WEIGHTS ON SPACES

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INTRODUCTION

Boolean σ-algebras and probability measures arise in the study of the logic of propositions associated with a single experiment, as was pointed out by Kolmogorov [15]. Recently [4, 16, 17] a program has been initiated to generate models for the logic of propositions associated with multiple experiments. These investigations and others on the quantum logic approach to quantum mechanics (e.g., [10, 12, 14]) have motivated us to study generalized probability measures, called weights on a space $(X,\mathcal{E})$.

Intuitively $\mathcal{E}$ represents a collection of experiments. Each experiment E in $\mathcal{E}$ is identified with a set of possible outcomes which determine the experiment, usually the elementary outcomes. $X=\bigcup\mathcal{E}$ is the set of all outcomes under consideration. Two outcomes x, $y \in X$ are said to be orthogonal, denoted $x \perp y$, if there is some experiment E in $\mathcal{E}$ such that x and y are mutually exclusive outcomes of E. (An expanded account of these heuristics may be found in [4].) When $\mathcal{E}$ is the set of all cliques in the graph $(X,\perp)$, $(X,\mathcal{E})$ is called an orthogonality space. We are most interested in

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orthogonality spaces which give rise to orthomodular posets $\mathcal{L}(X,\mathcal{E})$ (as defined in Section 1) for a logic of propositions; these are the Dacey spaces [2]. The combinatorial properties of the collection $\mathcal{E}$ influence the structure of $\mathcal{L}(X,\mathcal{E})$.

In Section (1) we give two characterizations of when a space is an orthogonality space and establish a correspondence between weights on an orthogonality space $(X,\mathcal{E})$ and states on the associated logic. It is this correspondence that allows us to restrict our attention to the study of weights on spaces.

In Section (2) we see that there are point closed spaces having a full set of weights for which the projection postulate of quantum mechanics fails. These spaces have the flavor of a theory of measurement for which the postulate does not apply. Such a situation could occur for other than academic reasons, for example when the "filtering-device" needed is socially, politically, economically or morally unacceptable or unfeasible.

In Section (3) we study the relationship between properties of a space $(X,\mathcal{E})$ and the geometry of the weight space Ω. Under topological hypotheses we obtain a bijection between the elements of certain spaces and the facets of the weight space. We obtain the result that an ortho-bijection between orthogonality spaces is equivalent to an affine structure map between the weight spaces for a nontrivial class of spaces. Section (4) continues this investigation for finite spaces.

Finally we consider symmetry groups of the space, its dual space and weight space showing, in certain cases, that these are isomorphic to the symmetry group of the logic of propositions.

This work shows a relationship between several fields. It is predominantly combinatorial in nature in that it is related to incidence structures, convex set theory and graph theory. Indeed the "stochastic functions on hypergraphs" of [1] are essentially the same as our weights on spaces. However our motivation was originally derived from the foundations of quantum mechanics [14, 20], in particular quantum logic [10, 12] and empirical logic [4, 16, 17]. It is precisely this willingness to depart from the well-honed paths of knowledge and the restrictive paradigms of accepted theory while adhering to the canons of mathematical rigor which is so much in the spirit of him to whom this paper is dedicated.

1. SPACES AND WEIGHTS

By a space we mean a pair $(X,\mathcal{E})$ where X is a nonempty set and $\mathcal{E}$ is a family of nonempty subsets of X. By a weight on a space $(X,\mathcal{E})$ we mean a mapping $\omega : X \to [0,1]$ such that $\sum_{a\in A} \omega(a) = 1$ for all $A \in \mathcal{E}$.

An orthogonality graph is a pair $(X,\perp)$ where $\perp$ is a nonempty symmetric irreflexive relation on the non empty set X. Given an orthogonality graph $(X,\perp)$ we obtain a space $(X,\mathcal{E})$ by taking $\mathcal{E}_\perp = \{E \mid E\subseteq X,\ E \text{ is a maximal} \perp\text{-set}\}$. (Recall that $M\subseteq X$ is a $\perp$-set if $x,y\in M$ and $x\neq y$ imply $x\perp y$). Conversely, given a space $(X,\mathcal{E})$ we obtain an orthogonality space $(X,\perp_\mathcal{E})$ by defining $x\perp_\mathcal{E} y$ to mean that $x \neq y$ and there is a set $A\in\mathcal{E}$ such that $x \in A$ and $y\in A$. We now characterize when $\mathcal{E}$ is the family of maximal orthogonal sets of an orthogonality space.

Let $(X,\mathcal{E})$ be any space. For each $x\in X$ we define

$$\mathcal{E}_x = \{A \mid A\in\mathcal{E} \text{ and } x \in A\} \text{ and } \mathcal{E}(x) = \bigcup\{A \mid A \in \mathcal{E}_x\}.$$

Lemma 1.1. Let $(X,\perp)$ be an orthogonality graph and $\mathcal{E} = \mathcal{E}_\perp$. Then $\mathcal{E}$ satisfies

(1) If $E\in\mathcal{E}$ and $E\notin\mathcal{E}_x$ then $E\nsubseteq\mathcal{E}(x)$.

(2) If $A \subseteq X$ and $A\subseteq \bigcap\{\mathcal{E}(y) \mid y \in A\}$ then there exists $E\in\mathcal{E}$ such that $A \subseteq E$.

Proof. Ad(1). Assume $E\in\mathcal{E}$ and $E\notin\mathcal{E}_x$. Now E is a maximal $\perp$-set and $x\notin E$. Thus there is a $y\in E$ such that $x\perp y$ fails. $y\notin \mathcal{E}(x)$.

Ad(2). Assume $A\subseteq\bigcap\{\mathcal{E}(y) \mid y \in A\}$. Let $x \in A$, $y \in A$ and $x \neq y$. Then $x\in\mathcal{E}(y)$ so that $x\perp y$. Hence A is a $\perp$-set and the result follows.

Theorem 1.2. Let $(X,\mathcal{Q})$ be a space such that $X=\bigcup\{A \mid A \in \mathcal{Q}\}$, and $\mathcal{E}$ be the family of all maximal $\perp_\mathcal{Q}$-sets. Consider the following conditions.

(1) $A\in\mathcal{Q}$ and $A\notin\mathcal{Q}_x$ implies $A\nsubseteq\mathcal{Q}(x)$.

(2) $M\subseteq X$ and $M\subseteq\bigcap\{\mathcal{Q}(x) \mid x\in M\}$ implies that $M\subseteq A$ for some $A\in\mathcal{Q}$. Then, (1) implies that $\mathcal{Q}\subseteq\mathcal{E}$ and (2) implies that $\mathcal{E}\subseteq\mathcal{Q}$. Thus $\mathcal{Q}=\mathcal{E}$ if and only if (1) and (2) hold.

Proof. Ad(1). Assume (1) holds. Let $A\in\mathcal{Q}$. Then A is a $\perp_\mathcal{Q}$-set.

Suppose $y \notin A$ and $A \cup \{y\}$ is a $\perp_{\mathfrak{A}}$-set. For each $a \in A$ there is an $A_a \in \mathfrak{A}$ such that $\{a,y\} \subseteq A_a$. Hence $A \subseteq \mathfrak{A}(y)$ but $A \notin \mathfrak{A}_y$, a contradiction. Thus A is maximal $\perp_{\mathfrak{A}}$-set and $A \in \mathcal{E}$.

