On Structures Related to States on an Empirical Logic

II. Weights and Duality on Finite Spaces

by

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## 0. Introduction

This paper continues the investigation of spaces begun in [ 3 ]. Undefined terms appear in [ 3 ].

In section 1 we study the dual space for space $(X,\mathcal{E})$. We introduce generalized weights, functions which sum to 1 over each element of $\mathcal{E}$ (but are not necessarily non-negative), and show that, for a finite space admitting a non-negative weight, the real number $\sum_{x \in X} \mu(x)$ does not depend on the weight $\mu$ if and only if there exists a generalized weight on the dual space. We also give some partial results on when the weight space of a finite space is a simplex. The <u>simplex conjecture</u> remains open, however: If $(X,\mathcal{E})$ is an orthogonality space and the set $\Omega$ of all weights on $(X,\mathcal{E})$ is full and is a simplex, then $\mathcal{L}(X,\mathcal{E})$ is a Boolean algebra. We conclude the section with the remark that pure weights on finite orthogonality spaces take on only rational values.

In section 2 we study those spaces $(X,\mathcal{E})$ which are self-dual in the sense that there exist mappings $\alpha : X \to \mathcal{E}$ and $\beta : \mathcal{E} \to X$ such that $x \in E$ if and only if $\beta(E) \in \alpha(x)$. We present two interesting notions of subspace, called type I and type II. In a self-dual space, the dual of a subspace of type I (resp., II) may be identified with a subspace of type II (resp., I).

In section 3 we define an automorphism of a space $(X,\mathcal{E})$. The main (expected and not terribly deep) result proved is that the automorphism group of a distinguishing space is isomorphic to the automorphism group of its dual space. We conclude by mentioning Schrag's theorem: Every finite group is the automorphism group of some (complete) Dacey space.

## 1. Weights and Duality

Let $(X,\mathcal{E})$ be any space.* We define the <u>dual space</u> $(X^*,\mathcal{E}^*)$ by $X^* = \mathcal{E}$ and $\mathcal{E}^* = \{\mathcal{E}_x \mid x \in X\}$. We say that $(X,\mathcal{E})$ is <u>separating</u> if $\mathcal{E}_x \not\subseteq \mathcal{E}_y$ or $\mathcal{E}_x = \mathcal{E}_y$ whenever $x \neq y$. We say that $(X,\mathcal{E})$ is <u>distinguishing</u> if $\mathcal{E}_x \neq \mathcal{E}_y$ whenever $x \neq y$. Note that the logic of a separating and distinguishing D-space need not be hyper-irreducible [1].

<u>Proposition 1.1.</u>

1) If $(X,\mathcal{E})$ is any space then $(X^*,\mathcal{E}^*)$ is a distinguishing space.

2) If $E \in \mathcal{E}$, $F \in \mathcal{E}$, $E \neq F$ implies $E \not\subseteq F$ then $(X^*,\mathcal{E}^*)$ is also a separating space.

3) $(X,\mathcal{E})$ is distinguishing if and only if $(X^{**},\mathcal{E}^{**}) \cong (X,\mathcal{E})$.

<u>Proof.</u> Ad1). Suppose that $E_1 \in X^*$, $E_2 \in X^*$, and $E_1 \neq E_2$. Either $E_1 \setminus E_2 \neq \phi$ or $E_2 \setminus E_1 \neq \phi$. If the former is true choose $x \in E_1 \setminus E_2$. Then $\mathcal{E}_x \in \mathcal{E}^*_{E_1}$ but $\mathcal{E}_x \notin \mathcal{E}^*_{E_2}$ so that $\mathcal{E}^*_{E_1} \neq \mathcal{E}^*_{E_2}$. Similarly, $E_2 \setminus E_1 \neq \phi$ implies that $\mathcal{E}^*_{E_2} \setminus \mathcal{E}^*_{E_1} \neq \phi$. Ad2). The hypothesis of 2) shows that both $E_1 \setminus E_2 \neq \phi$ and $E_2 \setminus E_1 \neq \phi$ whenever $E_1 \neq E_2$. Thus we get $\mathcal{E}^*_{E_1} \not\subseteq \mathcal{E}^*_{E_2}$ and $\mathcal{E}^*_{E_2} \not\subseteq \mathcal{E}^*_{E_1}$.

Ad3). $X^{**} = \mathcal{E}^* = \{\mathcal{E}_x \mid x \in X\} \cong X$ since $(X,\mathcal{E})$ is distinguishing. $\mathcal{E}^{**} = \{\mathcal{E}^*_E \mid E \in \mathcal{E}\} \cong \mathcal{E}$ by part 1. Now $\mathcal{E}_x \in \mathcal{E}^*_E$ if and only if $E \in \mathcal{E}_x$ if and only if $x \in E$. ∎

We now characterize those spaces $(X,\mathcal{E})$ which have the property that $(X^*,\mathcal{E}^*)$ is an orthogonality space. Note that separating orthogonality spaces which have the property that $\text{card}(E_1 \cap E_2) \leq 1$ for all distinct $E_1, E_2 \in \mathcal{E}$ are such.

*We will always assume that $X = \bigcup \mathcal{E}$.

Proposition 1.2. Let $(X,\mathcal{E})$ be a finite space. Then $(X^*,\mathcal{E}^*)$ is an orthogonality space if and only if $(X,\mathcal{E})$ is a separating space and $E \cap \bigcap_{i=1}^{n} E_i \neq \emptyset$ whenever $\{E, E_1, E_2, \ldots, E_n\} \subseteq \mathcal{E}$ with $\bigcap_{i=1}^{n} E_i \neq \emptyset$ and $E \cap E_i \neq \emptyset$ for each $i = 1, 2, \ldots, n$.

Proof. Clearly $X^* = \mathcal{E} = \bigcup_{x \in X} \mathcal{E}_x$. Now suppose that $\mathcal{G} \subseteq \mathcal{E} = X^*$ and for each $E_1, E_2$ in $\mathcal{G}, E_1 \cap E_2 \neq \emptyset$. Choose $E_1, E_2$ distinct elements of $\mathcal{G}$ and $x_1 \in E_1 \cap E_2$. Suppose that we have $E_1, \ldots, E_n$ distinct elements of $\mathcal{G}$ and $x_{n-1} \in \bigcap_{i=1}^{n} E_i$. If $E_{n+1}$ is a different element of $\mathcal{G}$ the condition allows us to choose $x_n \in \bigcap_{i=1}^{n+1} E_i$. Continuing until $\mathcal{G}$ is exhausted we obtain some $x_n$ such that $\mathcal{G} \subseteq \mathcal{E}_{x_n}$. Finally, suppose that $\mathcal{E}_x \neq \mathcal{E}_y$. Then $x \neq y$ and since $(X,\mathcal{E})$ is separating and $\mathcal{E}_x \neq \mathcal{E}_y$ we have $\mathcal{E}_x \nsubseteq \mathcal{E}_y$. By Corollary 1.3 [3] $(X^*,\mathcal{E}^*)$ is an orthogonality space. The converse is immediate. ∎

Let $(X,\mathcal{E})$ by any space. Recall that a weight on $(X,\mathcal{E})$ is a mapping $\mu: X \to [0,1]$ such that $\sum_{x \in E} \mu(x) = 1$ for each $E \in \mathcal{E}$. $\mu$ is said to be dispersion free if $\operatorname{im}(\mu) \subseteq \{0,1\}$. An indication of the relationship between weights and duality is given by Lemma 1.3.

