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A REMARK ON ADAMS' AND FINCH'S PAPER

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Abstract

In establishing a tensor product of Boolean algebras B_1 and B_2, Adams and Finch state (Lemma 5.1) a fundamental lemma. While the statement is correct, there is a critical error in the proof. We supply a correct proof for this lemma.

§1. Introduction

In an attempt to obtain a tensor product of two Boolean algebras B_1 and B_2, Adams and Finch construct a Boolean algebra $B_1 \boxplus B_2$ from an orthogonality relation on $B_1 \times B_2$ and realize the tensor product $B_1 \otimes B_2$ as the set of finitely generated elements in $B_1 \boxplus B_2$. This product could be constructed by regarding B_1 and B_2 as Boolean algebras of sets and using the field product of Sikorski [ ]. However, as Adams and Finch point out, "By regarding B_1 as the Boolean algebra of propositions about an experiment $\mathcal{E}_1$ and B_2 as the Boolean algebra of propositions about an experiment $\mathcal{E}_2$, the tensor product construction given below is motivated by the need to construct the appropriate Boolean algebra of propositions for the combined experiment $(\mathcal{E}_1, \mathcal{E}_2)$." Here is a sketch of their argument.

Let B_1, B_2, and B be Boolean algebras. Let $B_1 \times B_2$ denote the cartesian product of <u>the sets</u> B_1 and B_2. For $(x_1,x_2), (y_1,y_2) \in B_1 \times B_2$ write $(x_1,x_2) \perp (y_1,y_2)$ if $x_1 \perp y_1$ or $x_2 \perp y_2$. For $M \subseteq B_1 \times B_2$, let $M^\perp = \{(x_1,x_2) \in B_1 \times B_2 \mid (x_1,x_2) \perp (y_1,y_2) \text{ for all } (y_1,y_2) \in M\}$ and let $M^{\perp\perp} = (M^\perp)^\perp$. Let $B_1 \boxtimes B_2 = \{M \subseteq B_1 \times B_2 \mid M = M^{\perp\perp}\}$. Then $B_1 \boxtimes B_2$ is a complete Boolean algebra.

Recall that $\phi : B_1 \to B$ is a <u>lattice morphism</u> if ϕ preserves join and meet (but not necessarily orthocomplementation).

Call $\beta : B_1 \times B_2 \to B$ a <u>bimorphism</u> if

(i) for all $x_2 \in B_2$, $x_1 \mapsto \beta(x_1, x_2)$ is a lattice morphism,

(ii) for all $x_1 \in B_1$, $x_2 \mapsto \beta(x_1, x_2)$ is a lattice morphism,

(iii) $\beta(1_1, 1_2) = 1$, and

(iv) $(x_1, x_2) \perp (y_1, y_2)$ implies $\beta(x_1, x_2) \perp \beta(y_1, y_2)$ in B.

If $x \in B$, let $I(x)$ denote the principal order ideal generated by x.

An element $M \in B_1 \boxtimes B_2$ is said to be <u>finitely generated</u> if there exists a finite subset $F \subseteq B_1 \times B_2$ with $F^{\perp\perp} = M$. The set $B_1 \otimes B_2$ of all finitely generated elements of $B_1 \boxtimes B_2$ is a Boolean subalgebra of $B_1 \boxtimes B_2$. It is clearly closed under join. That it is closed under orthocomplementation is the content of the following:

<u>Lemma 4.1</u>. Let $N = \{1, 2, \ldots, n\}$, (n finite) and let $(x_{j,1}, x_{j,2})$ be n elements of $B_1 \times B_2$. Write $x_{0,1} = 0_1$, $x_{0,2} = 0_2$, and $\Phi = \{1,2\}^N$. For each $\phi \in \Phi$, define

$$y_{\phi,1} = \bigwedge_{j \in N} x^{\perp}_{(2-\phi(j))j,1}$$

and

$$y_{\phi,2} = \bigwedge_{j \in N} x^{\perp}_{(\phi(j)-1)j,2}$$

If $F = \{(x_{j,1}, x_{j,2}) \mid j \in N\}$, then

$$F^{\perp} = \bigcup_{\phi \in \delta} I_1(y_{\phi,1}) \times I_2(y_{\phi,2})$$

and

$$F^{\perp} = \Box^{\perp\perp} \cup \bigcup_{\phi \in \lambda} (I_1(y_{\phi,1}) \times I_2(y_{\phi,2}))$$

Remark. For $(x_1,x_2) \in B_1 \times B_2$, define $\beta^*((x_1,x_2)) = \{(x_1,x_2)\}^{\perp\perp}$. Then $\beta^*: B_1 \times B_2 \to B_1 \otimes B_2$ is a bimorphism.