Ad(2). Assume (2) holds. Let $E \in \mathcal{E}$. Then E is a $\perp_{\mathfrak{A}}$-set so that $E \subseteq \bigcap \{\mathfrak{A}(y) \mid y \in E\}$. But this means that there is an $A \in \mathfrak{A}$ such that $E \subseteq A$. Since A is a $\perp_{\mathfrak{A}}$-set and $E \in \mathcal{E}$ we have $E = A \in \mathfrak{A}$. Finally, we have proved that (1) and (2) imply $\mathfrak{A} = \mathcal{E}$. If $\mathfrak{A} = \mathcal{E}$ the previous lemma shows that (1) and (2) hold.

Corollary 1.3. Let $(X, \mathfrak{A})$ be a space and $\mathcal{E}$ be the family of all maximal $\perp_{\mathfrak{A}}$-sets. Then $\mathfrak{A} = \mathcal{E}$ if and only if the following conditions are all satisfied.

(1) $\bigcup \{A \mid A \in \mathfrak{A}\} = X$.

(2) If $M, N \in \mathfrak{A}$ and $M \neq N$ then $M \nsubseteq N$.

(3) If $M \subseteq X$ and for all $x,y \in M$ there is an $N_{x,y}$ in $\mathfrak{A}$ with $\{x,y\} \subseteq N_{x,y}$, then there exists $N \in \mathfrak{A}$ with $M \subseteq N$.

We will use the name orthogonality space to refer to a space $(X, \mathcal{E})$ such that $X = \bigcup \mathcal{E}$, and (1) and (2) of Theorem 1.2. hold. (Note that this definition is equivalent to the usual one given earlier.) In this situation we will freely use the symbol $\perp$ to refer to $\perp_{\mathcal{E}}$.

Let $(X, \mathcal{E})$ be an orthogonality space and $\mathcal{O}(X, \perp)$ denote the set of all $\perp$-sets. For each $M \subseteq X$ define $M^{\perp} = \{x \in X \mid x \perp m$ for all $m \in M\}$ and $M^{\perp\perp} = (M^{\perp})^{\perp}$. By the quasilogic of $(X, \mathcal{E})$ (as of $(X, \perp)$) we mean the set $\mathcal{L} = \{D^{\perp\perp} \mid D \in \mathcal{O}(X, \perp)\}$ partially ordered by set theoretic inclusion [2]. If ambiguity threatens we sometimes write $\mathcal{L}(X)$, $\mathcal{L}(X, \mathcal{E})$, or $\mathcal{L}(X, \perp)$ for $\mathcal{L}$. A quasilogic $\mathcal{L}$ is said to be a logic if $M \in \mathcal{L}$ implies $M^{\perp} \in \mathcal{L}$. An orthogonality space $(X, \mathcal{E})$ such that $\mathcal{L}(X, \mathcal{E})$ is a logic is called an orthocomplemented space. Note that if M and N are elements of a logic $\mathcal{L}$ and $M \subseteq N^{\perp}$ then the join of M and N exists in $\mathcal{L}$ and $M \vee N = (D_1 \cup D_2)^{\perp\perp}$ where $D_1^{\perp\perp} = M$ and $D_2^{\perp\perp} = N$. An orthogonality space $(X, \mathcal{E})$ is said to be point closed if $\{x\}^{\perp\perp} = \{x\}$ for all $x \in X$.

An orthogonality space $(X, \mathcal{E})$ is called a Dacey space if, for all $x,y \in X$, $x^{\perp} \cup y^{\perp} \supseteq A$ for some $A \in \mathcal{E}$ implies $x \perp y$. It can be shown that each Dacey space $(X, \mathcal{E})$ is an orthocomplemented space and that $\mathcal{L}(X)$ is an orthomodular poset [2]. We denote the cardinality of a set M by $|M|$.

Proposition 1.4. Suppose that $(X, \mathcal{E})$ is an orthogonality space such that $|A| \geq 3$ for each $A \in \mathcal{E}$ and $|(A_1 \cap A_2)| \leq 1$ for all $A_1 \neq A_2 \in \mathcal{E}$. Then $(X, \mathcal{E})$ is a Dacey space. Such a space is also point closed.

Proof. Assume that $x, y \in X$, $A \in \mathcal{E}$ and $x^\perp \cup y^\perp \supseteq A$. Then $x \neq y$. If $x \in A$ then $x \in y^\perp$ so that $x \perp y$. Similarly, $y \in A$ implies $x \perp y$. Suppose $x \perp y$ fails. Then $x \notin A$ and $y \notin A$. Now $|A| \geqslant 3$ so that

$|A \cap x^\perp| = |A \cap \mathcal{E}(x)|$ or $|A \cap y^\perp| = |A \cap \mathcal{E}(y)|$ is at least 2.

We may assume $|A \cap \mathcal{E}(x)| \geqslant 2$. Choose z_1, z_2 in $A \cap \mathcal{E}(x)$. Now $z_1 \perp z_2$ so that $\{x, z_1, z_2\}$ is a $\perp$-set. Choose $A_1 \in \mathcal{E}$ such that $A_1 \supseteq \{x, z_1, z_2\}$. Since $x \notin A$ we have $A_1 \neq A$. But $|A \cap A_1| \geqslant 2$, a contradiction. The last sentence follows from the previous observation.

By a state on a logic $\mathcal{L} = \mathcal{L}(X, \mathcal{E})$ we mean a mapping $\alpha : X \to [0,1]$ such that $\alpha(X) = 1$ and if $M \subseteq N^\perp$ then $\alpha(M \vee N) = \alpha(M) + \alpha(N)$ for any $M, N \in \mathcal{L}$. A set $\mathcal{S}$ of states on $\mathcal{L}$ is said to be full if, for any $M, N \in \mathcal{L}$, $M \subseteq N$ if and only if $\alpha(M) \leqslant \alpha(N)$ for all $\alpha \in \mathcal{S}$. A set $\mathcal{W}$ of weights on an orthogonality space $(X, \mathcal{E})$ is said to be full if, for any $x, y \in X$, $x \perp y$ if and only if $\omega(x) + \omega(y) \leqslant 1$ for all $\omega \in \mathcal{W}$. A pair $(\mathcal{L}, \mathcal{S})$ where $\mathcal{S}$ is a full set of states on $\mathcal{L}$ is called a quantum logic [10].

Remark 1.5. Every weight on a Dacey space $(X, \mathcal{E})$ gives rise to a state $\overline{\omega}$ on $\mathcal{L}(X)$ by defining, for $M \in \mathcal{L}(X)$, $\overline{\omega}(M) = \sum_{d \in D} \omega(d)$ where $D \in \mathcal{O}(X, \perp)$ and $D^{\perp\perp} = M$.

Theorem 1.6. Let $(X, \mathcal{E})$ be a Dacey space and $\mathcal{L} = \mathcal{L}(X, \mathcal{E})$. Then the mapping $\Phi : \Omega_X \to \mathcal{S}_0$ is an injection where Ω_X is the set of all weights on $(X, \mathcal{E})$, $\mathcal{S}_0$ is the set of all states on $\mathcal{L}$, and $\Phi(\omega) = \overline{\omega}$ as defined in Remark 1.5. Now let $\mathcal{S}$ be any subset of $\Phi(\Omega)$. Then the following are equivalent :

(1) $\mathcal{S}$ is full.