Lemma 1.3. Let $(X,\mathcal{E})$ by any space. Then there is a one-to-one correspondence between disjoint covers of $X$ by members of $\mathcal{E}$ and dispersion free weights on the space $(X^*,\mathcal{E}^*)$.

Proof. By Theorem 3.1 [3] the dispersion free weights of $(X^*,\mathcal{E}^*)$ are in one-to-one correspondence with certain subsets $\mathcal{G} \subseteq X^* = \mathcal{E}$. It is easy to see that these are exactly the subsets which give disjoint covers of $X$. ∎

Let $(X,\mathcal{E})$ be any finite space and $\Omega$ be the set of weights on $(X,\mathcal{E})$. In [3] $T: F(\mathcal{E}) \to F(X)$, called the linear realization of $(X,\mathcal{E})$, was defined by

$T(E) = \sum_{x \in E} x$ and linear extension. We then have $T^*: F(X)^* \to F(\mathcal{E})^*$ which is such that $T^*(\delta_x) = \sum_{x \in E} \delta_E$. If $K = \{\mu \mid \mu \in F(X)^* \text{ and } \mu(x) \geq 0 \text{ for each } x \in X\}$ then $\Omega = T^{*^{-1}}\left(\sum_{E \in \mathcal{E}} \delta_E\right) \cap K$. The existence of weights can thus be considered in two steps; first, the condition that $T^{*^{-1}}\left(\sum_{E \in \mathcal{E}} \delta_E\right) \neq \emptyset$, and, second, that this set meets K. The first condition has a simple combinatorial interpretation using duality.

By a <u>generalized</u> <u>weight</u> we mean any element of $\Omega_g = T^{*^{-1}}\left(\sum_{E \in \mathcal{E}} \delta_E\right)$. Recall that a perturbation is any element of Ker $T^*$. It is clear that $\mu_1$ and $\mu_2$ in $\Omega_g$ implies $\mu_1 - \mu_2 \in$ Ker $T^*$ and if $\mu$ is any element in $\Omega_g$ then $\Omega_g = \mu +$ Ker $T^*$.

<u>Lemma 1.4</u>. Let $(X, \mathcal{E})$ be any space. Define an equivalence relation, $\rho$, on X by $x \,\rho\, y$ if and only if $\mathcal{E}_x = \mathcal{E}_y$. For each $x \in X$ let $\hat{x} = \{y \mid y \in X \text{ and } x \,\rho\, y\}$ and for each $Y \subseteq X$ let $\hat{Y} = \{\hat{x} \mid x \in Y\}$. Then

1) For each $E \in \mathcal{E}$, $x \in E$ implies $\hat{x} \subseteq E$.
2) $(\hat{X}, \hat{\mathcal{E}}) \cong (X^{**}, \mathcal{E}^{**})$ where $\hat{\mathcal{E}} = \{\hat{E} \mid E \in \mathcal{E}\}$.
3) If $(X, \mathcal{E})$ is an orthogonality space then $(\hat{X}, \hat{\mathcal{E}})$ is an orthogonality space.
4) If $(X, \mathcal{E})$ is a D-space then $(\hat{X}, \hat{\mathcal{E}})$ is a D-space.

<u>Proof</u>. 1) Let $E \in \mathcal{E}$. Suppose $x \in E$ and $x \,\rho\, y$. Then $E \in \mathcal{E}_x = \mathcal{E}_y$ so that $y \in E$. Thus $\hat{x} \subseteq E$. 2) Define $\varphi: X \to X^{**}$ by $\varphi(x) = \mathcal{E}_x$. Then $\varphi$ is onto and $x \,\rho\, y$ if and only if $\varphi(x) = \varphi(y)$. Thus $\varphi$ induces a bijection, $\hat{\varphi}: \hat{X} \to X^{**}$. Now $\mathcal{E}^{**} = \{\mathcal{E}_E^* \mid E \in \mathcal{E}\}$ where $\mathcal{E}_E^* = \{\mathcal{E}_x \mid E \in \mathcal{E}_x\} = \{\mathcal{E}_x \mid x \in E\}$. Thus $\mathcal{E}_E^* = \{\hat{\varphi}(\hat{x}) \mid \hat{x} \subseteq E\} = \{\hat{\varphi}(\hat{x}) \mid \hat{x} \in \hat{E}\} = \hat{\varphi}(\hat{E})$. 3) and 4) are routine and are left as exercises. ∎

Let $\Omega_g^{**}$ and $\Omega^{**}$ denote the generalized weights and weights, respectively, on $(X^{**}, \mathcal{E}^{**})$. $\hat{\Omega}_g$ and $\hat{\Omega}$ denote the corresponding sets on $(\hat{X}, \hat{\mathcal{E}})$ .

Theorem 1.5. There is a natural mapping $\theta:\Omega_g \to \Omega_g^{**}$ which preserves convex combinations and satisfies $\theta(\Omega_g) = \Omega_g^{**}$, $\theta(\Omega) = \Omega^{**}$. Thus there is a generalized weight (weight) on $(X,\mathcal{E})$ if and only if there is a generalized weight (weight) on $(X^{**},\mathcal{E}^{**})$.

Proof. We will prove the theorem using $(\hat{X},\hat{\mathcal{E}})$ for $(X^{**},\mathcal{E}^{**})$ (see lemma 1.4). Let $\mu \in \Omega_g$. Define $\theta(\mu) = \hat{\mu}:\hat{X} \to R$ by $\hat{\mu}(\hat{x}) = \sum_{x\rho y} \mu(y)$. Let $E \in \mathcal{E}$. Then by part 1 of lemma 1.4, $E = \dot{\bigcup}\{P \mid P \in E\}$. Thus we get $\sum_{P\in E} \hat{\mu}(P) = \sum_{P\in \hat{E}} \sum_{x\in P} \mu(x)$ $= \sum_{x\in E} \mu(x) = 1$ which shows that $\hat{\mu} \in \hat{\Omega}_g$. Suppose $\mu \in \Omega$. Then $\mu(x) \geq 0$ for each $x \in X$ so that $\hat{\mu}(P) = \sum_{x\in P} \mu(x) \geq 0$ for each $P \in \hat{X}$. Thus $\hat{\mu} \in \hat{\Omega}$.

