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COMBINATORIAL QUANTUM LOGIC

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The purpose of this talk is to illustrate some connections between block designs and quantum logic. We begin by defining a quantum logic, motivating this definition, and expounding the connection with graph theory. We then pass from "orthogonality graphs" to "orthogonality spaces" and their corresponding "logics". Certain natural conditions on the logic induce new and interesting conditions on the underlying graphs. (The way in which these conditions interact with the more familiar graph theoretic conditions, such as chromatic number, connectivity, etc., has not yet been investigated.)

We then study weights on orthogonality graphs, giving special emphasis to dispersion free weights and relating these to perfect matchings of the dual space.

§1. Definitions and the Prototypical Example. A <u>quantum logic</u> is a pair $(\mathcal{L}, \mathcal{S})$ where $\mathcal{L}$ is a σ-orthocomplete orthomodular poset and $\mathcal{S}$ is an order determining (full) set of states on $\mathcal{L}$. We define each of these terms and then give some motivation for the terminology.

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An orthomodular poset $\mathcal{L}$ consists of a non-empty set $\mathcal{L}$ together with a partial ordering $\leq$ and an orthocomplementation $' : \mathcal{L} \to \mathcal{L}$ which satisfy the following: For all $x,y,z \in \mathcal{L}$

(1) There exist $0,1 \in \mathcal{L}$ with $0 \leq x \leq 1$ for all $x \in \mathcal{L}$;

Moreover for all $x,y,z \in \mathcal{L}$

(2) $x \leq x$,

(3) $x \leq y$ and $y \leq x$ imply $x = y$,

(4) $x \leq y$ and $y \leq z$ imply $x \leq z$,

(5) $(x')' = x$,

(6) $x \leq y$ implies $y' \leq x'$,

(7) If $x \leq y'$ then $x \vee y$, the least upper bound of x and y, exists,

(8) $x \vee x' = 1$ and $x \wedge x' = 0$,

(9) $x \leq y$ implies $y = x \vee (y \wedge x')$.

The $\wedge$'s in (8) and (9) exist because of (7) and the fact that the (generalized) de Morgan Laws hold in the setting defined by (1) - (6).

Two elements x,y in $\mathcal{L}$ are said to be orthogonal, written $x \perp y$, in case $x \leq y'$. $\mathcal{L}$ is called σ-orthocomplete if every countable family of pairwise orthogonal elements of $\mathcal{L}$ have a least upper bound in $\mathcal{L}$.

A state (or generalized probability measure) on a σ-orthocomplete orthomodular poset is a mapping α from $\mathcal{L}$ into the real unit interval satisfying the following:

(1) $\alpha(1) = 1$,

(2) If $\{x_i : i \in I\}$ is a countable family of pairwise orthogonal elements of $\mathcal{L}$ then $\alpha(\bigvee_{i \in I} x_i) = \sum_{i \in I} \alpha(x_i)$.

A set $\mathcal{S}$ of states $\mathcal{L}$ is order determining or full in case

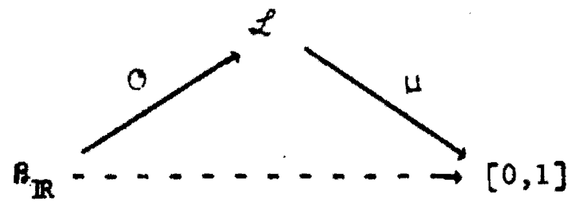
(*) if $\alpha(x) \leq \alpha(y)$ for every $\alpha \in \mathcal{S}$, then $x \leq y$.

The converse of (*) always obtains because of (9). Let $\mathcal{S}(\mathcal{L})$ denote the set of all states on $\mathcal{L}$.

The usual model for quantum mechanics, herein called the <u>standard quantum logic</u>, is the pair $(\mathcal{L}, \mathcal{S})$ where $\mathcal{L}$ is the complete orthomodular lattice of all projections on a separable infinite dimensional complex Hilbert space $\mathcal{H}$ and $\mathcal{S}$ is the set of all states on $\mathcal{L}$. It turns out that every $\mu \in \mathcal{S}$ is induced by a density operator, i.e. a trace class operator T of trace 1. In fact, each $\mu \in \mathcal{S}$ is of the form μ_T where $\mu_T(P) =$ trace(TP) where P is a projection and trace(T) = 1 .