Lemma 5.1. Let B_1 and B_2 be Boolean algebras, F a finite subset of $B_1 \times B_2$. Then there exists a finite indexing set A and a subset $\{(z_{\alpha,1}, z_{\alpha,2}) : \alpha \in A\}$ of $B_1 \times B_2$ such that

$$F^{\perp\perp} = \Box^{\perp\perp} \cup [\cup_{\alpha \in A} I_1(z_{\alpha,1}) \times I_2(z_{\alpha,2})].$$

Moreover, for all $\alpha \in A$ there exist finite indexing sets P_α and Q_α and subsets $\{s_{p,1} : p \in P_\alpha\}$ and $\{s_{q,2} : q \in Q_\alpha\}$ of B_1 and B_2 respectively such that

$$z_{\alpha,1} = \vee\{s_{p,1} : p \in P_\alpha\},\ z_{\alpha,2} = \vee\{s_{q,2} : q \in Q_\alpha\},$$

and for all $(p,q) \in P_\alpha \times Q_\alpha$ there exists $(x_1,x_2) \in F$ such that $s_{p,1} \leq x_1$, and $s_{q,2} \leq x_2$.

Using this lemma it follows in a relatively easy fashion that $(\beta^*, B_1 \otimes B_2)$ is a tensor product of the Boolean algebras B_1 and B_2.

Theorem 6.1. If $\gamma : B_1 \times B_2 \to B$ is a bimorphism and if $\beta^* : B_1 \times B_2 \to B_1 \otimes B_2$ is defined by $\beta^*((x_1,x_2)) = (x_1,x_2)^{\perp\perp}$

then there exists a unique morphism $\phi : B_1 \otimes B_2 \to B$ such that $\gamma = \phi\, \beta^*$.

This is the main result of the paper. It leans heavily on lemma 5.1 the proof of which in [ ] depends on the false statement "$S_\phi \cap T_\psi = \square$ implies $\phi = \psi$." (The notation will be established below.) Our purpose is to supply a valid proof for lemma 5.1. In order to make this proof reasonably self-contained we supply the computational proof of lemma 4.1.

§2. The proof

Proof of Lemma 4.1:

$$F^\perp = \bigcap_{j\in N} (x_{j,1}, x_{j,2})^\perp = \bigcap_{j\in N} [(B_1 \times I_2(x^\perp_{j,2})) \cup (I_1(x^\perp_{j,1}) \times B_2)]$$

$$= \bigcap_{j\in N} [(I_1(x^\perp_{0,1}) \times I_2(x^\perp_{j,2})) \cup (I_1(x^\perp_{j,1}) \times I_2(x^\perp_{0,2}))].$$

Letting $S_{1,j} = I_1(x^\perp_{j,1}) \times I_2(x^\perp_{0,2})$ and $S_{2,j} = I_1(x^\perp_{0,1}) \times I_2(x^\perp_{j,2})$ we have

$$F^\perp = \bigcap_{j\in N} \bigcup_{i\in\{1,2\}} S_{i,j} = \bigcup_{\phi\in\{1,2\}^N} \bigcap_{j\in N} S_{\phi(j),j}$$

$$= \bigcup_{\phi\in\{1,2\}^N} \left(\left(\bigcap_{\phi(j)=1} S_{1,j}\right) \cap \left(\bigcap_{\phi(j)=2} S_{2,j}\right)\right)$$

$$= \bigcup_{\phi\in\Phi} \left(\left(I_1\left(\bigwedge_{\phi(j)=1} x^\perp_{j,1}\right) \times I_2\left(x^\perp_{0,2}\right)\right) \cap \left(I_1\left(x^\perp_{0,1}\right) \times I_2\left(\bigwedge_{\phi(j)=2} x^\perp_{j,2}\right)\right)\right),$$