Conversely, suppose $\omega \in \hat{\Omega}_g$. Choose a choice function $\chi:\hat{X} \to X$ so that $\chi(P) \in P$ for each $P \in \hat{X}$. Define $\mu:X \to R$ by $\mu(x) = \begin{cases} 0 \text{ if } x \notin \text{im } \chi \\ \omega(\chi^{-1}x) \text{ if } x \in \text{im } \chi \end{cases}$.
Now let $E \in \mathcal{E}$. Then $\sum_{x\in E} \mu(x) = \sum_{x\in E\cap \text{im}\chi} \omega(\chi^{-1}x) = \sum_{P\in \hat{E}} \omega(P) = 1$ so that $\mu \in \Omega_g$. Clearly, $\omega \in \hat{\Omega}$ implies $\mu \in \Omega$. Now $\hat{\mu}(P) = \sum_{x\in P} \mu(x) = \omega(\chi^{-1}(\chi P)) = \omega(P)$ so that $\theta(\mu) = \omega$. ∎

We can now give a combinatorial description of generalized weights. For each $A \subseteq X$ ($\mathcal{G} \subseteq \mathcal{E}$) consider $A \in F(X)$ by $A = \sum_{x\in A} x$ $\left(\mathcal{G} \in F(\mathcal{E}) \text{ by } \mathcal{G} = \sum_{E\in \mathcal{G}} E\right)$.

Theorem 1.6. Let $(X,\mathcal{E})$ be any space.

1) A generalized weight on $(X,\mathcal{E})$ determines a representation $X^* = \sum_{E^*\in \mathcal{E}^*} \lambda(E^*)E^*$. Conversely, such an equation determines a set of generalized weights on $(X,\mathcal{E})$.

2) A generalized weight on $(X^*,\mathcal{E}^*)$ determines a representation $X = \sum_{E\in\mathcal{E}} \lambda(E)E$. Conversely, such an equation determines a unique generalized weight on $(X^*,\mathcal{E}^*)$.

3) Suppose that $\Omega_g \neq \phi$. Then $\sum_{x\in X} \mu_1(x) = \sum_{x\in X} \mu_2(x)$ for all $\mu_1,\mu_2$ in $\Omega_g$ if and only if there exists a generalized weight on $(X^*,\mathcal{E}^*)$.

Proof. 1) Let $\mu \in \Omega_g$. By definition $\mathcal{E}^* = X^{**}$ and by lemma 1.4, $X^{**} = \{\mathcal{E}_{\hat{x}} \mid \hat{x} \in \hat{X}\}$, $\hat{x} \neq \hat{y}$ implies $\mathcal{E}_{\hat{x}} \neq \mathcal{E}_{\hat{y}}$, and $\mathcal{E}_{\hat{x}} = \{E \mid \hat{x} \subseteq E\}$. Let $\theta:\Omega_g \to \Omega_g^{**}$ be the mapping given by theorem 1.5. Define $\lambda(\mathcal{E}_{\hat{x}}) = \theta(\mu)(\mathcal{E}_{\hat{x}})$ for each $\mathcal{E}_{\hat{x}} \in X^{**}$. Then $\sum_{E^*\in\mathcal{E}^*} \lambda(E^*)E^* = \sum_{\hat{x}\in\hat{X}} \theta(\mu)(\mathcal{E}_{\hat{x}})\mathcal{E}_{\hat{x}} = \sum_{\hat{x}\in\hat{X}} \theta(\mu)(\mathcal{E}_{\hat{x}}) \sum_{E\supseteq\hat{x}} E$ $= \sum_{E\in\mathcal{E}} \left(\sum_{\hat{x}\subseteq E} \theta(\mu)(\mathcal{E}_{\hat{x}})\right) E = \sum_{E\in\mathcal{E}} E = \mathcal{E} = X^*$. Conversely, given an equation $X^* = \sum_{E^*\in\mathcal{E}^*} \lambda(E^*)E^*$ we get $\sum_{E\in\mathcal{E}} E = \sum_{E^*\in\mathcal{E}^*} \lambda(E^*) \sum_{E\in E^*} E = \sum_{E\in\mathcal{E}} \left(\sum_{E\in E^*} \lambda(E^*)\right) E$ so that $\sum_{E\in E^*} \lambda(E^*) = 1$ for each $E \in \mathcal{E}$. That is $\sum_{E^*\in\mathcal{E}_E} \lambda(E^*) = 1$ so that $\lambda \in \Omega_g^{**}$. By theorem 1.5 $\lambda$ determines a set of generalized weights on $(X,\mathcal{E})$.

2) This is similar to 1). The uniqueness follows from the fact that $(X^*,\mathcal{E}^*)$ is distinguishing (see proposition 1.1).

3) Suppose $\Omega_g \neq \phi$. Suppose there exists a generalized weight on $(X^*,\mathcal{E}^*)$. Then we can write $X = \sum_{E\in\mathcal{E}} \lambda(E)E$. Let $\mu \in \Omega_g$. Then $\sum_{x\in X} \mu(x) = \mu\left(\sum_{x\in X} x\right)$ $= \mu\left(\sum_{E\in\mathcal{E}} \lambda(E) \sum_{x\in E} x\right) = \sum_{E\in\mathcal{E}} \lambda(E) \sum_{x\in E} \mu(x) = \sum_{E\in\mathcal{E}} \lambda(E)$ and this is independent of $\mu$. Conversely, suppose $\Omega_g \neq \phi$ and the "constant sum" property holds. Consider $F(X) = F(X)^{**}$. Suppose $\mu \in F(X)^*$ is such that $\sum_{x\in E} \mu(x) = 0$ for all $E \in \mathcal{E}$.

Choose $\mu_1 \in \Omega_g$ and let $\mu_2 = \mu_1 + \mu$. Then $\sum_{x\in X} \mu_1(x) = \sum_{x\in X} \mu_2(x)$ shows that $\sum_{x\in X} \mu(x) = 0$. Considering $E \in F(X) = F(X)^{**}$ and $X\in F(X) = F(X)^{**}$ as above, we have have shown that $\bigcap_{E\in\mathcal{E}} \operatorname{Ker} E \subseteq \operatorname{Ker} X$. By a well know theorem of linear algebra we get an equation $X = \sum_{E\in\mathcal{E}} \lambda(E)E$. ∎

Remark. Part 2) of the preceeding theorem generalizes lemma 1.3. The equation $X = \sum_{E\in\mathcal{E}} \lambda(E)E$ becomes $X = \dot{\bigcup}_{E\in \operatorname{Supp}\lambda} E$ if $\lambda$ is a dispersion free weight in $(X^*,\mathcal{E}^*)$.

We now consider more closely, the relationship between generalized weight, weights, and perturbations.

