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ON GENERATING PATHOLOGICAL ORTHOMODULAR STRUCTURES

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Introduction. By an orthomodular structure we mean an orthomodular poset or an orthomodular lattice. Viewed as a generalization of the prototypical orthomodular structures generated by contemporary functional analysis (e.g., the projection lattice of a von Neumann algebra), all of the structures cited in this paper are indeed pathological. Viewed as a general setting for recent work on empirical logic [16], most of the structures cited here are still pathological. But viewed as structures of a combinatorial interest, the examples presented indicate that the study of "orthomodularity" in structures of "small" cardinality may very well yield insight into as yet unformulated concepts pertinent to the theory. From this point of view we consider these structures to be anything but pathological.

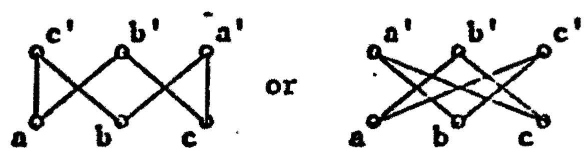
Essentially we depict an orthomodular structure as a union of simpler such structures, usually Boolean lattices, and investigate how the simpler structures are intertwined. From this point of view the Boolean lattices are trivial and the projection lattices quite complicated. Thus our goal is to exhibit some simple but non-trivial orthomodular structures. We will select our examples to be of maximal interest to quantum logicians.

I. Pasting Together Subsets of Two Structures

(1) Adjunction of a Crown to an Orthomodular Structure Having an Atom.

By a crown [5] we mean a copy of $2^3 \setminus \{0,1\}$, see Figure I.

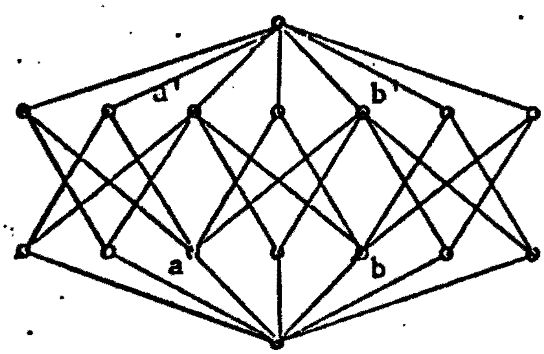
Figure I.



Let L be an orthomodular structure and let a be an atom of L; let C be a crown disjoint from L and let a_1 be a minimal element of C. By adjoining the crown C to L at a we mean the process of identifying a with a_1 and a' with a_1'. The resulting structure $\hat{L}$, with the induced ordering (for $0,1 \in L$, $x \in C$, define $0 \leq x \leq 1$) and orthocomplementation, is orthomodular; it is a lattice if L is a lattice, and only a poset if L is only a poset.

If b is an atom of $\hat{L}$ with $b \neq a$ and if we adjoin another crown D (disjoint from $\hat{L}$) to $\hat{L}$ at b, then we obtain a non-modular orthomodular (irreducible) structure $\hat{\hat{L}}$. If L were 2^3, then $\hat{\hat{L}}$ would be D_{16}[4], viz., Figure II.

Figure II.



(2) Pasting Orthomodular Posets to Obtain an Orthomodular Poset.

Definition: Recall that a sub-orthomodular poset R of an orthomodular poset Q is a non-empty subset of Q which satisfies the following conditions for all x, y $\in$ R:

(i) $x' \in R$,

(ii) $x \leq y$ in R if and only if $x \leq y$ in Q,

(iii) if $x \leq y'$ then $x \vee_R y$ exists and equals $x \vee_Q y$.

A non-empty subset S of an orthomodular poset Q is a <u>section</u> of Q in case S is a sub-orthomodular poset of Q and $S = I \cup F$ where I is an order ideal in Q and $F = \{x' : x \in I\}$. If S is a section and $S = Q[x,1] \cup Q[0,x']$ for some $x \in Q$, then S is called a principal section; in this case we write $S = S_x$.

<u>Convention</u>. Assume that $(Q_1, \leq_1, \#)$ and $(Q_2, \leq_2, +)$ are two disjoint orthomodular posets, $S_i = I_i \cup F_i$ is a proper section of $Q_i (i = 1,2)$, and that there exists an order ortho-isomorphism $\theta: S_1 \to S_2$ with $I_1\theta = I_2$.

