—	β_{10}	<	β_1	<	<	<	<	—	<	—	<	β_{10}	<	<	<	<
h'	<	—	<	<	—	<	—	<	—	<	<	<	<	<	<	<	<	β_7	<	<	<	<	<	<	β_4
i'	<	<	—	<	<	—	—	—	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<
j'	<	<	—	<	<	<	β[illegible]	<	<	<	—	<	—	<	<	—	<	<	<	<	<	<	<	<	<
k'	<	<	—	<	<	<	<	<	<	—	<	<	<	—	<	<	—	.	<	<	<	<	<	<	<
m_0'	<	<	—	<	<	<	β_1[illegible]	<	<	<	<	<	—	—	—	.	<	<	<	<	<	<	<	<	<
m'	<	<	<	<	—	<	<	<	<	—	.	—	.	—	.	—	<	<	<	<	—	<	<	<	<
n'	<	<	<	<	<	<	<	<	<	<	—	—	—	.	<	<	—	β[illegible]	<	<	<	<	<	<	β_3
p'	<	<	—	<	<	<	<	<	<	<	<	—	<	.	<	—	—	<	<	<	<	<	<	<	<
q'	<	<	<	<	<	<	<	<	<	—	<	<	—	<	—	<	—	β_1	<	<	<	<	<	<	β_3
r'	<	β_9	<	<	<	β_7	—	<	<	<	—	<	<	—	—	—	<	<	—	<	β_9	<	<	<	<
s'	<	<	—	β_1	<	<	<	β_7	<	<	<	<	<	β[illegible]	<	β_8	<	<	—	<	—	<	<	—	<
t'	<	<	—	<	<	<	—	<	<	<	<	<	<	.	<	<	—	—	<	<	<	—	<	<	—
u'	<	<	—	<	<	<	<	<	<	<	<	.	<	.	.	<	<	<	<	<	—	—	—	<	<
v'	<	<	β_5	<	—	<	β_5	<	<	<	<	.	—	<	<	<	β_6	—	<	—	<	—	<	—	β_3
w'	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	<	—	—	—	<	<	<	—
x'	<	<	—	<	<	<	<	<	<	<	<	.	.	.	.	<	<	<	<	—	<	<	<	—	—
y'	<	<	<	<	<	<	<	<	<	<	<	.	.	.	<	<	<	—	<	<	—	<	—	<	—
z'	<	<	β_3	β_3	<	<	<	β_4	<	<	<	.	.	β_1	<	β_3	<	<	—	<	β_3	—	—	—	<

Table 1 illustrates the fact that for every non-zero atom a of G_{52} there exist (dispersion free) states α, β such that a α = 1 and a β = 0. Hence this property obtains for every non-zero, non-unit element of G_{52}. Hence we may assume $0 < x < 1$ and $0 < y < 1$ in (I). Table 3 illustrates the fact that if x and y are both atoms (or both coatoms), then (I) obtains; Table 4 illustrates the fact that if x is a coatom and y is an atom then (I) obtains. If x is an atom and y is a coatom, then the state β_{13} provides the necessary inequality.

3. Conclusion.

The poset G_{52} has the property that if the atom c is mapped by a state α to 1, then the atom w is also mapped by α to 1. If we did not require that both elements be atoms, then v' would also satisfy the requirement for all states α, c α = 1 implies $(v')\alpha = 1$; but $c \nleq v'$. Moreover, the poset G_{42}, given in Figure 2, which is a suborthomodular poset of G_{52} (and hence admits a full set of states), would provide the necessary counterexample.

Figure 2. G_{42}.



We have proved the following:

Theorem. There exists an orthomodular poset admitting a full set of states which is not embeddable in the lattice $\mathcal{L}(H)$ of closed subspaces of any complex Hilbert space H.

We have not proved the following:

Conjecture 1. There exists an orthomodular lattice admitting a full set of states which is not embeddable in $\mathcal{L}(H)$ for any Hilbert space H.

Conjecture 2. If an orthomodular poset admits a strongly order determinin set of states, then it is embeddable in $\mathcal{L}(H)$ for some Hilbert space H.

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