Proposition 1.7. Let $(X,\mathcal{E})$ be any finite space, $\Omega \subseteq F(X)^*$ be the weight space of $(X,\mathcal{E})$, and $\Omega_p = \{P_1,P_2,\ldots,P_n\}$ be the set of pure weights. Let $R = \{\nu \in F(X)^* \mid \text{There is a number } r > 0 \text{ and weights } \mu_1,\mu_2 \text{ such that } r\nu = \mu_1 - \mu_2\}$. Let d be the number of vectors in a maximal linearly independent subset of $\{P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1\}$. Then

1) $R \subseteq \operatorname{Ker} T^*$ and $R$ is a vector subspace.

2) dimension $R = d$

3) If $(X,\mathcal{E})$ possesses a positive weight then $R = \operatorname{Ker} T^*$.

Proof. (We assume $\Omega \neq \emptyset$).

1). Choose $\mu \in \Omega$. Then $1\cdot 0 = \mu - \mu$ so that $0 \in R$. Suppose $\nu \in R$ and s is any number. If $s = 0$ then $s\cdot\nu = 0 \in R$. Suppose $s \neq 0$. Choose $r > 0$ and weights $\mu_1,\mu_2$ so that $r\nu = \mu_1 - \mu_2$. If $s > 0$ we get $(\frac{r}{s})(s\nu) = \mu_1 - \mu_2$. If $s < 0$ we get $(-\frac{r}{s})(s\nu) = \mu_2 - \mu_1$. Thus $s\nu \in R$. Suppose $0 \neq \nu_1 \in R$ and $0 \neq \nu_2 \in R$. Choose $r_1 > 0$, $r_2 > 0$, $\mu_1$, $\mu_2$, $\delta_1$, $\delta_2$ so that $r_1\nu_1 = \mu_1 - \mu_2$

and $r_2\nu_2 = \delta_1 - \delta_2$. Let $s = \frac{r_1 r_2}{r_1 + r_2} > 0$. Then $0 < \frac{s}{r_1} = \frac{r_2}{r_1 + r_2} < 1$,

$0 < \frac{s}{r_2} = \frac{r_1}{r_1 + r_2} < 1$, and $\frac{s}{r_1} + \frac{s}{r_2} = 1$. Now

$$s(\nu_1 + \nu_2) = s\left\{\frac{1}{r_1}(\mu_1 - \mu_2) + \frac{1}{r_2}(\delta_1 - \delta_2)\right\}$$

$$= \left(\frac{s}{r_1}\mu_1 + \frac{s}{r_2}\delta_1\right) - \left(\frac{s}{r_1}\mu_2 + \frac{s}{r_2}\delta_2\right).$$

This proves that $\nu_1 + \nu_2 \in R$ so that $R$ is a vector subspace.

2) Clearly $\{P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1\} \subseteq R$. Let $\nu \in R$. Write $r\nu = \mu_1 - \mu_2$ where $r > 0$ and $\mu_1, \mu_2$ are weights. We can write $\mu_1 = \sum_{i=1}^{n} \alpha_i P_i$ and $\mu_2 = \sum_{i=1}^{n} \beta_i P_i$, where $\sum_1^n \alpha_i = 1 = \sum_1^n \beta_i$. Thus

$$r\nu = \sum_1^n (\alpha_i - \beta_i)P_i = \left((1 - \sum_2^n \alpha_i) - (1 - \sum_2^n \beta_i)\right) P_1 + \sum_2^n (\alpha_i - \beta_i)P_i$$

$$r\nu = \left(\sum_2^n (\beta_i - \alpha_i)\right) P_1 + \sum_2^n (\alpha_i - \beta_i)P_i = \sum_2^n (\alpha_i - \beta_i)(P_i - P_1).$$

Thus $\nu = \sum_2^n \left(\frac{\alpha_i - \beta_i}{r}\right)(P_i - P_1)$ and we have shown that $\{P_i - P_1 \mid 2 \le i \le n\}$ generates $R$. The result now follows.

3) Suppose there is a weight $\mu$ such that $\mu(x) > 0$ for each $x \in X$. Let $\nu \in \operatorname{Ker} T^*$. Let $m = \min\{\mu(x) \mid x \in X\}$. Choose $r > 0$ so that $|r\nu(x)| < m$ for all $x \in X$. Let $\mu_1 = \mu + r\nu$. Then $\mu_1 \in \Omega$ and since $r\nu = \mu_1 - \mu$ we have that $\nu \in R$. Thus $\operatorname{Ker} T^* = R$. ∎

<u>Corollary 1.8</u>. Let $(X,\mathcal{E})$ be any finite space and suppose there exists $\mu \in \Omega$ such that $\mu(x) \neq 0$ for all $x \in X$. Then $\sum_{x\in X}\mu_1(x) = \sum_{x\in X}\mu_2(x)$ for all $\mu_1,\mu_2 \in \Omega$ if and only if there exists a generalized weight on $(X^*,\mathcal{E}^*)$.

Proof. Proposition 1.7 and part 3 of Theorem 1.6.

Let $(X,\mathcal{E})$ be any finite space and let $\Omega_p \subseteq \Omega$ be the set of pure weights. We know by [ 3; Lemma 3.5 ] that $\Omega_p$ is a finite set. Let $\Omega_p = \{\mu_0, \mu_1, \mu_2, \ldots, \mu_n\}$. It is a well known fact that $\Omega$ is a simplex if and only if $\{\mu_k - \mu_0 \mid 1 \le k \le n\}$ is a linearly independent subset of $F(X)^*$. $(X,\mathcal{E})$ is said to be a classical space if $\mathcal{E} = \{X\}$. In this case $\Omega$ is a simplex and $|\Omega_p| = |X|$. We now obtain a converse to this result.

Let $(X,\mathcal{E})$ be any finite space. Let $E \subseteq F(X)^*$ be a complement of Ker $T^*$. That is $F(X)^* = \text{Ker } T^* \oplus E$. Then $F(X) = F(X)^{**} = (\text{Ker } T^*)^* \oplus E^*$ so that $(\text{Ker } T^*)^{\perp} = E^*$. Now $\text{Ker } T^* = (\text{im } T)^{\perp}$ so that we get $\text{im } T = (\text{im } T)^{\perp\perp} = (\text{Ker } T^*)^{\perp} = E^*$. Now $|X| = \dim \text{Ker } T^* + \dim E = \dim \text{Ker } T^* + \dim E^*$ so that $|X| = \dim \text{Ker } T^* + \dim \text{im } T$.

Theorem. Let $(X,\mathcal{E})$ be any finite space which has a positive weight. Then $\Omega$ is a simplex if and only if $|X| + 1 = \dim (\text{im } T) + |\Omega_p|$.

Proof. Suppose $|X| + 1 = \dim (\text{im } T) + |\Omega_p|$. Then $|\Omega_p| - 1 = \dim \text{Ker } T^*$ and by Proposition 1.7 we can conclude that $\Omega$ is a simplex.

Suppose $\Omega$ is a simplex. Then $\dim \text{Ker } T^* = |\Omega_p| - 1$ and the result follows from $|X| = \dim \text{Ker } T^* + \dim (\text{im } T)$. ∎

Corollary. Suppose $(X,\mathcal{E})$ is a finite space, $\Omega$ is a simplex, and there is a positive weight. If $|\Omega_p| \ge |X|$ then $(X,\mathcal{E})$ is a classical space.