<u>Definition</u>:

(1) Let $Q_0 = Q_1 \dot{\cup} Q_2$

(2) Let $P_1 = \{(x,y) \in Q_0 \times Q_0 | y = x\theta\}$

and let $\Delta = \{(x,x) | x \in Q_0\}$

(3) Let $P = \Delta \cup P_1 \cup P_1^{-1}$ (P is an equivalence relation)

(4) $Q = Q_0/P$

(5) For i = 1, 2, let $R_i = \{([x], [y]) \in Q \times Q |$ there exists $x_i \in [x]$ and $y_i \in [y]$ such that $x_i \leq_i y_i\}$

(6) Let $\leq$ be the relation $(R_1 \dot{\cup} R_2)^2$

(7) Let $[0] = [0_1]$, $[1] = [1_1]$

(8) Define $'$: $Q \to Q$ as follows: for $[x] \in Q$,

$$[x]' = \begin{cases} [x_1^{\#}] & \text{if there exists } x_1 \in Q_1 \text{ with } x_1 \in [x] \\ \\ [x_2^{+}] & \text{if there exists } x_2 \in Q_2 \text{ with } x_2 \in [x] \end{cases}$$

Theorem. $(Q, \leq, ')$, as defined above, is an orthomodular poset.

We write $Q = P(Q_1, Q_2; S_1, S_2)$ and say that Q is obtained from Q_1 and Q_2 by pasting S_1 to S_2 (according to the prescription θ). The proof of the theorem follows closely that of [6], Theorem 3.4 and is therefore omitted.

(3) Pasting Two Orthomodular Lattices to Obtain an Orthomodular Lattice.

Recall that a sublattice M of an orthomodular lattice L is sub-complete in case all existing suprema and infima, as computed in L, of subsets of M fall back in M. We adopt the notation and convention of the preceeding section (while replacing the symbols Q_1, Q_2, and Q by L_1, L_2, and L respectively) to obtain Theorem 3.4 of [6]. (Note that the convention of (2) incorporates a part of the hypothesis of [6] Theorem 3.4.)

Theorem. Let L_i be a complete orthomodular lattice and let S_i be a subcomplete sub-orthomodular lattice of L_i $(i = 1,2)$. Then L is a complete orthomodular lattice.

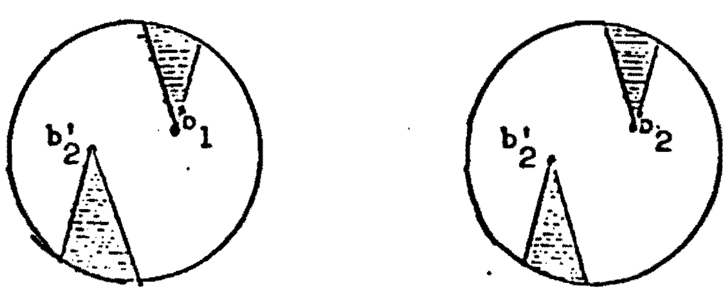
We now introduce two examples in order to illustrate the utility of this theorem.

Example. Let L_1 and L_2 be disjoint copies of a non-trivial irreducible modular lattice $\bar{L}$ which is not isomorphic to the horizontal sum of copies of 2^2.

Fix $b \in \bar{L}\setminus\{0,1\}$, b not an atom of $\bar{L}'$. There exist natural isomorphisms $\phi_i: \bar{L} \to L_i$; let $b_i = b\,\phi_i$. Let $S_i = S_{b_i}$ and let $L = P(L_1, L_2; S_1, S_2)$.

Figure III is a diagrammatical representation of L in which the shaded areas indicate where the "pasting" has occurred.

Figure III.



Remark. L (in the above example) is a complete irreducible orthomodular lattice which is not M-symmetric.

Proof. We claim that there exists $a \in L_2$ such that $a \wedge_2 b_2 = 0$ and $a \not\leq b_2'$.

Suppose this were false, then $a \wedge_2 b_2 = 0$ implies $a \leq b_2'$. Hence every complement of b_2 is $\leq b_2'$; since L_2 is irreducible there is such a complement distinct from b_2'. But this contradicts the modularity of L_2. We have established the claim. It follows that $a \nmid S_2$ (and also that $a \nmid b_2$).