(2) $M, N \in \mathcal{L}$ and $\alpha(M) + \alpha(N) \leqslant 1$ for all $\alpha \in \mathcal{S}$ imply $M \subseteq N^\perp$.

(3) $\mathcal{W} = \Phi^{-1}(\mathcal{S})$ is full.

Furthermore if X is finite Φ is a bijection.

Proof. See [4] or [8].

We have seen that the condition for an arbitrary space $(X, \mathcal{E})$ to be orthogonality space is a relationship between $\mathcal{E}$ and $\perp_{\mathcal{E}}$; namely, that $\mathcal{E}$ is the collection of maximal $\perp_{\mathcal{E}}$ sets. If one wishes to obtain the classical measure algebras as logics of spaces, and hence represent measures as weights on a space, this relationship must be generalized. Such a generalization has been given in [4].

We say that $(X,\mathcal{E})$ is a generalized sample space if conditions (1) and (2) of corollary 1.3. are satisfied together with

(3') If $A,B\in\mathcal{E}$, $C\subseteq A\cup B$, and $x\perp_{\mathcal{E}} y$ holds for all x,y in C with $x\neq y$ then there exists $D\in\mathcal{E}$ so that $C\subseteq D$.

Remark 1.7.

(1) Every orthogonality space is a generalized sample space.

(2) Every finite generalized sample space is an orthogonality space.

The quasilogic of a generalized sample space $(X,\mathcal{E})$ is given by $\mathcal{L}(X,\mathcal{E}) = \{D^{\perp\perp} \mid \exists\, A\in\mathcal{E} \ni D\subseteq A\}$. $(X,\mathcal{E})$ is said to be a Dacey space if $A\in\mathcal{E}$ and $A\subseteq x^{\perp}\cup y^{\perp}$ implies $x\perp y$. Then theorem 1.6. holds for generalized sample spaces. For more details see [4]. The important fact is that the (regular) states on the logic of a generalized sample space are included in our investigations of weights on spaces.

2. SPECTRUM OF AN ELEMENT

Let $(X,\mathcal{E})$ be any space and Ω be the set of weights. If $\omega_1, \ldots, \omega_n\in\Omega$ and $\lambda_1, \ldots, \lambda_n$ are positive real numbers so that $\Sigma\lambda_i = 1$ then $\omega(x) = \sum_{i=1}^{n}\lambda_i\omega_i(x)$ defines an element of Ω so that Ω is a convex set. Let

$$\underline{\underline{A}}(X,\mathcal{E}) = \{f : X\to R \mid \sum_{x\in E}|f(x)| < \infty \text{ for all } E\in\mathcal{E}\}\subseteq R^X$$

with the product topology. Then $\Omega\subseteq\underline{\underline{A}}(X,\mathcal{E})$ is a convex set in the locally convex topological vector space $\underline{\underline{A}}(X,\mathcal{E})$. If we define $S:\underline{\underline{A}}(X,\mathcal{E})\to R^{\mathcal{E}}$ by $S(f)(E) = \sum_{x\in E} f(x)$ then $\bar{\Omega} = S^{-1}(u)\cap K$ where $u\in R^{\mathcal{E}}$ by $u(E) = 1$ for all E and $K = \{f\in\underline{\underline{A}}(X,\mathcal{E}) \mid f(x)\geq 0 \ \forall x\in X\}$

For each $x\in X$ we define the spectrum of x, spec(x) as follows : spec(x) = closure $\{\omega(x) \mid \omega\in\Omega\}$; spec(x) is a closed bounded convex subset of R, thus a closed interval. Define numbers m_x, M_x by spec(x) = $[m_x, M_x]$ if spec(x) $\neq\emptyset$, and $m_x = 1$, $M_x = 0$ if spec(x) $= \emptyset$. The consideration of spaces in which spec(x) $\neq [0,1]$ is important because of the possibility of developing a theory of measurement which utilizes a full set of weights rather than a strong set (cf.3.1.) of weights. In such a theory the well-known projection postulate of Quantum Mechanics may not be valid. Such a theory would not have at its disposal a class of filters which produce with certainty the properties of interest. Such spaces may suggest an extension of the domain of theoretical physics and appear necessary in the behavioral sciences where subjective judgements restrict the purely academic possibilities in experimentation.

For $Y \subseteq X$ we define the induced space $(Y, \mathcal{E}_Y)$ by $\mathcal{E}_Y = \{Y \cap E \mid E \in \mathcal{E} \text{ and } Y \cap E \neq \emptyset\}$.

Proposition 2.1. Let $(X, \mathcal{E})$ be any finite space.

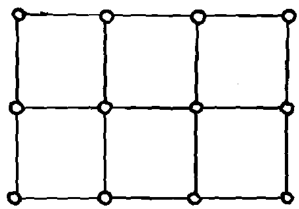
(1) $M_x < 1$ if and only if the space $(X \setminus \mathcal{E}(x), \mathcal{E}_{X \setminus \mathcal{E}(x)})$ admits no weights.

(2) $m_x > 0$ if and only if the space $(X \setminus \{x\}, \mathcal{E}_{X \setminus \{x\}})$ admits no weights.

Proof. We leave the straightforward proof to the reader.

We can now construct orthogonality spaces exhibiting various properties of spec(x). Let $(X_I, \mathcal{E}_I)$ be given by Figure I.

Figure I.



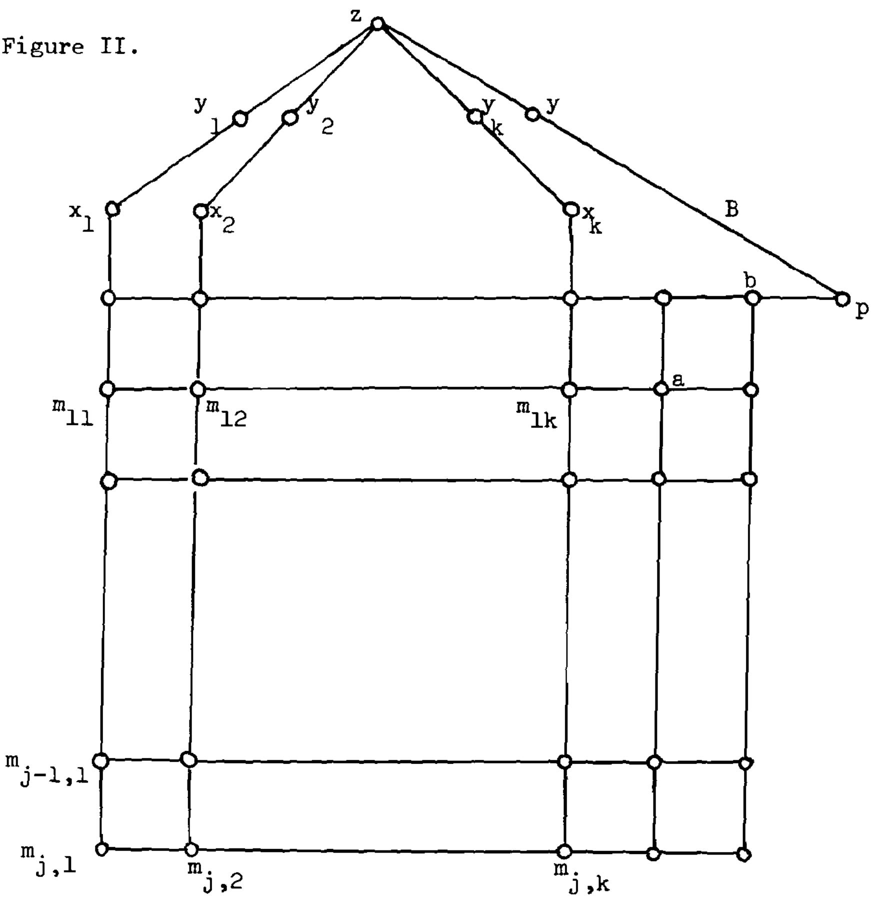
where X_I consists of the twelve points and $\mathcal{E}_I$ consists of the sets of points on maximal line segments, cf. [7,13]. By Proposition 2.1. $(X_I, \mathcal{E}_I)$ is a point closed Dacey-space, so that $(X_I, \mathcal{E}_I)$ is a logic. However, there are no weights on $(X_I, \mathcal{E}_I)$. Suppose that μ is a weight on $(X_I, \mathcal{E}_I)$. Let E_i, $1 \leq i \leq 3$ be the three horizontal members of $\mathcal{E}_I$ and E_i, $4 \leq i \leq 7$ be the four vertical members of $\mathcal{E}_I$. Then