Recall that a space $(X, \mathcal{E})$ is <u>n-regular</u> if $|E| = n$ for all $E \in \mathcal{E}$.

<u>Remark</u>. Let $(X, \mathcal{E})$ be a finite <u>n-regular</u> space such that there exist dispersion free weights $\delta_1, \delta_2$ and a point $x \in X$ such that $\delta_1 \neq \delta_2$ and $\delta_1(x) = \delta_2(x) = 1$. Then $\Omega(X, \mathcal{E})$ is not a simplex.

<u>Proof</u>. Let K = the set of all bounded probability measures on $(X, \mathcal{E})$ $E = K - K$, $B = \Omega(X, \mathcal{E})$. Then B is a base for E and by the Choquet-Kendell Theorem ([ 4 ], Pg. 30), E is a vector lattice iff B is a linearly compact simplex.

Suppose E were a vector lattice. Let $\alpha = \delta_1 \vee \delta_2$, $\beta = \delta_1 + \delta_2$.

Since $\delta_1 \neq \delta_2$ $\exists E \in \mathcal{E}$ $a, b \in E \ni$. $\delta_1(a) = \delta_2(b) = 1$ and $\delta_i(y) = 0$. Otherwise $\forall y \in E$ $i = 1, 2$.

Since $\beta \geq \delta_1, \delta_2$, we have $\beta \geq \alpha$.

Let $E_1 \in \mathcal{E}_x$. For $y \in E_1 \setminus \{x\}$, $\beta(y) = 0$, so $\alpha(y) = 0$.

Thus $\alpha(x) = \bar{\alpha}(1)$.

Now $2 = \bar{\beta}(1) \geq \bar{\alpha}(1) \geq \bar{\alpha}(a) + \bar{\alpha}(b) \geq \delta_1(a) + \delta_2(b) = 2 \therefore \alpha(1) = 2$.

Since $(X, \mathcal{E})$ is a n-regular $\delta : X \to \mathbb{R}$ defined by $\delta(x) = 1$ $\forall x \in X$ is a prob. measure. $\delta \geq \delta_1, \delta_2$ so $\gamma \geq \alpha$.

Therefore, $1 = \gamma(x) \geq \alpha(x) = 2$. But $1 \not\geq 2$. Contradiction.

Thus, $\delta_1 \vee \delta_2$ does not exist.

Since B is linearly compact, $B = \Omega$ is not a simplex.

<u>Corollary</u>. $(X, \mathcal{E})$ be a finite n-regular space with a full set of dispersion free weights. If $\Omega$ is a simplex, then $(X, \mathcal{E})$ is a classical space.

We conclude this section with a result which was inadvertently omitted from [ 3 ]. It is a corollary to Proposition 3.6 of [3].

Remark. If $\omega$ is a pure weight on the finite orthogonality space $(X, \mathcal{E})$, then $\omega(x)$ is rational for all $x \in X$.

Proof. Let $Y = X - Z$ , $F = \{E \cap Y | E \in \mathcal{E}\}$ . By Theorem 3.4 and Proposition 3.6 of [ 3 ] , $(Y,F)$ is an orthogonality space and $\mu_{df} = \omega|Y$ is the only weight on $(Y,F)$; moreover $\mu(y) > 0$ for all $y \in Y$ .
Let $Y = \{y_1, \ldots, y_k\}$ , $F = \{E_1, \ldots, E_n\}$ and $w_{ij} = \begin{cases} 1 & \text{if } y_j \in E_i \\ 0 & \text{if } y_j \notin E_i \end{cases}$ .

By Proposition 3.7 of [ 3 ], $k \leq n$ . Let $W = (w_{ij})$ . Then

$W \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = (W')^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}$ admits a unique solution $x_i = \mu(i)$ , $i = 1, \ldots, k$ .

Thus there exists a $k \times k$ submatrix $W'$ of $W$ such that

$W' \begin{pmatrix} x \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{k \times 1}$ has a unique solution. We conclude that $(W')^{-1}$ exists and has rational entries. Since $\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = (W')^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ each $x_i = \mu(y_i)$ is rational.

## 2. Self-dual Spaces

Let $(X,\mathcal{E})$ be any space. We say that $(X,\mathcal{E})$ is <u>self dual</u> if there are bijections $X \overset{\alpha}{\to} \mathcal{E} \overset{\beta}{\to} X$ such that $x \in E$ if and only if $\beta(E) \in \alpha(x)$.

<u>Proposition 2.1</u>. Let $(X,\mathcal{E})$ be any space (not necessarily finite), and $X \overset{\alpha}{\to} \mathcal{E} \overset{\beta}{\to} X$ be any mappings. Then

1) $\alpha$ 1-1 and $\alpha(E) = \mathcal{E}_{\beta(E)}$ for all $E \in \mathcal{E}$ implies $\beta$ is 1-1.
2) $\beta$ onto and $\alpha(E) = \mathcal{E}_{\beta(E)}$ for all $E \in \mathcal{E}$ implies $\alpha$ is onto.
3) $\alpha$ onto and $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in \mathcal{E}$ implies $\beta$ is onto.
4) $(X,\mathcal{E})$ distinguishing, $\beta$ 1-1, and $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in E$ implies $\alpha$ is 1-1.

<u>Proof</u>. Ad (1). Suppose that $\beta(E) = \beta(F)$. This gives $\alpha(E) = \mathcal{E}_{\beta(E)} = \mathcal{E}_{\beta(F)} = \alpha(F)$. Since $\alpha$ is 1-1 we get $E = \alpha^{-1}(\alpha(E)) = \alpha^{-1}(\alpha(F)) = F$. This $\beta$ is 1-1. Ad (2). Let $E \in \mathcal{E}$. Choose $x \in E$ and since $\beta$ is onto choose $F \in \mathcal{E}$ so that $\beta(F) = x$. Now $\alpha(F) = \mathcal{E}_{\beta(F)} = \mathcal{E}_x$ so there is $y \in F$ for which $\alpha(y) = E$. Thus $\alpha$ is onto. Ad (3). Let $x \in X$. Choose $E \in \mathcal{E}$ so that $x \in E$. Since $\alpha$ is onto we can choose $y \in X$ so that $\alpha(y) = E$. Now $E = \alpha(y) = \beta(\mathcal{E}_y)$ so there is an $F \in \mathcal{E}_y$ for which $\beta(F) = x$. Thus $\beta$ is onto. Ad (4). Suppose that $\alpha(x) = \alpha(y)$. Then $\beta(\mathcal{E}_x) = \beta(\mathcal{E}_y)$ which gives $\mathcal{E}_x = \mathcal{E}_y$ since $\beta$ is 1-1. But if $(X,\mathcal{E})$ is distinguishing this implies $x = y$.

