ON A GENERALIZED PARTIAL ORDER ON SEMIGROUPS AND GROUPS

By

L. M. CHAWLA AND R. J. GREECHIE

0. Introduction and Summary. At its present stage of development the theory and applications of n-ary relations appears significantly more meager than that of binary relations——perhaps because of the plurality of possible generalizations of the more familiar concepts. We present here a generalization of partial order and apply it to semigroups and groups.

Let A be a given non-empty set and A^n the set of all ordered n-tuples $(a_1, \dots, a_n)$, $a_i \in A$, $n \geq 2$. A subset R of A^n is said to be an *n-ply partial order* on A, if R is reflexive, transitive, and anti-symmetric as defined below :

(*i*) R is said to be *reflexive*, if $(a, \dots, a) \in R$, for all $a \in A$, *i.e.* the diagonal $D_n(A) \subseteq R$.

(*ii*) R is said to be *transitive* if each row of an $n \times n$ matrix $\alpha = (a_{ij})$ is in R and each row of its transpose $\alpha' = (a_{ji})$ is in R imply that their common diagonal $(a_{11}, \dots, a_{nn})$ is in R.

(*iii*) R is said to be *anti-symmetric* if $(a_1, \dots, a_n) \in R$ and $(a_n, a_1, \dots, a_{n-1}) \in R$ imply $a_1 = a_2 = \dots = a_n$.

A set A with an n-ply partial order R defined on it shall be called an n-ply partially ordered set and shall be denoted by $(G. R)$. A groupoid (G, R) is called *n-ply partially ordered* if an n-ply partial order R is defined on G such that if $(a_1, \dots, a_n) \in R$ then $(xa_1, \dots, xa_n) \in R$ and $(a_1 x, \dots, a_n x) \in R$ for all $x \in G$. The same definition holds for an n-ply partially ordered semigroup or a group (G, R).

In Section 1, we prove that the set T of all monotonic transformations on a partially ordered set (A, R) is a partially ordered sub-semi-group (T, R') in the symmetric semigroup S of all transformations on A, where R' is an n-ply partial order on T induced by the n-ply partial order R on A. This enables us to generalize Krishnan's Theorem [1] on two-ply partially ordered semigroups to the case of n-ply partially ordered semigroups ; see *Theorem* 1.3 below.

In Section 2, we define certain positively ordered $(n-1)$-tuples of elements of an n-ply partially ordered group (G, R) and prove that the set P of all such $(n-1)$-tuples is a sub-semigroup of G^{n-1}, the direct product of $n-1)$ copies of G. Finally, we establish the necessary and sufficient conditions for a sub-semigroup of G^{n-1}, to be the semi-group of positively ordered $(n-1)$-tuples of (G, R). The case $n = 2$ is detailed in [2]. pp. 256-257.

1. **Lemma 1.1.** Let (G, R) be a partially ordered groupoid. If $(a_1, \ldots, a_n) \in R$ and $(b_1, \ldots, b_n) \in R$, then $(a_1 b_1, \ldots, a_n b_n) \in R$ and $(b_1 a_1, \ldots, b_n a_n) \in R$.

Let $(a_1, \ldots, a_n) \in R$, $(b_1, \ldots, b_n) \in R$ then by the definition of R on G, the rows of the matrices

$$\alpha = \begin{bmatrix} a_1b_1 & a_2b_1 & \cdots & a_nb_1 \\ a_1b_2 & a_2b_2 & \cdots & a_nb_2 \\ a_1b_n & a_2b_n & \cdots & a_nb_n \end{bmatrix}, \quad \alpha' = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}$$

are all in R. Hence their common diagonal $(a_1 b_1, \ldots, a_n b_n) \in R$. Similarly we prove $(b_1 a_1, \ldots, b_n a_n) \in R$.

Definition 1. Two partially ordered groupoids (semigroups or groups) (G, R) and (G', R') are said to be *isomorphic*, if there exists a one-to-one mapping ϕ of G on to G' such that for all a, b, a_i in G

(*i*) $(ab)\phi = a\phi \cdot b\phi$.