$$3 = \sum_{i=1}^{3} \sum_{x \in E_i} \mu(x) = \sum_{x \in X_I} \mu(x) = \sum_{i=4}^{7} \sum_{x \in E_i} \mu(x) = 4,$$ a contradiction.

We now exhibit, in Figure II, a point closed Dacey-space $(X_{II}, \mathcal{E}_{II})$ admitting an element z such that $\text{spec}(z) = [0, j/k]$

where j and k are arbitrary pre-assigned integers with $1 \leq j < k$.

Figure II.



Let ω be a weight on $(X_{II}, \mathcal{E}_{II})$. Summing the images under ω of the elements of X_{II} in two rays we obtain

$$j + 2 + k - (k-1)\omega(z) + \omega(y) = k + 2 + \omega(z) + \omega(y) + \omega(p) + \sum_{i=1}^{k} \omega(y_i)$$

so that $j = k\omega(z) + \omega(p) + \sum_{i=1}^{k} \omega(y_i)$. Thus $\omega(z) \leq j/k$ and $\mathrm{spec}(z) \subseteq [0, j/k]$.

To see that $\mathrm{spec}(z) = [0, j/k]$ we need only exhibit two weights ω_1 and ω_2 such that $\omega_1(z) = 0$ and $\omega_2(z) = j/k$. For $x \in X_{II}$ and $h, i = 1, \ldots, k$ and $g = 1, \ldots, j$ define ω_1 and ω_2 as follows :

$$\omega_1(x) = \begin{cases} 1 - j/k & \text{if } x = x_i \\ j/k & \text{if } x = y_i \\ 1 & \text{if } x \in \{a,b,y\} \\ 1/k & \text{if } x = m_{g,h} \\ 0 & \text{otherwise} \end{cases}$$

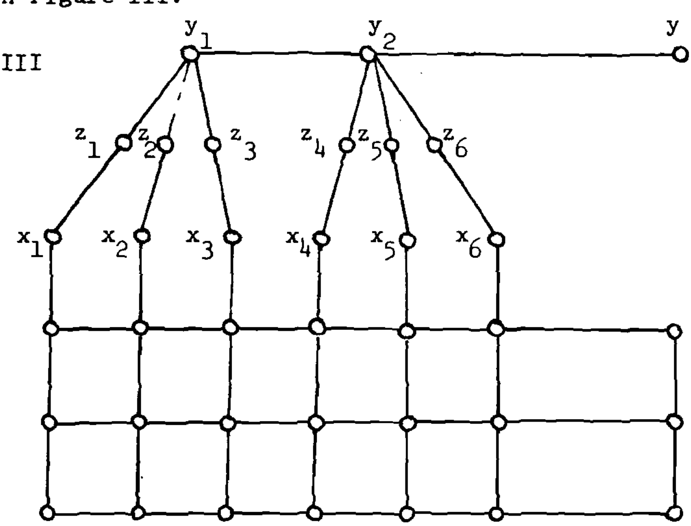
$$\omega_2(x) = \begin{cases} j/k & \text{if } x = z \\ 1 - j/k & \text{if } x = y \\ 0 & \text{if } x = y_i \\ \omega_1(x) & \text{otherwise} \end{cases}$$

Proposition 2.2. Let $(X_{II}, \mathcal{E}_{II})$ be the point closed Dacey-space given in Figure II. Then $\Omega_{X_{II}}$ is full if and only if $1/2 < j/k$.

We omit the easy but lengthy proof. The interested reader will note that the line B and the elements p and y are essential only if $k - j = 1$. If $j < k - 1$ then this line and these two points may be deleted from $(X_{II}, \mathcal{E}_{II})$. The result on fullness and the restriction on spec(z) remain valid.

We have observed, via Figure II, that it is possible for a point closed Dacey-space X to admit an element x such that $M_x<1$ and Ω_x be full. From X we may construct a Dacey-space Y such that Ω_Y is full and $m_y >0$ for some $y \in Y$. Simply let $Y = \mathcal{L}(X)\setminus\{\emptyset\}$with the induced orthogonality relation and let $y = x^{\perp}$. It is somewhat more difficult to obtain a point closed Dacey-space admitting a full set of weights and an element y with $m_y > 0$. In fact we find the existence of such a space rather surprising. One such space is given in Figure III.

Figure III



We outline the argument that $m_y > 0$. Let ω be any weight on the space $(X_{III}, \mathcal{E}_{III})$ of Figure III,

$$\sigma = \sum_{x \in X_{III}} \omega(x), \ \tau = \sum_{i=1}^{6} \omega(x_i), \ \zeta = \sum_{i=1}^{6} \omega(z_i).$$ Then counting σ

three different ways we get

(1) $\sigma = 4 + \tau + \zeta$

(2) $\sigma = 8 + \zeta$

(3) $\sigma = 7 + 3\omega(y)$

and hence

(4) $\tau = 4$ and $1 + \zeta = 3\omega(y)$.

Therefore $\omega(y) \geqslant 1/3$. Keeping (4) in mind it is easy to list enough states to show that $\Omega_{X_{III}}$ is full and spec(y) = $[1/3,1]$.

3. TOPOLOGICAL-GEOMETRIC PROPERTIES OF SPACES

Let $(X, \mathcal{E})$ be any space and Ω be the set of weights on $(X, \mathcal{E})$. In section 2 we obtained $\Omega \subseteq \underline{\underline{A}}(X, \mathcal{E})$ making Ω a convex set in a topological vector space. In this section we will define two topologies for X, which are nontrivial even in the finite case, and we will see that the sets spec(x) defined in section 2 relate X to the geometry of Ω .