<u>Proposition 2.2</u>. Let $(X,\mathcal{E})$ be any space (not necessarily finite) and $X \overset{\alpha}{\to} \mathcal{E} \overset{\beta}{\to} X$ be any mappings. The following are equivalent.

1) $\alpha$ and $\beta$ are bijections and $x \in E \Leftrightarrow \beta(E) \in \alpha(x)$.
2) $\alpha$ is 1-1, $\beta$ is onto, and $\alpha(E) = \mathcal{E}_{\beta(E)}$ for all $E \in \mathcal{E}$.
3) $\beta$ is 1-1, $\alpha$ is 1-1, $\alpha$ is onto, and $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in X$.

If the space $(X,\mathcal{E})$ is distinguishing then 3) is equivalent to 3') $\beta$ is 1-1, $\alpha$ is onto, and $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in X$.

Proof. 1) $\Rightarrow$ 2). Assume that 1) is true. Let $\alpha(x) \in \alpha(E)$, $x \in E$. Then $\beta(E) \in \alpha(x)$ so that $\alpha(x) \in \mathcal{E}_{\beta(E)}$. Thus $\alpha(E) \subseteq \mathcal{E}_{\beta(E)}$. Let $F \in \mathcal{E}_{\beta(E)}$. Choose x so that $\alpha(x) = F$. Then $\beta(E) \in \alpha(x)$ so we let $x \in E$. Thus $F \in \alpha(E)$. Thus $\mathcal{E}_{\beta(E)} \subseteq \alpha(E)$.

2) $\Rightarrow$ 3). Assume that 2) is true. By proposition 2.1 $\alpha$ and $\beta$ are bijections. Let $y \in \alpha(x)$. Choose E so that $\beta(E) = y$. Now $\alpha(x) \in \mathcal{E}_{\beta(E)} = \alpha(E)$ so that $x \in E$. Thus $E \in \mathcal{E}_x$ so that $y \in \beta(\mathcal{E}_x)$. Thus $\alpha(x) \subseteq \beta(\mathcal{E}_x)$. Let $\beta(E) \in \beta(\mathcal{E}_x)$, $x \in E$. Then $\alpha(x) \in \alpha(E) = \mathcal{E}_{\beta(E)}$ so that $\alpha(x) \in \mathcal{E}_{\beta(E)}$ which gives $\beta(E) \in \alpha(x)$. Thus $\beta(\mathcal{E}_x) \subseteq \alpha(x)$.

3) $\Rightarrow$ 1). Assume 3) is true. By proposition 2.1 $\alpha$ and $\beta$ are bijections. Let $x \in E$. Then $E \in \mathcal{E}_x$ so that $\beta(E) \in \beta(\mathcal{E}_x) = \alpha(x)$. Conversely, suppose $\beta(E) \in \alpha(x)$. Then $\beta(E) \in \beta(\mathcal{E}_x)$ so that $E \in \mathcal{E}_x$ which gives $x \in E$.

Finally, Proposition 2.1 shows that 3') is equivalent to 3) whenever $(X,\mathcal{E})$ is distinguishing.

Proposition 2.3. Suppose that $(X,\mathcal{E})$ is any space (not necessarily finite), and $X \overset{\alpha}{\rightarrow} \mathcal{E} \overset{\beta}{\rightarrow} X$ are bijections which make $(X,\mathcal{E})$ self dual. Then

1) $(X,\mathcal{E})$ is distinguishing.

2) $\beta(E) \in \bigcap_{x \in E} \alpha(x)$ for all $E \in \mathcal{E}$.

3) Suppose that $E \in \mathcal{E}$, $F \in \mathcal{E}$, $E \neq F \Rightarrow E \not\subseteq F$. Then $(X,\mathcal{E})$ is separating. Also, $\bigcap_{x \in E} \alpha(x) = \{\beta(E)\}$ so that $\beta$ is uniquely determined by $\alpha$ for such spaces.

4) If $(X,\mathcal{E})$ is an orthogonality space then $(X^*,\mathcal{E}^*)$ is an orthogonality space.

Proof. By definition of self duality we have that $x \in E \Leftrightarrow \beta(E) \in \alpha(x)$. Ad(1). Suppose $\mathcal{E}_x = \mathcal{E}_y$. By 3) of Proposition 2.2 we get that $\alpha(x) = \alpha(y)$. Since $\alpha$ is one-to-one $x = y$. Ad (2). Let $E \in \mathcal{E}$. Then $x \in E$ implies $\beta(E) \in \alpha(x)$ so that $\beta(E) \in \bigcap_{x \in E} \alpha(x)$. Ad (3). By proposition 2.2 we have $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in X$. Suppose $x \neq y$. Then $\alpha(x) \neq \alpha(y)$ so that $\alpha(x) \not\subseteq \alpha(y)$ so that $\beta(\mathcal{E}_x) \not\subseteq \beta(\mathcal{E}_y)$ so that $\mathcal{E}_x \not\subseteq \mathcal{E}_y$. Thus $(X,\mathcal{E})$ is separating. Finally, let $y \in \bigcap_{x \in E} \alpha(x)$. Choose $F \in \mathcal{E}$ so that $\beta(F) = y$. Let $x \in E$. Then $\beta(F) \in \alpha(x)$ so that $x \in F$. Thus $E \subseteq F$ which gives $E = F$ so that $y = \beta(E)$. Ad (4). Suppose that $(X,\mathcal{E})$ is an orthogonality space. By part 3 $(X,\mathcal{E})$ is separating. We will show that the hypothesis of 1.2 is satisfied. Suppose we have $x \in \bigcap_{i=1}^{n} E_i$ and $x_i \in E_i \cap E$ $1 \leq i \leq n$. Then $\{\beta(E),\beta(E_i)\} \subseteq \alpha(x_i)$ $1 \subseteq i \subseteq n$ and $\{\beta(E_i) \mid 1 \leq i \leq n\} \subseteq \alpha(x)$ so that $\{\beta(E_i) \mid 1 \leq i \leq n\} \cup \{\beta(E)\}$ is a $\perp_{\mathcal{E}}$ set. Choose $F \in \mathcal{E}$ containing this set and let $\alpha(y) = F$. Then $\beta(E_i) \in \alpha(y)$ $1 \leq i \leq n$ and $\beta(E) \in \alpha(y)$ gives $y \in \bigcap_{i=1}^{n} E_i \cap E$. By proposition 1.2 $(X^*,\mathcal{E}^*)$ is an orthogonality space.

Corollary 2.4. Let $(X,\mathcal{E})$ be a finite space and $X \overset{\alpha}{\to} \mathcal{E} \overset{\beta}{\to} X$ be bijections. Let $\alpha$ and $\beta$ also denote the extensions $F(X) \overset{\alpha}{\to} F(\mathcal{E}) \overset{\beta}{\to} F(X)$. $\alpha$ and $\beta$ make $(X,\mathcal{E})$ self dual if and only if we have the operator equations $T\alpha = \beta S$, where $S$ is the linear realization of $(X^*, \mathcal{E}^*)$.