(*ii*) $(a_1, \ldots, a_n) \in R$, if and only if $(a_1\phi, \ldots, a_n\phi) \in R'$.

Definition 2. A mapping ϕ of a partially ordered set (A, R) into itself is called a *monotonic transformation* if $(a_1, \ldots, a_n) \in R$ implies $(a_1\phi, \ldots, a_n\phi) \in R$.

We now prove the following

Theorem 1.2. The set T of all monotonic transformations on a partially ordered set (A, R) is a partially ordered sub-semigroup (T, R') in the symmetric semigroup S of all transformations on A, where R' is an n-ply partial order on T induced by the n-ply partial order R on A.

Proof. Since the identity mapping E of A is a monotonic transformation on (A, R), it follows that T is non-empty. Let $\phi, \psi \in T$, then for any $(a_1, \ldots, a_n) \in R$, we have $(a_1\phi, \ldots, a_n\phi) \in R$ and hence $((a_1\phi)\psi, \ldots, (a_n\phi)\psi) \in R$ and thus $\phi\psi \in T$. It is immediate that T is a sub-semigroup of S and has E as a unit.

We now define an n-ply partial order R' on T as follows: Let $(\phi_1, \ldots, \phi_n) \in R'$ if and only if $(a\phi_1, \ldots, a\phi_n) \in R$ for all $a \in A$.

It is evident that R' is an n-ply partial order on T as a set. That R' is a partial order on T as a semigroup is proved as follows: Let θ be an arbitrary montonic transformation on (A,R), then $(\phi_1, \ldots, \phi_n) \in R'$ and $(a\phi_1, \ldots, a\phi_n) \in R$, for all $a \in A$, imply $(a(\phi_1\theta), \ldots, a(\phi_n\theta)) \in R$, for all $a \in A$. Hence $(\phi_1\theta, \ldots, \phi_n\theta) \in R'$. Also $((a\theta)\phi_1, \ldots, (a\theta)\phi_n) \in R$, for all $a \in A$, it follows that $(\theta\phi_1, \ldots, \theta\phi_n) \in R'$. Hence R' is an n-ply partial order on the semigroup T. This completes the proof.

We are now in a position to generalize Krishnan's Theorem [1] on two-ply partially ordered semigroups to the case of n-ply partially ordered semigroups as follows:

Theorem 1.3. Every n-ply partially ordered semigroup (G, R) with a unit element e can be isomorphically embedded in the n-ply partially ordered semigroup (T, R') of all the monotonic transformations of (G, R).

Proof. Assign the transformation ϕ_a to every element a of G by the rule, $x\phi_a = x a$, for all $x \in G$. Let $(x_1, \ldots, x_n) \in R$ then since $(x_1 a, \ldots, x_n a) \in R$ or $(x_1 \phi_a, \ldots, x_n \phi_a) \in R$ it follows that ϕ_a is a monotonic transformation. Further since G has a unit, it follows that the mapping $\psi : a \to \phi_a$ is an isomorphic mapping of G into the symmetric semigroup of G and consequently into the sub-semigroup T of all monotonic transformations of G. That ψ is an isomorphism of (G, R) into (T, R') as n-ply partially ordered sets can be seen as follows: Let $(a_1, \ldots, a_n) \in R$, then for all $x \in G$, we have $(x a_1, \ldots, x a_n) \in R$ or $(x\phi_{a_1}, \ldots, x\phi_{a_n}) \in R$ and hence $(\phi_{a_1}, \ldots, \phi_{a_n}) \in R'$. Conversely if $(\phi_{a_1}, \ldots, \phi_{a_n}) \in R'$, it follows in particular that $(e\phi_{a_1}, \ldots, e\phi_{a_n}) \in R$, i.e., $(e a_1, \ldots, e a_n) = (a_1, \ldots, a_n) \in R$.