In the usual axiomatic framework [6] an observable $\mathcal{O}$ is a σ-homomorphism from the Real Borel sets, $\mathcal{B}_{\mathbb{R}}$, into a Boolean subalgebra of $\mathcal{L}$. $\mathcal{L}$ is interpreted as the set of yes-no questions which may be asked about a physical system. Given an observable $\mathcal{O}$, a Borel set $E \subseteq \mathbb{R}$, and a state μ, the composition $\mu(\mathcal{O}(E))$ is interpreted as the probability that a measurement of the observable $\mathcal{O}$ will yield a result in the Borel set E if the system is in the state μ. The situation is given diagramatically in Figure 1.

Figure 1.



One of the goals of quantum logic is to explain why the Hilbert space model works so well. One way to do this is to give a set of physically motivated axioms from which Hilbert space may be derived. The main feature of the combinatorial approach to quantum logic appears to be the ability to give models of quantum logics which show that given axioms or properties are independent. However there are two other aspects of this approach which may be equally important. The first is the theory which has begun to surface--such results as Schrag's theorem which states that every finite group is the automorphism group of some quantum logic. The second (closely related) aspect is the connection with graph theory and combinatorial designs.

§2. Graphs, Spaces, and Logics. Henceforth we assume that X is a finite set, $\perp$ an irreflexive symmetric relation on X, i.e. $(X,\perp)$ is a graph admitting no loops or multiple edges. For $M \subseteq X$, let $M^{\perp} = \{x \in X \mid x \perp m \text{ for all } m \in M\}$, $M^{\perp\perp} = (M^{\perp})^{\perp}$. $M \subseteq X$ is called an orthogonal set or a $\perp$-set if $x \perp y$ for all $x,y \in M$ with $x \neq y$. A maximal $\perp$-set is called a clique. The logic, $\mathcal{L}$ or $\mathcal{L}(X,\perp)$, over $(X,\perp)$ is the set $\{M^{\perp\perp} \mid M \subseteq X \text{ and } M \text{ is a } \perp\text{-set}\}$ partially ordered by set theoretic inclusion and orthocomplemented by $^{\perp}$. The logic $\mathcal{L}(X,\perp)$ is an orthomodular poset if and only if the following condition, due to J. C. Dacey [1], obtains in $(X,\perp)$:

(D) If E is a clique and $x,y \in X$ with $E \subseteq x^{\perp} \cup y^{\perp}$, then $x \perp y$.

A graph satisfying condition (D) is called a Dacey graph.

A space is a pair $(X,\mathcal{E})$ where X is a nonempty set and $\mathcal{E}$ is a family of nonempty subsets of X. Every (finite) space induces a graph $(X,\perp_{\mathcal{E}})$ by defining $x \perp_{\mathcal{E}} y$ to mean that $x \neq y$ and $\{x,y\} \subseteq E$ for some $E \in \mathcal{E}$. $(X, \mathcal{E})$ is called an <u>orthogonality space</u> if the set of cliques of the graph $(X, \perp_{\mathcal{E}})$ is $\mathcal{E}$. $(X, \mathcal{E})$ is called a <u>Dacey space</u> in case it is an orthogonality space and the corresponding graph $(X, \perp_{\mathcal{E}})$ is a Dacey graph.

The graph given in Figure 2 generates the logic whose Hasse diagram is given in Figure 3; the associated orthogonality space is given in Figure 4.

Figure 2.

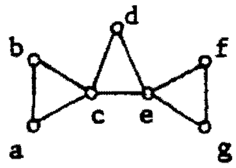


Figure 3.