Proof. By proposition 2.2 $\alpha$ and $\beta$ make $(X,\mathcal{E})$ self dual if and only if $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in X$. Now let $x \in X$. Then $T\alpha(x) = \sum_{y \in \alpha(x)} y$ and

$$\beta S(x) = \sum_{x \in E} \beta(E) = \sum_{E \in \mathcal{E}_x} \beta(E) = \sum_{y \in \beta(\mathcal{E}_x)} y. \quad \sum_{y \in \alpha(x)} y = \sum_{y \in \beta(\mathcal{E}_x)} y$$

for all $x \in X$ if and only if $\alpha(x) = \beta(\mathcal{E}_x)$ for all $x \in X$.

Proposition 2.5 . Let $(X,\mathcal{E})$ be a finite space. Suppose that there is an integer n such that $|E| = |\mathcal{E}_x| = n$ for all $E \in \mathcal{E}$ and $x \in X$. Then $|X| = |\mathcal{E}|$; moreover, $(X,\mathcal{E})$ is self dual if and only if there is a bijection $\alpha : X \to \mathcal{E}$ such that $\left| \bigcap_{x \in E} \alpha(x) \right| = 1$ for all $E \in \mathcal{E}$.

Proof. Let $I = \{(x,E) \mid x \in X,\ E \in \mathcal{E}\}$. Then $|I| = \sum_{x \in X} |\mathcal{E}_x| = n|X|$ and $|I| = \sum_{E \in \mathcal{E}} |E| = n|\mathcal{E}|$. Thus $|X| = |\mathcal{E}|$. Now $|E| = |F| = n$ for all $E,F \in \mathcal{E}$ so that $E \neq F$ implies $E \not\subseteq F$. Part 3) of proposition 2.3 now shows that the existence of such an $\alpha$ is a necessary condition for self duality. Conversely, suppose such an $\alpha$ exists. Define $\beta : \mathcal{E} \to X$ by $\beta(E) \in \bigcap_{x \in E} \alpha(x)$. Let $E \in \mathcal{E}$. By definition $\alpha(E) \subseteq \mathcal{E}_{\beta(E)}$. But $|\alpha(E)| = |E| = |\mathcal{E}_{\beta(E)}|$ so that $\alpha(E) = \mathcal{E}_{\beta(E)}$. By part 1 of proposition 2.1 $\beta$ is 1-1. Now $\beta : \mathcal{E} \to X$ is 1-1 and $|\mathcal{E}| = |X|$ so that $\beta$ is onto. By proposition 2.2 part 2 $(X,\mathcal{E})$ is self dual.

Let $(X,\mathcal{E})$ be any distinguishing space. By a subspace of type I we mean a space $(Y,\mathcal{F})$ such that

1) $Y = X$, $\mathcal{F} \subseteq \mathcal{E}$

2) $\cup \mathcal{F} = X$

3) $\mathcal{F}_x = \mathcal{F}_y \Rightarrow x = y$, for all x,y in X.

By a subspace of type II we mean a space $(Y,\mathcal{F})$ such that

1) $Y \subseteq X$, $\mathcal{F} = \mathcal{E}_y$

2) Y is $\mathcal{E}$-supporting

3) $E \cap Y = E_1 \cap Y \Rightarrow E = E_1$, for all $E,E_1$ in $\mathcal{E}$.

Theorem 2.6. Let $(X,\mathcal{E})$ be any space and suppose $X \overset{\alpha}{\to} \mathcal{E} \overset{\beta}{\to} X$ establishes $(X,\mathcal{E})$ as being self dual.

1) If $(X,\mathcal{F})$ is a subspace of type I then $(X,\mathcal{F})^*$ can be identified with a subspace of type II.

2) If $(Y,\mathcal{E}_Y)$ is a subspace of type II then $(Y,\mathcal{E}_Y)^*$ can be identified as a subspace of type I.

<u>Proof</u>.

1) Let $(X,\mathcal{R})$ be a subspace of type I. Then $(X,\mathcal{R})^* = (\mathcal{R},\mathcal{R}^*)$ where $\mathcal{R}^* = \{\mathcal{R}_x \mid x \in X\}$. Define $Y = \beta(\mathcal{R}) \subseteq X$. Then $\beta$ gives a bijection between $\mathcal{R}$ and Y. Let $x \in X$. Then $\beta(\mathcal{R}_x) = \{\beta(F) \mid x \in F, F \in R\}$ $= \{\beta(F) \mid \beta(F) \in \alpha(x), F \in R\} = \alpha(x) \cap \beta(\mathcal{R}) = \alpha(x) \cap Y$. This shows that $\beta$ induces a bijection between $\mathcal{R}^*$ and $\mathcal{E}_y$, and also shows that Y is supporting. Suppose $E \cap Y = E_1 \cap Y_1$. Choose $x,x_1$ so that $\alpha(x) = E$ and $\alpha(x_1) = E_1$. Then $\alpha(x) \cap Y = \alpha(x_1) \cap Y$ shows that $\beta(\mathcal{R}_x) = \beta(\mathcal{R}_y)$. Since $\beta$ is injective we get $\mathcal{R}_x = \mathcal{R}_y$ and thus we can conclude $x = y$. This establishes 1.

2) Let $(Y,\mathcal{E}_y)$ be a subspace of type II. Now $(Y,\mathcal{E}_y)^* = (\mathcal{E}_Y,\mathcal{E}_Y^*)$ where $\mathcal{E}_Y^* = \{\mathcal{E}_{Yy} \mid y \in Y\}$. By conditions 2) and 3) the mapping $E \cap Y \to E$ is a bijection between $\mathcal{E}_y$ and $\mathcal{E}$. We apply $\alpha^{-1}$ to get a bijection $E \cap Y \overset{\varphi}{\to} \alpha^{-1}(E)$ between $\mathcal{E}_y$ and X. We now show that $\varphi(\mathcal{E}_{Yy}) = \beta^{-1}(y)$ for each $y \in Y$. Suppose $E \in \mathcal{E}_{Yy}$. Then $y \in E \cap Y$. $\varphi(E) = x$ where $\alpha(x) = E$. Since $y \in \alpha(x)$ we get $x \in \beta^{-1}(y)$. Thus $\varphi(\mathcal{E}_{Yy}) \subseteq \beta^{-1}(y)$. Suppose $x \in \beta^{-1}(y)$. Then $y \in \alpha(x)$ so that $\alpha(x) \cap Y \in \mathcal{E}_{Yy}$ and $x = \varphi(\alpha(x) \cap Y) \in \varphi(\mathcal{E}_{Yy})$. Thus $\varphi(\mathcal{E}_{Yy}) = \beta^{-1}(y)$ for each $y \in Y$. Therefore $(Y,\mathcal{E}_Y)^*$ can be identified with the space $(X,\beta^{-1}(Y))$. The proof just given shows that $\bigcup_{y\in Y} \beta^{-1}(y) = X$. Finally, assume $\beta^{-1}(Y)_x$ $= \beta^{-1}(Y)_y$. Then $\alpha(x) \cap Y = \beta(\beta^{-1}(Y)_x) = \beta(\beta^{-1}(Y)_y) = \alpha(y) \cap Y$ which implies $\alpha(x) = \alpha(y)$ by condition 3) so that $x = y$. Thus $(X,\beta^{-1}(Y))$ is a subspace of type I.