Corollary. Every n-ply partially ordered group (G, R) can be isomorphically embedded in n-ply partially ordered group (T, R') of all monotonic permutations ϕ_a defined by $x\phi_a = x a$, for all $x \in G$.

2. Let (G, R) be a group with an n-ply partial order R on G. Then the ordered $(n-1)$-tuple $(a_1, \ldots, a_{n-1})$, $a_i \in G$, is said to be *positively ordered relative to* R, if and only if $(a_1, a_2, \ldots, a_{n-1}, e) \in R$ where e is the identity in G. Since the n-tuple $(e, \ldots, e) \in R$ it follows that the $(n-1)$-tuple $(e, \ldots, e)$ is positively ordered. Hence the set P of all positively ordered $(n-1)$-tuples is non-empty.

Lemma 2.1. If (G, R) is a group with an n-ply partial order R on G, then the set P of all positively ordered $(n-1)$-tuples relative to R is a sub-semigroup of G^{n-1}, the direct product of $n-1$ copies of G.

This follows immediately from lemma 1.1 and the definition of P. It may be noted that P admits the $(n-1)$-tuple $(e, \ldots, e)$ as its identity.

In the next theorem, we establish the necessary and sufficient conditions for a sub-semigroup P of G^{n-1} to be the semigroup of positively ordered $(n-1)$-tuples of G, relative to a suitable n-ply partial order R on G.

Theorem 2.2. Let G be a group, then sub-semigroup P of G^{n-1} is the semigroup of positively ordered $(n-1)$-tuples of G, relative to a suitable n-ply partial order R on G, if and only if

(*i*) $(e, \ldots, e) \in P$.

(*ii*) $(a_1, a_2, \ldots, a_{n-1}) \in P$ and $(a_{n-1}^{-1}, a_{n-1}^{-1} a_1, \ldots, a_{n-1}^{-1} a_{n-2}) \in P$ imply $a_1 = a_2 = \ldots = a_{n-1} = e$.

(*iii*) $(a_1, a_2, \ldots, a_{n-1}) \in P$, $x \in G$, then $(x^{-1} a_1 x, \ldots, x^{-1} a_{n-1} x) \in P$.

(*iv*) If (a_{ij}) is an $n \times n$ matrix with $(a_{il} a_{in}^{-1}, \ldots, a_{i,n-1} a_{in}^{-1}) \in P$ and $(a_{1j} a_{nj}^{-1}, \ldots, a_{n-1,j} a_{nj}^{-1}) \in P$, $i, j = 1, \ldots, n$ then $(a_{11} a_{nn}^{-1}, \ldots, a_{n-1,n-1} a_{nn}^{-1}) \in P$.

Proof. To prove that the conditions are necessary, let (G, R) be a group with an n-ply partial order R defined on it. We know by lemma 2.1 that the set P of positively ordered $(n-1)$-tuples is a sub-semigroup of G^{n-1} with the $(n-1)$-tuple $(e, \ldots, e)$, as its identity. This proves (*i*).

Now let $(a_1, a_2, \dots, a_{n-1})$ and $(a_{n-1}^{-1}, a_{n-1}^{-1} a_1, \dots, a_{n-1}^{-1} a_{n-2}) \in P$. Hence $(a_1, a_2 \dots, a_{n-1}, e)$ and $(a_{n-1}^{-1}, a_{n-1}^{-1} a_1, \dots, a_{n-1}^{-1} a_{n-2}, e) \in R$. Hence $(a_1, a_2, \dots, a_{n-1}, e)$ and $(e, a_1, \dots, a_{n-1}) \in R$. Hence by the anti-symmetry of R, we have $a_1 = a_2 = \dots = a_{n-1} = e$. This proves (*ii*). For (*iii*), let $(a_1, \dots, a_{n-1}) \in P$. Hence $(a_1, \dots, a_{n-1}, e) \in R$. Hence by definition of R, $(x^{-1} a_1 x, \dots, x^{-1} a_{n-1} x, x^{-1} ex) \in R$. Hence $(x^{-1} a_1 x, \dots, x^{-1} a_{n-1} x) \in P$.