For each $\omega \in \Omega$ let $E(\omega) = \{x \in X \mid \omega(x) > m_x\}$. Recall spec(x) = $[m_x, M_x]$. Assume that $X = \bigcup_{\omega \in \Omega} E(\omega), \Omega \neq \emptyset$ and let $\tau_{\mathcal{E}}$ be the topology on X which is generated by the sets $E(\omega), \omega \in \Omega$, as a sub-base.

<u>Remark 3.1.</u> (1) If $m_x = 0$ for all $x \in X$ then the closed sets of $\tau_{\mathcal{E}}$ are generated (as a sub-base) by the sets $Z_\omega = \{x \mid \omega(x)=0\}$, $\omega \in \Omega$.

(2) $(X, \tau_{\mathcal{E}})$ is a T1-topological space if and only if $(\forall x \neq y)(\exists \omega \in \Omega)(\omega(x)=m_x \text{ and } \omega(y) > m_y)$.

(3) Let $(X, \tau_{\mathcal{E}})$ be a T1-topological space. If $(X, \mathcal{E})$ is a Dacey space then $(X, \mathcal{E})$ is point closed. If X is finite then $m_x=0$ for each x in X.

<u>Definition 3.2.</u> Let $F \subseteq \Omega$. We say that F is a <u>face</u> of Ω if

(1) $\emptyset \subsetneq F \subsetneq \Omega$ and F is convex.

(2) $\omega_1 \in \Omega$, $\omega_2 \in \Omega$, $(\omega_1,\omega_2) \cap F \neq \emptyset \implies [\omega_1,\omega_2] \subseteq F$.

A face maximal under set theoretic inclusion is called a <u>facet</u>.

For each $x \in X$ let $S(x) = \{\omega \in \Omega | \omega(x) = m_x\}$. Now we have

<u>Lemma 3.3.</u> Let $(X,\mathcal{E})$ be any space and assume that $(X,\tau_{\mathcal{E}})$ is a T1-topological space. Then, for each $x \in X$, $S(x)$ is a face of Ω.

<u>Proof.</u> Let $x \in X$. Since $\tau_{\mathcal{E}}$ is a topology there exists $\omega \in \Omega$ so that $x \in E(\omega)$. Thus $\omega \in S(x)$ and we have $S(x) \subsetneq \Omega$. Since $\tau_{\mathcal{E}}$ is T1, if we choose $y \neq x$, there is $\omega \in \Omega$ so that $y \in E(\omega)$ and $x \notin E(\omega)$. But then $\omega \in S(x)$ so that $S(x) \neq \emptyset$. Now suppose $\omega_1 \neq \omega_2 \in \Omega$, $0<\lambda<1$, and $\lambda\omega_1 + (1-\lambda)\omega_2 \in S(x)$. Then $\lambda\omega_1(x) + (1-\lambda)\omega_2(x) = m_x$. Since $\omega_1(x) > m_x$ and $\omega_2(x) > m_x$ we must have $\omega_1(x) = \omega_2(x) = m_x$. Thus $[\omega_1,\omega_2] \subseteq S(x)$.

We define a second topology $S_{\mathcal{E}}$ on X as follows : $S_{\mathcal{E}}$ is generated (as a sub-base) by the sets $E(\omega,n) = \{x | \omega(x)>1/n\}$ for $\omega \in \Omega$ and n is a positive integer. Clearly if $\tau_{\mathcal{E}}$ is a topology on X then so is $S_{\mathcal{E}}$. Other interesting topologies can be defined on X using $\mathcal{E}$ including the one used in [5]. The properties and relationships between these topologies give a class of combinatorial problems for infinite spaces.

<u>Lemma 3.4.</u> Let $(X,\mathcal{E})$ be any space and $\omega \in \Omega$ such that $\omega(x) \geqslant m > 0$ for some m. If $\omega_1 \neq \omega$ then there exists ω_2 so that $\omega \in (\omega_1,\omega_2)$. Thus if F is a face of Ω we must have $\omega \notin F$.

<u>Proof.</u> Let $\mu(t) = \omega + t(\omega_1-\omega)$. Then $\mu(t) \in \Omega$ for $-m \leqslant t \leqslant 1$.

<u>Theorem 3.5.</u> Let $(X,\mathcal{E})$ be a space so that $(X,\tau_{\mathcal{E}})$ is a T1-topological space and $S_{\mathcal{E}}$ is a compact topology . Then $m_x=0$ for each x in X and the mapping $x \to S(x)$ is a bijection between the points of X and the facets of Ω.

<u>Proof.</u> Suppose $F \subseteq \Omega$ is a face and $\nexists$ x in X so that $F \subseteq S_0(x) = \{\omega \in \Omega | \omega(x)=0\}$. Then, for each $x \in X$, we can choose $\omega_x \in F$ so that $\omega_x(x) > 0$. Also choose an integer n_x so that $\omega_x(x)>\frac{1}{n_x}$.

Now $X = \bigcup_{x \in X} E(\omega_x, \frac{1}{n_x})$ and using $S_{\mathcal{E}}$ -compactness we can choose $x_1,\ldots,x_n$ so that $X = \bigcup_{i=1}^{n} E(\omega_{x_i}, \frac{1}{n_{x_i}})$. Let $\omega = \sum_{i=1}^{n} \frac{1}{n} \omega_{x_i}$.

then for each $x \in X$ we have $\omega(x) \geqslant \frac{1}{n} \min(\frac{1}{n_{x_i}}) > 0$ and $\omega \in F$.

But this contradicts lemma 3.4. Thus there is an x so that $F \subseteq S_0(x)$. Now if $y \in X$, by Lemma 3.3, $S(y)$ is a face. Thus there exists an x so that $S(y) \subseteq S_0(x)$; $m_x=0$; and the T1-property of $\tau_{\mathcal{E}}$ gives x=y. Hence $m_y=0$. Since we have shown that any face of Ω is contained in an $S(x)$ the T1-property of $\tau_{\mathcal{E}}$ gives the bijection.

Definition 3.6. Let $(X,\mathcal{E})$ be any space and Ω the set of weights on $(X,\mathcal{E})$. Define $S_1(x) = \{\omega \in \Omega | \omega(x) = 1\}$

$$S_0(x) = \{\omega \in \Omega | \omega(x) = 0\}.$$

Definition 3.7. The set of weights Ω is strong on an orthogonality space $(X,\mathcal{E})$ when $x \perp y \longleftrightarrow S_1(x) \subseteq S_0(y)$. If Ω is strong then it is also full and hence $(X,\mathcal{E})$ is a Dacey space. For the rest of this section $|X|>1$.

Lemma 3.8. If $(X,\mathcal{E})$ is a point closed orthogonality space with Ω strong on $(X,\mathcal{E})$ then

(1) $S_1(x) \neq \emptyset$ and $S_0(x) \neq \emptyset$.

(2) $(X,\tau_{\mathcal{E}})$ is a T1-topological space.

Proof. Both parts are straightforward and the proofs are left to the reader.

Corollary 3.9. Let $(X,\mathcal{E})$ be a point closed orthogonality space. If Ω is strong and $S_{\mathcal{E}}$ is a compact topology, then the mapping $x \to S_0(x)$ is a bijection between the points of X and the facets of Ω.