Now $([b],[a])M$ since $[y] \leq [a]$ implies there exists $y_2 \in [y]$, $y_2 \in L_2$ and therefore

$$[y] \vee ([b] \wedge [a]) = [y \vee_2 (b \wedge_2 a)] = [(y \vee_2 b) \wedge_2 a]$$
$$= ([y] \vee [b]) \wedge [a].$$

But $([a],[b])M$ fails. To see this select any $x \in L_1 \setminus \{0\}$ such that $x < b_1$. Note that $[x] = \{x\} \subset L_1$. Now $[x] < [b]$; we must show that

$$(*) \qquad [x] \vee ([a] \wedge [b]) \neq ([x] \vee [a]) \wedge [b].$$

It follows from Lemma 3.3 of [6] that $([x] \vee [a]) \geq [b]$ so that $([x] \vee [a]) \wedge [b] = [b]$. But

$$[x] \vee ([a] \wedge [b]) = [x] \vee [a \wedge_2 b] = [x] \vee [0] = [x] < [b]$$

Hence (*) obtains and L is not M-symmetric.

If $\tilde{L}$ happens to be totally non-atomic, e.g. a type II factor, then L is a complete totally non-atomic orthomodular lattice which is non M-symmetric. The existence of such a lattice was conjectured by S. Maeda [12].

Example. Let X and Y be disjoint sets each having cardinal number 4, let M be a 2-element subset of X and let N be a 2-element subset of Y. Let $B_1 = \mathcal{P}(X)$, $B_2 = \mathcal{P}(Y)$, let $S_1 = S_{X \setminus M}$ and let $S_2 = S_{Y \setminus N}$.

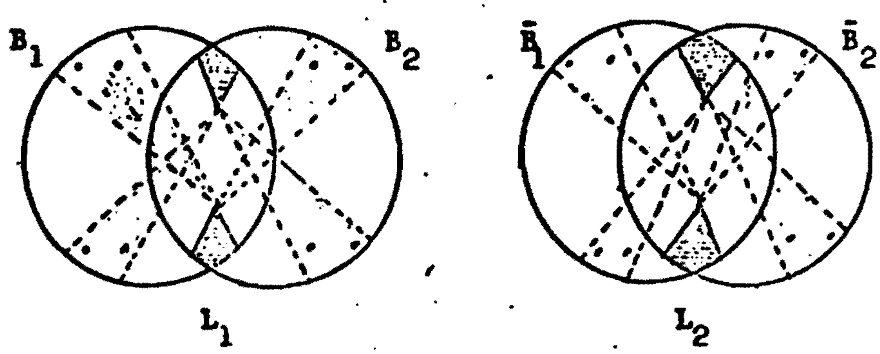
Let $L_1 = P(B_1, B_2; S_1, S_2)$. Repeat this process by finding $\bar{X}$, $\bar{Y}$, $\bar{M}$, $\bar{N}$, $\bar{B}_1$, $\bar{B}_2$, $\bar{S}_1$, and $\bar{S}_2$ and obtaining $L_2 = P(\bar{B}_1, \bar{B}_2, \bar{S}_1, \bar{S}_2)$ where $L_1 \cap L_2 = \phi$. Let

$$R_{36} = P(L_1, L_2; S^1, S^2)$$

where $S^1 \subset L_1$ is defined by $S^1 = S_{[M]}$ and $S^2 \subset L_2$ is defined by $S^2 = S_{[\bar{M}]}$.

Then R_{36} is an orthomodular lattice (named after Charles H. Randall [15] who has noted its significance as an operational logic). This lattice appears as one of a large class of finite orthomodular lattices in Randall's (unpublished) "catologue of operational logics" We take this opportunity to baptize it with Randall's initial in tribute to his revolutionary work in empirical logic. (It was independently discovered by J.C. Dacey and by the author.) We sketch a diagrammatic representation of R_{36} in Figure IV.

Figure IV



II. Notation

At this point it is necessary to develop a simpler notation. There are several in use. Since each notation is in some way superior to the others, we present a brief survey with references to more complete expositions. Throughout this section, unless otherwise stated, we assume that we are dealing with a finite orthomodular lattice L with A as its set of atoms.

(0) The Hasse Diagram. This notation is standard and has already been taken for granted in Figure I and II. Figures III and IV are abstract Hasse Diagrams.