$$= \bigcup_{\phi\in\Phi} I_1\left(\bigwedge_{\phi(j)=1} x^\perp_{j,1}\right) \times I_2\left(\bigwedge_{\phi(j)=2} x^\perp_{j,2}\right)$$

$$= \bigcup_{\phi\in\Phi} I_1\left(\bigwedge_{j\in N} x^\perp_{(2-\phi(j))j,1}\right) \times I_2\left(\bigwedge_{j\in N} x_{(\phi(j)-1)j,2}\right)$$

$$= \bigcup_{\phi \in \Phi} I_1(y_{\phi,1}) \times I_2(y_{\phi,2})$$

But clearly $\square^{\perp\perp} \subseteq F^{\perp}$ so that

$$F^{\perp} = \square^{\perp\perp} \cup \bigcup_{\phi \in \Phi} I_1(y_{\phi,1}) \times I_2(y_{\phi,2}). \qquad \text{q.e.d.}$$

Proof of Lemma 5.1:

Write $F = \{(x_{j,1}, x_{j,2}) : j \in N\}$ and use the notation of lemma 4.1. Then

$$F^{\perp\perp} = \Big[\square^{\perp\perp} \cup \bigcup_{\phi \in \Phi} I_1(y_{\phi,1}) \times I_2(y_{\phi,2})\Big]^{\perp}$$

$$= B_1 \times B_2 \cap \bigcap_{\phi \in \Phi} (y_{\phi,1}, y_{\phi,2})^{\perp}$$

$$= \bigcap_{\phi \in \Phi} (y_{\phi,1}, y_{\phi,2})^{\perp}$$

$$= \bigcap_{\phi \in \Phi} [(B_1 \times I_2(y^{\perp}_{\phi,2})) \cup (I_1(y^{\perp}_{\phi,1}) \times B_2)]$$

Hence, as in the proof of lemma 4.1,

$$F^{\perp\perp} = \square^{\perp\perp} \cup \bigcup_{\alpha \in A_0} I_1(z_{\alpha,1}) \times I_2(z_{\alpha,2})$$

where $A_0 = \{1,2\}^{\Phi}$, $z_{\alpha,1} = \bigwedge_{\phi \in \Phi} y^{\perp}_{(2-\alpha(\phi))\phi,1}$,

$z_{\alpha,2} = \bigwedge_{\phi \in \Phi} y_{(\alpha(\phi)-1)\phi,2}$ where $0\phi \underset{\text{df.}}{=} 0$, $1\phi \underset{\text{df.}}{=} 1$,

$y_{0,1} = 0_1$ and $y_{0,2} = 0_2$.

For $\phi \in \Phi$, let $S_{\phi} = \{j \in N \mid \phi(j) = 1\}$, $T_{\phi} = \{j \in N \mid \phi(j) = 2\}$.

For $\alpha \in A_0$, let $\Phi_{\alpha,1} = \{\phi \in \Phi \mid \alpha(\phi) = 1\}$,

$\Phi_{\alpha,2} = \{\phi \in \Phi \mid \alpha(\phi) = 2\}$.

Let $S_1^{\alpha} = \bigcup_{\phi \in \Phi_{\alpha,1}} S_{\phi}$, $T_2^{\alpha} = \bigcup_{\phi \in \Phi_{\alpha,2}} T_{\phi}$.

Define $\psi_i : N \to \{1,2\}$ by $\psi_i(j) \equiv i$ for all $j \in N$, $i = 1,2$.

Define $\delta_i : \Phi \to \{1,2\}$ by $\delta_i(\phi) \equiv i$ for all $\phi \in \Phi$, $i = 1,2$.

Claim (1). $F^{\perp\perp} = \Box^{\perp\perp} \cup \bigcup_{\alpha\in A_1} I_1(z_{\alpha,1}) \times I_2(z_{\alpha,2})$

where $A_1 = A_o \setminus \{\delta_1, \delta_2\}$.