Proof. The proof is given by theorem 3.5 and the previous lemma.

Corollary 3.10. Let $(X,\mathcal{E})$ be a finite Dacey space with weights Ω. If $(X,\tau_{\mathcal{E}})$ is a T1-topological space then $\Omega(X,\mathcal{E})$ is not a planar pentagon.

Proof. The proof is by inspection and theorem 3.5. In particular the result holds if Ω is a strong set of weights and thus answers a question posed by F.Shultz [19].

Definition 3.11. A space $(X,\mathcal{E})$ is (homogeneous) of dimension n if $|E| = n$ for all $E \in \mathcal{E}$. For a pair of weights ω and ν in Ω and $0 \leqslant \lambda \leqslant 1$ define $\langle \omega,\nu,\lambda \rangle = (1-\lambda)\omega + \lambda\nu$. If $(X,\mathcal{E})$ is of dimension n the weight e, given by $e(x) = \frac{1}{n}$ for all x, is called the unit of Ω and the pair (Ω,e) is called the weight structure for $(X,\mathcal{E})$. If Ω_1 and Ω_2 are two convex sets, an affine map f from Ω_1 to Ω_2 is a function for which $f(\lambda\nu + (1-\lambda)\omega) = \lambda f(\nu) + (1-\lambda)f(\omega)$ [11].

If $(X_1,\mathcal{E}_1)$ and $(X_2,\mathcal{E}_2)$ are two orthogonality spaces and $f : X_1 \to X_2$ is a bijection satisfying $f(x)\perp_2 f(y) \leftrightarrow x\perp_1 y$, we say f is an ortho-bijection and we write $(X_1,\perp_1) \simeq (X_2,\perp_2)$. If $f : (\Omega_1,e_1) \to (\Omega_2,e_2)$ is an affine map and $f(e_1) = e_2$, then f is called a structure map. If there is a bijective structure map from (Ω_1,e_1) onto (Ω_2,e_2) we write $(\Omega_1,e_1) \simeq (\Omega_2,e_2)$.

Lemma 3.12. Let (Ω,e) be the weight structure for the homogeneous orthogonality space $(X,\mathcal{E})$ of dimension n. If $x \in E$ and $S_o(x) \neq \emptyset$ then

$$S_1(x) = \{\nu \in \Omega \mid \langle\omega,\nu,\tfrac{1}{n}\rangle = e,\ \omega \in S_o(x)\}$$

and if $M_x = 1$,

$$\frac{1}{n} = \inf_{\nu,\omega} \{\lambda \mid \langle\omega,\nu,\lambda\rangle = e,\ \omega \in S_o(x),\ \nu \in \Omega\}.$$

Proof. The proofs are straightforward and left to the reader.

Lemma 3.13. Let $(X_i,\mathcal{E}_i)_{i=1,2}$ be homogeneous point closed orthogonality spaces of dimension n_i with weight structures $(\Omega_i,e_i)_{i=1,2}$ and Ω_i strong on $(X_i,\mathcal{E}_i)$ with $S_{\mathcal{E}_i}$ compact. A bijective structure map $f : (\Omega_1,e_1) \to (\Omega_2,e_2)$ induces a natural ortho-bijection $\bar{f} : X_1 \to X_2$.

Proof. By corollary 3.9. there is a bijection ψ_j between the points of X_j and the facets of Ω_j, where $\psi_j(x) = S_o(x) \subseteq \Omega_j$. It is easy to show that f induces a bijection $\tilde{f}$ between the facets of Ω_1 and those of Ω_2. Hence $\bar{f} = \psi_2^{-1}\,\tilde{f}\,\psi_1 : X_1 \to X_2$ is a bijection and $f(S_o(x)) = S_o(\bar{f}(x))$.

It remains to show that

$$x \perp_1 y \text{ if and only if } \bar{f}(x) \perp_2 \bar{f}(y).$$

We now show that the dimensions of X_1 and X_2 are equal.

By lemma 3.12 if $x \in X_j$ $(j=1,2)$ then

$$\frac{1}{n_j} = \inf_{\omega,\nu} \{\lambda \mid \langle\omega,\nu,\lambda\rangle = e_j,\ \omega \in S_o(x), \nu \in \Omega\}$$

Let $x \in X_1$ then there exist $\omega \in S_0(x)$ and $\nu \in S_1(x)$ such that $<\omega,\nu, \frac{1}{n_1}> = e_1$. Since f is a structure map $<f(\omega),f(\nu), \frac{1}{n_1}> = e_2$; but $f(\omega) \in S_0(\bar{f}(x))$ so $n_2 \geq n_1$. By symmetry $n_1 = n_2$. Let $n_1 = n_2 = n$ and so for $x \in X_j$ $(j = 1,2)$,

$$S_1(x) = \{\nu \in \Omega_j \mid <\omega,\nu, \tfrac{1}{n}> = e_j,\ \omega \in S_0(x)\} .$$

Since f is a bijective structure map, for $x \in X_1$, $f(S_1(x)) = S_1(\bar{f}(x))$.

Let $\{x,y\} \subseteq X_1$. Since Ω_1 and Ω_2 are strong

$$x \perp_1 y \leftrightarrow S_1(x) \subseteq S_0(y) \leftrightarrow f(S_1(x)) \subseteq f(S_0(y)) \leftrightarrow$$

$$S_1(\bar{f}(x)) \subseteq S_0(\bar{f}(y)) \leftrightarrow \bar{f}(x) \perp_2 \bar{f}(y).$$

Theorem 3.14. Let $(X_i,\mathcal{E}_i)_{i=1,2}$ and $(\Omega_i,e_i)_{i=1,2}$ satisfy the hypothesis of the previous lemma, then $(X_1,\perp_1) \simeq (X_2,\perp_2) \leftrightarrow (\Omega_1,e_1) \simeq (\Omega_2,e_2)$.

Proof. The forward implication is left to the reader and applying the previous lemma completes the proof.

4. EXISTENCE OF WEIGHTS

Throughout this section $(X,\mathcal{E})$ will be assumed finite. A weight ω on $(X,\mathcal{E})$ is said to be dispersion free or deterministic if $\omega(X) \subseteq \{0,1\}$.

Theorem 4.1. The set of dispersion free weights on a space $(X,\mathcal{E})$ is in one-to-one correspondence with the set of nonempty subsets M of X having the following properties.

(1) If $x,y \in M$ and $x \neq y$ then $\mathcal{E}_x \cap \mathcal{E}_y = \emptyset$.

(2) $\mathcal{E} = \bigcup \{\mathcal{E}_x \mid x \in M\}$.

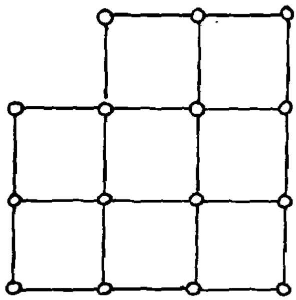
Proof. Let M be a set and $\psi_M: X \to \{0,1\}$ be the characteristic function of M. We claim that ψ_M is a weight. Let $E \in \mathcal{E}$. By (1) and (2) there is a unique $x \in M$ such that $E \in \mathcal{E}_x$. Let $y \in E$, $y \neq x$. Since $\mathcal{E}_x \cap \mathcal{E}_y \neq \emptyset$ we must have $y \in M$ by (1). Thus $\sum_{y \in E} \psi_M(y) = 1$.