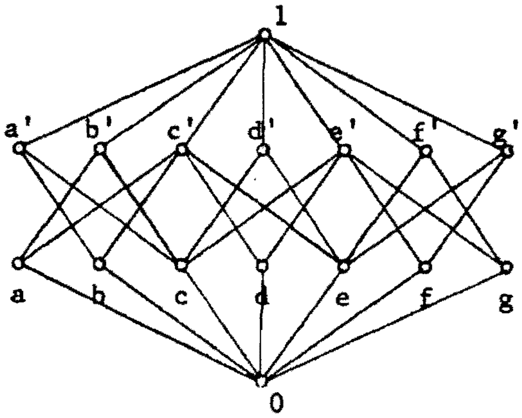
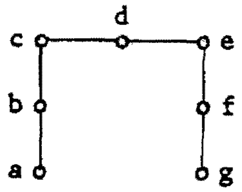


Figure 4.



A graph $(X,\perp)$ is called <u>point closed</u> in case $x^{\perp\perp} = x$ for all $x \in X$. The cliques of a point closed Dacey graph $(X,\perp)$ correspond to the maximal Boolean subalgebras of the logic $\mathcal{L}(X,\perp)$. Combinatorial quantum logic focusses on the intertwining (i.e. intersection) patterns of the cliques in order to glean information about the logic [2, 3, 4, 5]. These intertwining patterns are most perspicuous when the graph (or logic) is represented by an orthogonality space.

§3. Weights, Duality, and Perfect Matchings. A <u>weight</u> on a space $(X,\mathcal{E})$ is a mapping, $\omega : X \to [0,1]$, from X to the real unit interval such that $\sum_{x\in E} \omega(x) = 1$ for all $E \in \mathcal{E}$. Let $\Omega(X,\mathcal{E})$, or simply Ω, denote the set of all weights on $(X,\mathcal{E})$. Ω is said to be <u>full</u> in case $\omega(x) + \omega(y) \leq 1$ for all $\omega \in \Omega$ implies $x \perp_{\mathcal{E}} y$.

Let $(X,\mathcal{E})$ be a Dacey space. Each weight ω on $(X,\mathcal{E})$ induces a state $\bar{\omega}$ on $\mathcal{L}(X,\mathcal{E})$ by defining $\bar{\omega}(D^{\perp\perp}) = \sum_{d\in D} \omega(d)$ for each orthogonal set $D \subseteq X$. $\omega \mapsto \bar{\omega}$ is a bijection between the weight space $\Omega(X,\mathcal{E})$ and the state space $\mathcal{S}(\mathcal{L}(X,\mathcal{E}))$. Moreover $\Omega(X,\mathcal{E})$ is full if and only if $\mathcal{S}(\mathcal{L}(X,\mathcal{E}))$ is full [5].

There are two types of weights which are especially important: dispersion free weights and pure weights. A weight ω is <u>dispersion free</u> in case the range of ω is a subset of the set $\{0,1\}$; ω is <u>pure</u> in case ω may not be written as a non-trivial convex linear combination of weights, i.e. $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $0 \leq \lambda \leq 1$, $\omega_1, \omega_2 \in \Omega$ implies $\lambda = 0$ or $\lambda = 1$. Let Ω_p denote the set of pure weights in Ω, $\Omega_{d.f.}$ the set of dispersion free weights in Ω. Then $\Omega_{d.f.} \subseteq \Omega_p$.

We illustrate the combinatorial flavor of finite orthogonality spaces with three examples. The first, given in Figure 5, is a Dacey space with an empty weight space [3]. The second, given in Figure 6, is a Dacey space which admits exactly one weight (cf. [5]). This 23-element space appears here for the first time in print. No Dacey space admitting only one weight and having a smaller cardinality is known. The third space, given in Figure 7, is a Dacey space the logic of which is a quantum logic; it is not embeddable in the standard quantum logic [4].

Figure 5.

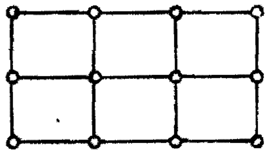


Figure 6.