For, $z_{\delta_1,1} = \bigwedge_{\phi\in\Phi} y^{\perp}_{(2-\delta_1(\phi))\phi,1} = \bigwedge_{\phi\in\Phi} y^{\perp}_{\phi,1}$

$= \bigwedge_{\phi\in\Phi} \left(\bigvee_{j\in N} x_{(2-\phi(j))j,1}\right) \leq \bigvee_{j\in N} x_{(2-\phi_2(j))j,1} = \bigvee_{j\in N} x_{0,1} = 0_1$

Hence $I_1(z_{\delta_1,1}) \times I_2(z_{\delta_1,2}) = I_1(0_1) \times I_2(z_{\delta_1,2}) \subseteq \Box^{\perp\perp}$.

Similarly $I_1(z_{\delta_2,1}) \times I_2(z_{\delta_2,2}) \subseteq \Box^{\perp\perp}$. Hence the claim.

¶ Note that, for $\alpha \in A_1$, $\Phi_{\alpha,1} \neq \Box$ and $\Phi_{\alpha,2} \neq \Box$.

Claim (2). For $\alpha \in A_1$, if $S_1^{\alpha} = \Box$ or $T_2^{\alpha} = \Box$,

then $I_1(z_{\alpha,1}) \times I_2(z_{\alpha,2}) \subseteq \Box^{\perp\perp}$.

Let $\alpha \in A_1$. We may assume $S_1^{\alpha} = \Box$.

Then $z_{\alpha,1} = \bigwedge_{\phi\in\Phi} y^{\perp}_{(2-\alpha(\phi))\phi,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} y^{\perp}_{\phi,1}$

$= \bigwedge_{\phi\in\Phi_{\alpha,1}} \bigvee_{j\in N} x_{(2-\phi(j))j,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} \bigvee_{j\in N} x_{0,1} = 0_1$.

As in claim (1), the result follows.

¶ Let $A_2 = \{\alpha \in A_1 \mid S_1^{\alpha} \neq \Box$ and $T_2^{\alpha} \neq \Box\}$. It follows that

$$F^{\perp\perp} = \Box^{\perp\perp} \cup \bigcup_{\alpha\in A_2} I_1(z_{\alpha,1}) \times I_2(z_{\alpha,2}).$$

Fix an arbitrary $\alpha \in A_2$, $z_{\alpha,1} = \bigwedge_{\phi\in\Phi} y^{\perp}_{(2-\alpha(\phi))\phi,1}$

$= \bigwedge_{\phi\in\Phi_{\alpha,1}} y^{\perp}_{\phi,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} \bigvee_{j\in N} x_{(2-\phi(j))j,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} \bigvee_{j\in S_\phi} x_{j,1}$.

Now, for $\phi \in \Phi_{\alpha,1}$, let $x_{j,1,\phi} = \begin{cases} x_{j,1} & \text{if } j \in S_\phi \\ 0 & \text{otherwise} \end{cases}$. (The following computation is valid even if some S_ϕ is empty.)

$$z_{\alpha,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} \bigvee_{j\in S_1^\alpha} x_{j,1,\phi} = \bigvee_{\lambda\in S_1^{\alpha\Phi_{\alpha,1}}} \bigwedge_{\phi\in\Phi_{\alpha,1}} x_{\lambda(\phi),1,\phi}$$

$$= \bigvee_{\substack{\lambda\in S_1^{\alpha\Phi_{\alpha,1}} \\ \lambda(\phi)\in S_\phi}} \bigwedge_{\phi\in\Phi_{\alpha,1}} x_{\lambda(\phi),1}$$

For $\lambda \in S_1^{\alpha\Phi_{\alpha,1}}$ with $\lambda(\phi) \in S_\phi$, and $\mu \in T_2^{\alpha\Phi_{\alpha,2}}$ with $\mu(\phi) \in T_\phi$, let $s_{\alpha,\lambda,1} = \bigwedge_{\phi\in\Phi_{\alpha,1}} x_{\lambda(\phi),1}$ and $s_{\alpha,\mu,2} = \bigwedge_{\phi\in\Phi_{\alpha,2}} x_{\mu(\phi),2}$.