Conversely, if ω is a dispersion free weight on X. let $M = \omega^{-1}(1)$.

Corollary 4.2. If $(X,\mathcal{E})$ is a space such that $|\mathcal{E}|$ is odd, and $|\mathcal{E}_x|$ is even for each $x \in X$ then there are no dispersion free weights on $(X,\mathcal{E})$.

Let $(X,\mathcal{E})$ be an orthogonality space with the relation $\perp = \perp_{\mathcal{E}}$. We say that $x \perp' y$ if and only if $x \neq y$ and not $x \perp y$. $(X,\mathcal{E})$ is said to be an F-space if $E \cap F \neq \emptyset$ for every maximal $\perp'$-set F and every $E \in \mathcal{E}$. The above theorem shows that in an F-space every maximal $\perp'$-set supports a dispersion free state and thus the set of dispersion free states is full. The converse, however is not true. An example of a point closed Dacey-space which has a full set of dispersion free states but is not an F-space is given by Figure IV.

Fig.IV



Suppose that $(X,\mathcal{E})$ is any space. The weights on $(X,\mathcal{E})$ are determined by a system of linear equations and this fact can be used to derive properties of the set of weight functions. We now give a realization of this structure which does not depend upon any labeling of the points of X.

For any set A, F(A) will denote the free real vector space over A, and $F(A)^*$ its dual vector space. For each $a \in A$ define $\delta_a \in F(A)^*$ by $\delta_a(b) = 1$ if $a = b$, $= 0$ if $a \neq b$, and linear extension. If A is finite, $F(A)^* \approx F(\{\delta_a | a \in A\})$. Let $(X,\mathcal{E})$ be a (finite) space. The linear transformation $T : F(\mathcal{E}) \to F(X)$ defined by $T(E) = \sum_{x \in E} x$ and linear extension, is called the linear realization of $(X,\mathcal{E})$. The transpose $T^* : F(X)^* \to F(\mathcal{E})^*$ is given by $T^*(\delta_x) = \sum_{E \in \mathcal{E}_x} \delta_E$, or in general,

$$\text{(I)} \quad T^*(\mu) = T^*\left(\sum_{x \in X} \mu(x)\delta_x\right) = \sum_{x \in X} \mu(x)\left(\sum_{E \in \mathcal{E}_x} \delta_E\right)$$

$$= \sum_{E \in \mathcal{E}}\left(\sum_{x \in E} \mu(x)\right)\delta_E .$$

Since X is finite the vector space $\underline{\underline{A}}(X,\mathcal{E})$, defined in section 2, is $R^X \approx F(X)^*$. With the identification it is easy to see that $S=T^*$ where $S : \underline{\underline{A}}(X,\mathcal{E}) \to R^{\mathcal{E}}$ was defined in section 2.

We will thus consider $\Omega_X \subseteq F(X)^*$. Since $F(X)^*$ is finite dimensional each element of Ω_X has a representation as a convex combination of the extreme points of Ω_X. Also, if μ_1 and μ_2 are in Ω_X then $\mu_2 - \mu_1 \in \ker T^*$ and since $\mu_2 = \mu_1 + (\mu_2 - \mu_1)$ we see that $\Omega_X = (\mu_1 + \ker T^*) \cap C$ for any $\mu_1 \in \Omega_X$. By a pure weight on $(X, \mathcal{E})$ we mean an extreme point of Ω_X and by a perturbation on $(X, \mathcal{E})$ we mean an element of $\ker T^*$.

Let Y be any subset of X. Recall that the space $(Y, \mathcal{E}_Y)$ is defined by $\mathcal{E}_Y = \{E \cap Y | E \in \mathcal{E} \text{ and } E \cap Y \neq \emptyset\}$. Note that this need not be an orthogonality space. We say that Y is supporting if $Y \cap E \neq \emptyset$ for each $E \in \mathcal{E}$. For each $\mu \in \Omega_X$ let $Z_\mu = \{x | x \in X \text{ and } \mu(x) = \emptyset\}$.

Theorem 4.4

(1) If $\mu \in \Omega_X$ then $X \setminus Z_\mu$ is supporting, and if $(X, \mathcal{E})$ is an orthogonality space then $(X \setminus Z_\mu, \mathcal{E}_{X \setminus Z_\mu})$ is an orthogonality space.

(2) If $X \setminus Y$ is supporting then the set of weights on $(X, \mathcal{E})$ which vanish on Y is in one-to-one correspondence with the weights on $(X \setminus Y, \mathcal{E}_{X \setminus Y})$.

(3) $Y = Z_\mu$ for some $\mu \in \Omega_X$ if and only if $X \setminus Y$ is supporting and $(X \setminus Y, \mathcal{E}_{X \setminus Y})$ admits a positive weight.

Proof. Ad(1). Let $\mu \in \Omega_X$ and $E \in \mathcal{E}$. Since $\sum_{x \in E} \mu(x) = 1$ there is an $x \in E$ such that $\mu(x) > 0$. Hence $E \cap (X \setminus Z_\mu) \neq \emptyset$. Suppose that $(X, \mathcal{E})$ is an orthogonality space. We will show that $\mathcal{E}_{X \setminus Z_\mu}$ satisfies the conditions of corollary 1.3. Clearly properties 1) and 3) are inherited from $\mathcal{E}$. Now suppose that $E_1 \cap (X \setminus Z_\mu) \subseteq E_2 \cap (X \setminus Z_\mu)$ and there is a point x in $E_2 \cap (X \setminus Z_\mu) \setminus E_1 \cap (X \setminus Z_\mu)$. We have

$$\sum_{y \in E_2} \mu(y) = \sum_{y \in E_2 \cap (X \setminus Z_\mu)} \mu(y) = \sum_{y \in E_1 \cap (X \setminus Z_\mu)} \mu(y) + \mu(x) = 1 + \mu(x) > 1, \text{ since}$$

$x \notin Z_\mu$. But this is a contradiction.

Ad(2). Let $\mu \in \Omega_X$ and $Y \subseteq Z_\mu$. For each $E \in \mathcal{E}$ $\sum_{y \in E \cap (X \setminus Y)} \mu(y) = \sum_{y \in E} \mu(y) = 1$ so that $\mu | X \setminus Y$ is a weight on $(X \setminus Y, \mathcal{E}_{X \setminus Y})$. Now suppose that $\mu: X \setminus Y \to [0,1]$ is a weight on $(X \setminus Y, \mathcal{E}_{X \setminus Y})$. Extend μ by 0 to all of X. Since $X \setminus Y$ is supporting we get a weight on $(X, \mathcal{E})$. Ad(3). (3) follows easily from (2).

We have seen that each weight on a space $(X, \mathcal{E})$ can be written as a convex combination of pure weights. Thus the existence of weights on $(X, \mathcal{E})$ is equivalent to the existence of pure weights.

Lemma 4.5. A weight μ on $(X, \mathcal{E})$ is pure if and only if $Z_\mu \nsubseteq Z_{\mu_1}$

for all weights $\mu_1 \neq \mu$.