(1) The Orthogonality Graph of A. In this notation the orthogonality graph on the atoms is depicted. To obtain the lattice, one considers the Galois auto-connection $\perp:\mathcal{P}(A) \to \mathcal{P}(A)$ defined as follows:

$$\text{for } M \subset A,\ M^{\perp} = \{x \in A \mid x \perp m \text{ for all } m \in M\}.$$

The lattice appears as the set of Galois-closed subsets of A, $\{M \mid M = M^{\perp\perp}\}$. The Boolean sub-orthomodular lattices correspond to the complete subgraphs. Thus D_{16}(cf., Figure II) is represented as in Figure V and P_{36} (cf., Figure IV) as in Figure VI.

Figure V.

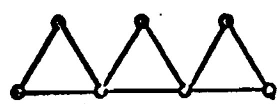
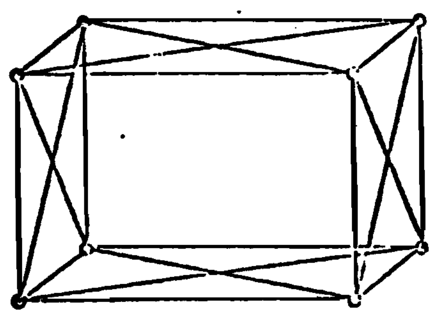


Figure VI.



This procedure may be applied, not only to the atoms of L, but to any join-dense subset of L. J.C. Dacey, [3] has utilized this fact to produce a significant number of phenomenologically interpretable orthomodular structures.

(2) The Operational Diagram.

C. H. Randall has developed a diagrammatical representation for "operational logics". Although not all finite orthomodular lattices are "operational", many important ones are, e.g., the free orthomodular lattice on two generators (cf. Figure XI). We illustrate Randall's notation by means of an example. Letting $\hat{x}$ denote an "operation" with elementary outcomes x and $\tilde{x}$ (not x), we interpret the symbols

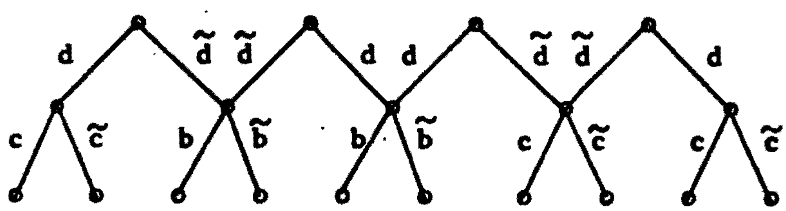
$$(\hat{b} \circ \hat{c} | \hat{d}$$

as instructions for an experiment in the following way:

> Perform the operation $\hat{d}$ obtaining one of the outcomes d or $\tilde{d}$; record the result; then perform one of the operations $\hat{b}$ or $\hat{c}$ obtaining one of the outcomes b, $\tilde{b}$, c, or $\tilde{c}$; record the result.

The resulting operational logic is R_{36} whose operational diagram appears in Figure VII.

Figure VII.



(3) <u>The Orthogonality Space of A</u>.

The orthogonality space $(A,\mathcal{E})$ is obtained from the orthogonality graph $(A,\perp)$ by defining $\mathcal{E}$ to be the set of all maximal complete subgraphs of $(A,\perp)$. The elements of $\mathcal{E}$ are called <u>cliques</u> or <u>frames</u> or <u>blocks</u>. Clearly, we may recapture the orthogonality graph $(A,\perp)$ from the orthogonality space $(A,\mathcal{E})$ by defining $x \perp y$ to mean that $x \neq y$ and $x,y \in E$ for some $E \in \mathcal{E}$. By drawing each clique as a smooth curve containing distinguished points we obtain a representation of the structure which is frequently simpler than any of the others mentioned. In this notation Figure V and VI (D_{16} and R_{36}, respectively) are translated into Figure VIII and IX, respectively,

Figure VIII.

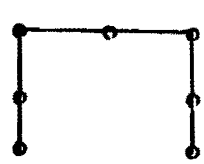
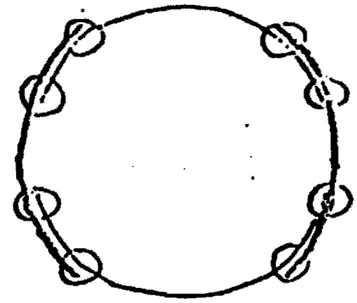


Figure IX.