To prove (*iv*), let (a_{ij}) be an $n \times n$ matrix with $(a_{i1} a_{in}^{-1}, \dots, a_{i,n-1} a_{in}^{-1}) \in P$ and $(a_{1j} a_{nj}^{-1}, \dots, a_{n-1,j} a_{nj}^{-1}) \in P$ for all $i, j = 1, \dots, n$. Then $(a_{i1} a_{in}^{-1}, \dots, a_{i,n-1} a_{in}^{-1}, e) \in R$ so that $(a_{i1}, \dots, a_{in}) \in R$. Similarly, $(a_{1j}, \dots, a_{nj}) \in R$. Since R is transitive $(a_{11}, \dots, a_{nn}) \in R$, $(a_{11} a_{nn}^{-1}, \dots, a_{n-1,n-1} a_{nn}^{-1}, e) \in R$, and $(a_{11} a_{nn}^{-1}, \dots, a_{n-1,n-1} a_{nn}^{-1}) \in P$, as desired.

To prove that the conditions (*i*) – (*iv*) are sufficient, suppose that a sub-semigroup P of G^{n-1} has these properties. We now define an n-ply order R on G as follows : $(a_1, \dots, a_n) \in R$ if and only if $(a_1 a_n^{-1}, \dots, a_{n-1} a_n^{-1}) \in P$.

To prove that R is an n-ply partial order on G, we first note that the n-tuples $(a, \ldots, a) \in R$, for all $a \in G$, since the $(n-1)$-tuple $(aa^{-1}, \ldots, aa^{-1})=(e, \ldots, e) \in P$ by (i). Thus R is reflexive. Next let $(a_1, \ldots, a_n)$ and $(a_n, a_1, \ldots, a_{n-1}) \in R$, so that $(a_1a_n^{-1}, \ldots, a_{n-1}a_n^{-1})$ and $(a_na_{n-1}^{-1}, a_1a_{n-1}^{-1}, \ldots, a_{n-2}a_{n-1}^{-1}) \in P$. Let $b_i=a_ia_n^{-1}$, $i=1, 2, \ldots, n-1$. Then the last two relations can be rewritten as $(b_1, b_2, \ldots, b_{n-1}) \in P$ and $(b_{n-1}^{-1}, b_1b_{n-1}^{-1}, \ldots, b_{n-2}b_{n-1}^{-1}) \in P$ and hence by (iii), $(b_1, \ldots, b_{n-1})$ and $(b_{n-1}^{-1}, b_{n-1}^{-1}b_1, \ldots, b_{n-1}^{-1}b_{n-2}) \in P$. Thus by (ii) we have $b_i=a_ia_n^{-1}=e$, $i=1, 2, \ldots, n-1$ or $a_1=a_2=\ldots=a_n$. This proves that R is anti-symmetric. The proof that R is transitive is left to the reader.

We next note that if $(a_1, \ldots, a_n) \in R$ then $(xa_1, \ldots, xa_n)$ and $(a_1x, \ldots, a_nx) \in R$, for all $x \in G$. For if $(a_1, \ldots, a_n) \in R$, then $(a_1a_n^{-1}, \ldots, a_{n-1}a_n^{-1}) \in P$. Hence by (iii), $(x(a_1a_n^{-1})x^{-1}, \ldots, x(a_{n-1}a_n^{-1})x^{-1}) \in P$ for all $x \in G$. Thus $(xa_1(xa_n)^{-1}, \ldots, xa_{n-1}(xa_n)^{-1}) \in P$. Hence $(xa_1, \ldots, xa_{n-1}, xa_n) \in R$, for all $x \in G$. Similarly $(a_1x, \ldots, a_{n-1}x, a_nx) \in R$, for all $x \in G$. Thus R is an n-ply partial order on G. Pinally the positively ordered $(n-1)$-tuples relative to R are just the elements of P. This completes the proof.