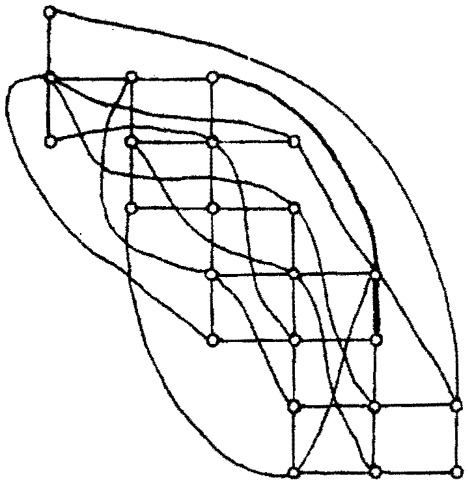
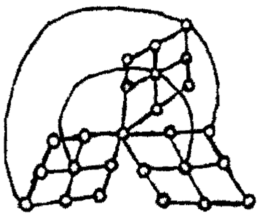


Figure 7.



Let $Z_\omega = \{x \in X \mid \omega(x) = 0\}$. Since X is a finite set, ω is pure if and only if $Z_\omega \not\subset Z_\mu$ for all weights μ distinct from ω [5]. Since the zero sets of the pure weights form an antichain in the power set of X, there are only finitely many pure weights on a finite space.

Let F(X) denote the free vector space over X; F(X)* the dual vector space, and $C = \{f \in F(X)^* \mid f(x) \geq 0$ for each $x \in X\}$. We may regard Ω as a subset of $F(X)^* \cap C$ by extending each ω in Ω by linearity. Ω is then a closed convex subset of F(X)* which is generated by its extreme points.

Conjecture: If $(X, \mathcal{E})$ is a finite orthogonality space and $\Omega(X, \mathcal{E})$ is full, then Ω is a simplex if and only if $\mathcal{L}(X, \mathcal{E})$ is a Boolean algebra.

There are two notable partial results on this conjecture. In order to state the first we must introduce the notion of the <u>linear</u> <u>realization</u> T of a space $(X, \mathcal{E})$ [5]. This is a mapping $T : F(\mathcal{E}) \to F(X)$ defined by $T(E) = \sum_{x \in E} x$ and linear extension. Note that, for any $\mu \in \Omega$, $\Omega = (\mu + (\ker T^*)) \cap C$. The author and F. R. Miller have proved that (for finite X) Ω is a simplex if and only if $|X| + 1 = \dim(\text{image } T) + |\Omega_p|$. The second result, also obtained while working with Miller (and also as yet unpublished), states that the conjecture is true if $(X, \mathcal{E})$ is a regular space and if $x \in X$, $\delta_1, \delta_2 \in \Omega_{d.f.}$ with $\delta_1 \neq \delta_2$ and $\delta_1(x) = \delta_2(x) = 1$. (A space $(X, \mathcal{E})$ is <u>regular</u> in case $|E| = |F|$ for all $E, F \in \mathcal{E}$.)

We now mention some selected results concerning duality and dispersion free weights.

The <u>dual</u> <u>space</u> of a space $(X, \mathcal{E})$ is the space $(X^*, \mathcal{E}^*)$ where $X^* = \mathcal{E}$ and $\mathcal{E}^* = \{\mathcal{E}_x \mid x \in X\}$. Here $\mathcal{E}_x = \{E \in \mathcal{E} \mid x \in E\}$.

Remark [5]: There is a one to one correspondence between disjoint covers of X by members of $\mathcal{E}$ and dispersion free weights on $(X^*, \mathcal{E}^*)$.

Regard $\perp$ as a set of 2-element subsets of X. Recall that a perfect matching of the graph $(X,\perp)$ is a partition of X by members of $\perp$. Assume that there are no isolated points in $(X,\perp)$. Then the following results obtain.

Remark: $(X,\perp)$ admits a perfect matching if and only if $(X^*, \mathcal{E}^*)$ admits a dispersion free weight.

Remark: $(X^*, \mathcal{E}^*)$ admits a full set of dispersion free weights if and only if every pair of disjoint edges of $(X,\perp)$ may be extended to a perfect matching.

Thus the search for full sets of dispersion free weights, which has been associated with the quest for hidden variables in quantum mechanics, is in this setting a perfectly natural combinatorial problem.

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