We include three additional examples. Figure X is the first known orthomodular poset which is not a lattice; it is due to M. F. Janowitz [11] and denoted by J_{18}. Figure XI is the free orthomodular lattice on two generators F_2, whose structure was first determined by J. Casey and the author. Figure XII due to the author and denoted by G_{32}, provides the motivation for the last theorem cited in this paper.

Figure X.

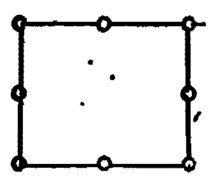


Figure XI.

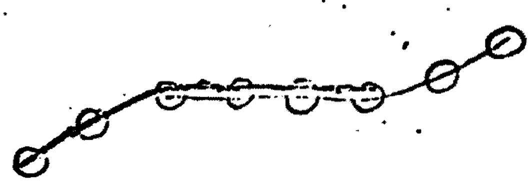
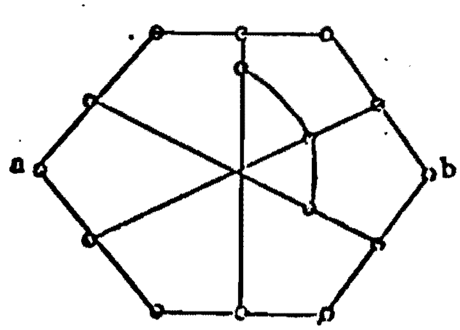


Figure XII.



III. Interactions of Blocks in an Orthomodular Structure.

(1) A Structure Theorem for Orthomodular Lattices of Finite Height [6]

Recall that a block in an orthomodular lattice L is a maximal Boolean sub-orthomodular lattice of L, and that $\mathcal{B}_L$ denotes the set of all blocks in L. It is clear that $L = \bigcup \mathcal{B}_L$ since every family of pairwise commuting elements of L may be extended to a maximal such family, i.e. to a block. Hence the structure of L ought to be determined once the blocks and their intersections are determined.

In the Randall formulation of empirical logic, the blocks correspond to those experiments no two of which may be refined into a single "grand canonical experiment". The intersection of blocks corresponds to the interaction of experiments thus motivating the title of this section.

A clue as to how the blocks of an orthomodular lattice may intersect is provided by the following result: If L is the union of two blocks B_1 and B_2, then these are the only blocks in L and their intersection is of the following form:

$$B_1 \cap B_2 = S_e \,, \quad \text{for some } e \in L.$$

This suggests the following conjecture: the intersection of any two blocks B_1, B_2 in an orthomodular lattice is of the form

$$B_1 \cap B_2 = S_e \cap (B_1 \cup B_2), \quad \text{for some } e \in B_1 \cup B_2.$$

This conjecture is false; R_{36} supplies the necessary counterexample. However, by restricting our attention to lattices satisfying the chain condition, we may obtain the following theorem ([6], Corollary 4.8):

Theorem: Let L be an orthomodular lattice admitting no infinite chain. If $B_1, B_2 \in \mathcal{B}_L$, then there exists a finite sequence of blocks $B^1, B^2, \cdots, B^{n+1}$ such that $B^1 = B_1$, $B^{n+1} = B_2$, and for $1 \le k \le n$

$$B^k \cap B^{k+1} = S_{c_k} \cap (B^k \cup B^{k+1})$$

where $c_1, \cdots, c_n$ are the distinct atoms of $B_1 \cap B_2$.

(2) Atomistic Loop Lemma

Convention. Throughout this section we let $L = \bigcup \{B_\alpha : \alpha \in I\}$ be such that

(1) $(B_\alpha, \le_\alpha, {}'^\alpha)$ is a Boolean lattice for all α in I.

(2) if $x \in B_\alpha \cap B_\beta$ $(\alpha, \beta \in I)$ then $x'^\beta = x'^\alpha$.

(3) if $\alpha \ne \beta$ $(\alpha, \beta \in I)$ then $B_\alpha \cap B_\beta = \{0,1\}$ or $= \{0,1,a,a'\}$ where a is an atom of both B_α and B_β, $a' = a'^\alpha = a'^\beta$.

(4) $B_\alpha \ne 2^1$, $B_\alpha \ne 2^2$ for all α in I.

The set $\{B_\alpha : \alpha \in I\}$ is called the set of initial blocks of L. If x and y are elements of the set L, we define $x \le y$ to mean that there exists an initial block B_α such that $\{x,y\} \subset B_\alpha$ and $x \le_\alpha y$. Define x' to be x'^α whenever x is an element of the initial block B_α. With these definitions, $(L, \le, ')$ is an orthocomplemented poset. We now define a concept which allows us to inject orthomodularity into L.