## 3. Automorphisms

Let $(X,\mathcal{E})$ be any space and $\varphi:X \to X$ a bijection. Then $\varphi$ induces a bijection $\tilde{\varphi}:2^X \to 2^X$ by $\tilde{\varphi}(E) = \varphi(E)$ for all $E \subseteq X$. If $\varphi$ is such that $\tilde{\varphi}(\mathcal{E}) = \mathcal{E}$ we say that $\varphi$ is an <u>automorphism</u> of the space $(X,\mathcal{E})$. We denote the set of all automorphisms of $(X,\mathcal{E})$ by Aut $(X,\mathcal{E})$. Since $\widetilde{\varphi\psi} = \tilde{\varphi}\tilde{\psi}$ and $\widetilde{\psi^{-1}} = \tilde{\psi}^{-1}$ Aut $(X,\mathcal{E})$ is a group. For any space $(X,\mathcal{E})$ and $\varphi \in$ Aut $(X,\mathcal{E})$ we will denote $\tilde{\varphi}|\mathcal{E}$ by $\bar{\varphi}$.

<u>Lemma 3.1</u>. Let $\varphi \in$ Aut $(X,\mathcal{E})$. For each $x \in X$ we have that $\bar{\varphi}(\mathcal{E}_x) = \mathcal{E}_{\varphi(x)}$.

<u>Proof</u>. $x \in E$ if and only if $\varphi(x) \in \varphi(E) = \bar{\varphi}(E)$. ∎

<u>Theorem 3.2</u>. Let $(X,\mathcal{E})$ be any space such that $\mathcal{E}$ is finite. If $\varphi:X \to X$ is a bijection such that $\varphi(E) \in \mathcal{E}$ for all $E$ in $\mathcal{E}$ then $\varphi$ is an automorphism of $(X,\mathcal{E})$.

<u>Proof</u>. By assumption $\tilde{\varphi}(\mathcal{E}) \subseteq \mathcal{E}$. $\varphi$ 1-1 and $\mathcal{E}$ finite implies $\tilde{\varphi}(\mathcal{E}) = \mathcal{E}$. ∎

<u>Theorem 3.3</u>. Let $\varphi \in$ Aut$(X,\mathcal{E})$. Then $\bar{\varphi} \in$ Aut $(X^*,\mathcal{E}^*)$.

<u>Proof</u>. We know that $\bar{\varphi}: \mathcal{E} \to \mathcal{E}$ is a bijection, and $X^* = \mathcal{E}$. $\tilde{\bar{\varphi}}:2^{\mathcal{E}} \to 2^{\mathcal{E}}$. $\mathcal{E}^* = \{\mathcal{E}_x \mid x \in X\}$. Let $\mathcal{E}_x \in \mathcal{E}^*$. By lemma 1 $\tilde{\bar{\varphi}}(\mathcal{E}_x) = \bar{\varphi}(\mathcal{E}_x) = \mathcal{E}_{\varphi(x)} \in \mathcal{E}^*$. Also, $\tilde{\bar{\varphi}}(\mathcal{E}_{\varphi^{-1}x}) = \bar{\varphi}(\mathcal{E}_{\varphi^{-1}x}) = \mathcal{E}_x$. Thus $\tilde{\bar{\varphi}}(\mathcal{E}^*) = \mathcal{E}^*$. ∎

We thus have a homomorphism $\psi:$Aut $(X,\mathcal{E}) \to$ Aut $(X^*,\mathcal{E}^*)$ by $\psi(\varphi) = \bar{\varphi}$. Now suppose that $(X,\mathcal{E})$ is distinguishing and $\varphi \in$ Aut $(X^*,\mathcal{E}^*)$. Then $\bar{\varphi} \in$ Aut $(X^{**},\mathcal{E}^{**})$. Since $(X,\mathcal{E})$ is distinguishing $(X^{**},\mathcal{E}^{**}) \cong (X,\mathcal{E})$ by $x \leftrightarrow \mathcal{E}_x$ and $E \leftrightarrow \mathcal{E}^*_E$. Let us denote by $\underline{\varphi}:X \to X$ the mapping obtained from $\bar{\varphi}$ by making these identifications. Then $\underline{\varphi}(x) = \bar{\varphi}(\mathcal{E}_x) = \varphi(\mathcal{E}_x)$. Now $\varphi(\mathcal{E}_x) \in \mathcal{E}^*$ so that there is an element $y$ in $X$ such that $\varphi(\mathcal{E}_x) = \mathcal{E}_y$ and since $(X,\mathcal{E})$ is distinguishing this element is unique. Thus $\underline{\varphi}(x) = \varphi(\mathcal{E}_x) = \mathcal{E}_y = y$. We thus have $\underline{\varphi} \in$ Aut $(X,\mathcal{E})$ defined by the equations $\varphi(\mathcal{E}_x) = \mathcal{E}_{\underline{\varphi}(x)}$, $x \in X$.

Theorem 3.4. If $(X, \mathcal{E})$ is a distinguishing space, then mapping $\psi: \text{Aut }(X, \mathcal{E}) \to \text{Aut }(X^*, \mathcal{E}^*)$ is an isomorphism.

Proof. Define $\theta: \text{Aut }(X^*, \mathcal{E}^*) \to \text{Aut }(X, \mathcal{E})$ by $\theta(\varphi) = \underline{\varphi}$. Suppose that $\varphi \in \text{Aut }(X^*, \mathcal{E}^*)$. Then by lemma 3.1 $\underline{\varphi}(E) = \overline{\varphi}(\mathcal{E}^*_E) = \varphi(E)$. The equation $\underline{\varphi}(E) = \varphi(E)$ shows that $\underline{\varphi\theta(\varphi)} = \overline{(\underline{\varphi})} = \varphi$. Let $\varphi \in \text{Aut }(X, \mathcal{E})$. Again using lemma 3.1, we get $\overline{\varphi}(\mathcal{E}_x) = \mathcal{E}_{\varphi(x)}$. By the definition of $\theta$ this gives $\theta\Psi(\varphi) = \underline{(\overline{\varphi})} = \varphi$. Thus $\theta = \psi^{-1}$.

In conclusion we mention that G. Schrag has integrated some of the results of this paper and of [ 2 ] with known results in graph theory to obtain the following result. Schrag's paper has not yet appeared. However, the proof may be found in [ 5 ].

Theorem (Schrag): Every finite group is the automorphism group of some finite orthomodular lattice.