Then $z_{\alpha,1} = \bigvee_{\substack{\lambda\in S_1^{\alpha\Phi_{\alpha,1}} \\ \lambda(\phi)\in S_\phi}} s_{\alpha,\lambda,1}$,

and similarly, $z_{\alpha,2} = \bigvee_{\substack{\mu\in T_2^{\alpha\Phi_{\alpha,2}} \\ \mu(\phi)\in T_\phi}} s_{\alpha,\mu,2}$.

Now let $I_1 = \{\lambda \in S_1^{\alpha\Phi_{\alpha,1}} \mid \lambda(\phi) \in S_\phi\}$ and $I_2 = \{\mu \in T_2^{\alpha\Phi_{\alpha,2}} \mid \mu(\phi) \in T_\phi\}$. Note that $I_1, I_2 \neq \Box$.

For $\lambda \in I_1$ and $\mu \in I_2$,

define $\phi_{\lambda,\alpha} \in \Phi$ by $\phi_{\lambda,\alpha}(j) = 1$ iff there exists $\phi \in \Phi_{\alpha,1}$ such that $\lambda(\phi) = j$,

define $\phi^{\mu,\alpha} \in \Phi$ by $\phi^{\mu,\alpha}(j) = 1$ iff there exists $\phi \in \Phi_{\alpha,2}$ such that $\mu(\phi) = j$.

Then $S_{\phi_{\lambda,\alpha}} = \{\lambda(\phi) \mid \phi \in \Phi_{\alpha,1}\}$, $T_{\phi^{\mu,\alpha}} = \{\mu(\phi) \mid \phi \in \Phi_{\alpha,2}\}$,

(*) $s_{\alpha,\lambda,1} = \bigwedge_{j\in S_{\phi_{\lambda,\alpha}}} x_{j,1}$, and $s_{\alpha,\mu,2} = \bigwedge_{j\in T_{\phi^{\mu,\alpha}}} x_{j,2}$.

Claim (3). There exists $\phi \in \Phi_{\alpha,1}$ and there exists $\psi \in \Phi_{\alpha,2} \ni S_\phi \cap T_\psi \neq \Box$.

Suppose not, then for all $\phi \in \Phi_{\alpha,1}$ and for all $\psi \in \Phi_{\alpha,2}$, $S_\phi \cap T_\psi = \Box$ so that $S_\phi \subseteq (N \setminus T_\psi) = S_\psi$. It follows that $S_1^\alpha = \bigcup_{\phi \in \Phi_{\alpha,1}} S_\phi \subseteq S_\psi$ for all $\psi \in \Phi_{\alpha,2}$. Since $\alpha \in A_2$, there exists $j_o \in S_{\phi_o}$ for some $\phi_o \in \Phi_{\alpha,1}$. Define $\theta \in \Phi$ by $\theta(j) = 2$ iff $j = j_o$. Then $S_\theta = N \setminus \{j_o\}$ and, since $S_{\phi_o} \not\subseteq S_\theta$, $\theta \in \Phi \setminus \Phi_{\alpha,2} = \Phi_{\alpha,1}$. Hence $S_1^\alpha = N$, so that $S_\psi = N$ and $T_\psi = \Box$ for all $\psi \in \Phi_{\alpha,2}$. But then $T_2^\alpha = \bigcup_{\phi \in \Phi_{\alpha,2}} T_\phi = \Box$ contradicting the fact that $\alpha \in A_2$. Hence the claim.

It follows that there exists $\phi \in \Phi_{\alpha,1}$ and there exists $\psi \in \Phi_{\alpha,2} \ni \lambda(\phi) = \mu(\psi)$. Hence $S_{\phi_{\lambda,\alpha}} \cap T_{\phi\mu,\alpha} = \{\lambda(\phi) \mid \phi \in \Phi_{\alpha,1}\} \cap \{\mu(\varphi) \mid \phi \in \Phi_{\alpha,2}\} \neq \Box$. Let $k \in S_{\phi_{\lambda,\alpha}} \cap T_{\phi\mu,\alpha}$. By (*) $s_{\alpha,\lambda,1} \leq x_{k,1}$ and $s_{\alpha,\mu,2} \leq x_{k,2}$. Let $A = A_2$, $P_\alpha = I_1$, and $Q_\alpha = I_2$; the proof is complete.