Definition. Let $n \in \mathbb{Z}$, $n \ge 3$. We call a set $\{B_0, B_1, \cdots, B_{n-1}\}$ of initial blocks of L an atomistic loop of order n in case for $0 \le j < i \le n-1$ we have

$$B_i \cap B_j = \begin{cases} \{0,1,a,a'\} & \text{if } i - j \in \{1, n-1\} \end{cases}$$

and for $0 \leq k < j < i \leq n - 1$ we have

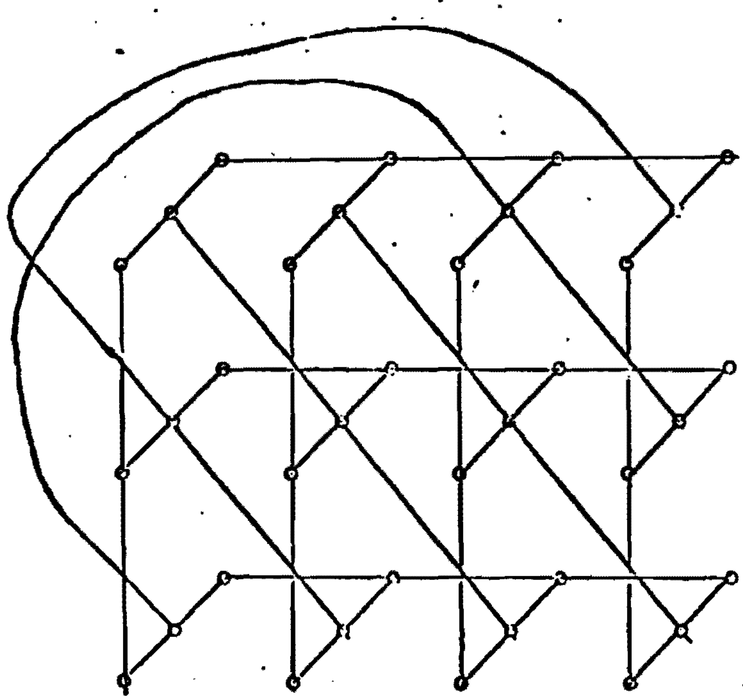
$$B_i \cap B_j \cap B_k = \{0,1\}.$$

We may now state the (atomistic) loop lemma [7].

Theorem. L is an orthomodular poset (resp., lattice) if and only if the order of every atomistic loop in L is at least 4 (resp., 5).

An immediate corollary of this theorem is that Figure XIII depicts an orthomodular lattice which we call $G_{3,4}$. The most interesting feature of this lattice is that its state space is empty, i.e. there exist no normalized (even finitely) orthogonaly additive measures defined on $G_{3,4}$. (For a class of such lattices see [7].)

Figure XIII



III. Additional Examples and Some Problems.

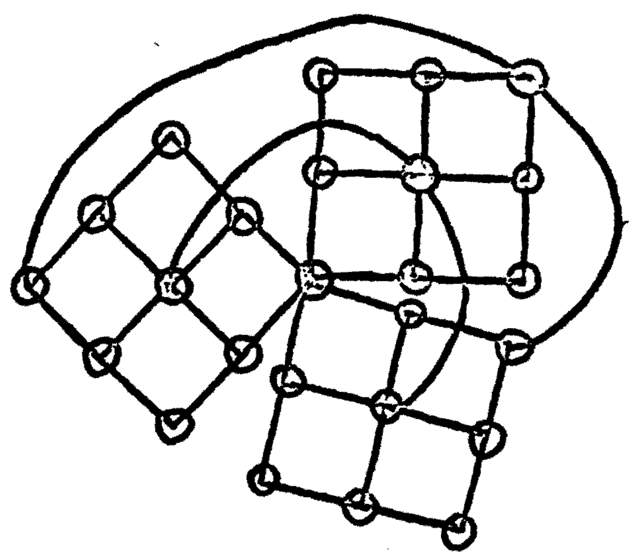
Let S_L denote the set of all states on an orthomodular structure L. S_L is said to distinguish the elements of L if for each pair of distinct elements x, y in L there is a state α on L such that $x\alpha \neq y\alpha$; S_L is said

$x\alpha > y\alpha$; S_L is strongly order determining [14] if $x \nleq y$ $(x,y \in L)$ implies there is a state α on L such that $x\alpha = 1$ and $y\alpha = 0$.

We have already observed that an orthomodular lattice (cf., Figure XIII) may admit no states. M. K. Bennett [1] has observed that $L = P(G_{32}, G_{32}; S_1, S_2$ where S_1 "=" $S_2 = \{0,1,a,a',b,b'\}$ in the notation of Figure XII, admits some states but S_L does not distinguish the elements of L. She [2] has also observed that G_{32} admits a state space which distinguishes the elements of G_{32} but is not full. The orthomodular poset G_{52} depicted in Figure XIV admits a full set of states but not a strongly order determining set of states [8]. Recently B. Colli has constructed an orthomodular lattice with a full but not strongly order determining set of states. Also G. Schrag has proved the following.

Theorem. Every finite group is the automorphism group of some finite orthomodular lattice.

Figure XIV



Additional examples of orthomodular structures, created by applications of the theorems cited in sections I.2 and III.2 in order to verify or refute certain natural conjectures, are presented in [9] and [10]. We conclude with four problems:

(1) In his book [13] O. M. Nikodym attempts to cast Boolean algebras

in a central role in the analysis associated with quantum theories. His success seems to hinge on the theorem which associates to every block in Hilbert Space a cyclic vector which yields a measure. Recall that a block B in an (arbitrary) orthomodular lattice L admits a cyclic vector in case there exists an atom a in L such that

$$\bigvee_{b \in B} a\phi_b = 1, \text{ where } a\phi_b = (a \vee b') \wedge b.$$

This condition is equivalent to the condition: $a \leq b$, $b \in B$ implies $b = 1$. Under what conditions on an orthomodular lattice does a block B (resp., every block) admit a cyclic vector?

(2) What is the structure of the free orthomodular lattice on n, $n > 2$, generators?

(3) Do there exist theorems on decompositions of Baer *-semigroups which correspond to the theorems of I.3 and/or III.2?

(4) Let G be a group. Does there exist an orthomodular lattice L with a full set of states such that G is isomorphic to the automorphism group of L?

Bibliography

[1] Bennett, M. K., States on Orthomodular Lattices, Journal of Natural Sciences and Mathematics, (October 1968).

[2] ____________, A Finite Orthomodular Lattice Which does not Admit a Full Set of States, to appear.

[3] Dacey, J. C., Orthomodular Spaces, Ph.D. Dissertation, University of Massachusetts, May 1968.

[4] Dilworth, R. P., On Complemented Lattices, Tohoku Math Journal 47 (1940) pp. 18-23.

[5] Greechie, R. J., A Class of Orthomodular Nonmodular Lattices, Abstract 64T-105 Notices Amer. Math. Soc. 11, No. 2, Issue No. 73 (Feb. 1966).

[6] ____________, On the Structure of Orthomodular Lattices Satisfying the Chain Condition, J. Comb. Theory 4, 210-218 (1968)

[7] ____________, Orthomodular Lattices Admitting No States, (to appear)

[8] ____________, An Orthomodular Poset with a Full Set of States Not Embeddable in Hilbert Space, (in preparation)

[9] ____________, A Particular Non-Atomistic Orthomodular Poset (in preparation).

[10] ____________, Hyper-Irreducibility in an Orthomodular Lattice, Journal of Natural Sciences and Mathematics, Vol VIII, No. 1, April 1968.

[11] Janowitz, M. F., Quantifiers on Quasi-orthomodular Lattices, Ph.D. Dissertation, Wayne State University, 1963.

[12] Maeda S.,The Theory of Symmetric Lattices,Conference on Orthomodular Lattices with Special Reference to Empirical Logic, Amherst 1968.

[13] Nikodým, O. M., The Mathematical Apparatus for Quantum Theories, Springer-Verlag, N. Y. Inc., 1966.

[14] Pool, J. C. T., Baer *-Semigroups and the logic of Quantum Mechanics, (to appear)

[15] Randall, C. H., A Mathematical Foundation for Empirical Science with Special Reference to Quantum Theory. Part I: A Calculus of Exper-

imental Propositions, Knolls Atomic Power Lab. Report KAPL-3147 June 1966.

[16] ____________, Empirical Logic and Orthomodular Lattices, unpublished notes, University of Massachusetts.

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