Proof. Assume there is a weight $\mu_1 \neq \mu$ for which $Z_\mu \subseteq Z_{\mu_1}$. Let $\nu = \mu - \mu_1$. Then ν is a perturbation. That is, $\sum_{x\in E} \nu(x) = 0$ for all $E \in \mathcal{E}$.

Recalling that X is finite, let $\delta = \min\{\mu(x) | \mu(x) \neq 0\}$. Suppose that $\lambda \in [-\delta, \delta]$ and $\mu_\lambda = \mu + \lambda\nu$. Clearly, $\sum_{x\in E} \mu_\lambda(x) = 1$ for all $E \in \mathcal{E}$. Let $x \in X$. If $\mu(x) = 0$ we have $\mu_1(x) = 0$ so that $\nu(x) = 0$ and $\mu_\lambda(x) = 0$. Suppose $\mu(x) \neq 0$. Then $|\lambda((\mu-\mu_1)(x))| = |\lambda||\nu(x)| < \delta \cdot 1 = \delta \leqslant \mu(x)$. Thus μ_λ is a weight for each $\lambda \in [-\delta, \delta]$ and, since $\mu_0 = \mu$, μ is not a pure weight. This establishes the necessity of the condition. Conversely, assume $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ where $\mu_1 \neq \mu_2$ are weights and $0 < \lambda < 1$. Then $Z_\mu = Z_{\mu_1} \cap Z_{\mu_2} \subseteq Z_{\mu_1}$ and $\mu_1 \neq \mu$.

Proposition 4.6. A weight μ on $(X, \mathcal{E})$ is pure if and only if the space $(X \setminus Z_\mu, \mathcal{E}_{X\setminus Z_\mu})$ has exactly one weight and that weight is positive (i.e., never vanishes).

Proof. By Theorem 4.4 the set of weights μ_1 on $(X, \mathcal{E})$ such that $Z_\mu \subseteq Z_{\mu_1}$ is in one-to-one correspondence with the weights on the space $(X \setminus Z_\mu, \mathcal{E}_{X\setminus Z_\mu})$. By Lemma 4.5 μ is pure if and only if $\mu_1 = \mu$ is the only such weight. Of course $\mu | X \setminus Z_\mu$ never vanishes.

Corollary 4.6. (1) Every dispersion free weight is pure. (2) The set of extreme points is finite. (3) A pure weight ω assumes only rational values.

Proof. (1) and (2) are immediate from proposition 4.6 and lemma 4.5 respectively. Ad(3). Let $X = Y - Z$ where $Z = \{x | \omega(x) = 0\}$ and $F = \{E \cap Y | E \in \mathcal{E}\}$. By 4.4 and proposition 4.6 $\mu = \omega | Y$ is the only weight on (Y, F). Thus the values of μ, listed as a vector, is the unique solution of a system of linear equations with integer coefficients. Such a vector has rational components.

The above results show the importance of orthogonality spaces which admit only one weight. An interesting question is the existence of orthogonality spaces which are also Dacey spaces and have only one weight. A construction of such a space is given in [8]. See [19] for the case when the associated logic is a lattice.

Let $(X, \mathcal{E})$ be any space and $Y \subseteq X$. We say that Y is connected if for any two distinct points x and y in Y there are sequences of points, $x_0, x_1, x_2, \ldots, x_n$ and of blocks $E_1, E_2, \ldots, E_n$ such that $x_0 = x$, $x_n = y$ and $\{x_{i-1}, x_i\} \subseteq E_i$, $1 \leqslant i \leqslant n$. A maximal connected is said to be a connected component.

Theorem 4.7. Let μ be a weight on $(X,\mathcal{E})$. Then $X\setminus Z_\mu$ has a decomposition $X\setminus Z_\mu = F\cup C_1\cup C_2\cup\ldots\cup C_n$ which is unique relative to the following conditions, (1), (2), and (3).

(1) $\mathcal{E}_F = \{\{x\}\mid x\in F\}$ so that F represents the dispersion free part of μ.

(2) Each $E\ \varepsilon$ intersects exactly one of the sets in $\{C_i \mid 1\leq i\leq n\}\cup\{\{x\}\mid x\in F\}$.

(3) Each C_i is connected and $E\in\mathcal{E}$, $E\cap C_i\neq\emptyset$ implies $|E\cap C_i|\geq 2$. That is, each element $\mathcal{E}_{C_i}$ has cardinality at least 2.

Moreover,

(4) If $(X,\mathcal{E})$ is an orthogonality space then each $(C_i, \mathcal{E}_{C_i})$ is an orthogonality space.

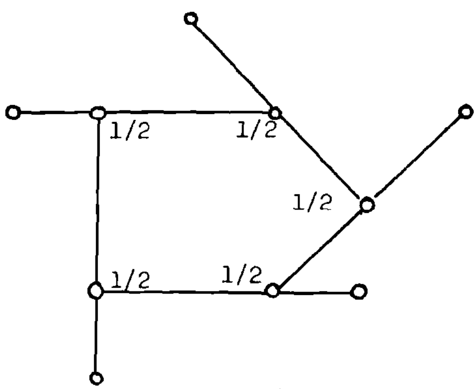
(5) μ is pure if and only if each $(C_i, \mathcal{E}_{C_i})$ admits exactly one weight and that weight is positive.

Proof. The C_i's are the connected components of $(X\setminus Z_\mu, \mathcal{E}_{X\setminus Z_\mu})$ which are not isolated points and $F = X\setminus(Z_\mu\cup\bigcup_{i=1}^{n} C_i)$. (1) and (2) are clear.

Suppose that $E\cap C_i = \{x\}$. Since $\{x\}$ is not isolated (otherwise it would be in F) there is an $E_1\in\mathcal{E}$ such that $E_1\cap C_i\supset\{x,y\}$ for some $y\neq x$. Now $E\cap C_i = \{x\}$ and by (2) we must have $\mu(x)=1$. But $\mu(y) > 0$ so that we get $\mu(x)+\mu(y) > 1$, $\{x,y\}\subseteq E_1$, a contradiction. Thus we get (3). (4) follows from the fact that $\mu|_{C_i}$ is a positive state on $(C_i,\mathcal{E}_{C_i})$ and (5) follows from Proposition 3.6.

We can easily give an example of a finite point closed Dacey-space which has a pure weight that is not dispersion free, cf. Figure V.

Fig.V



Note that the results of Theorem 4.1 concerning the existence of dispersion free weights are a special case of Theorem 4.7 corresponding to the situation $X\setminus Z_\mu = F$. Thus the existence of weights on $(X,\mathcal{E})$ depends upon the existence of a partition of $\mathcal{E}$ determined by a set $F \subseteq X$ as is described in Theorem 4.1 or upon the existence of orthogonality spaces $(C, \mathcal{E}_C)$ in $(X,\mathcal{E})$ which admit exactly one weight and that weight is positive.

By a <u>generalized weight</u> we mean any element of $\Omega_g = T^{*^{-1}}(\sum_{E\in\mathcal{E}} \delta_E)$. Recall that a <u>perturbation</u> is any element of Ker T^*. It is clear that μ_1 and μ_2 in Ω_g implies $\mu_1 - \mu_2$ Ker T^* and if μ is any element in Ω_g then $\Omega_g = \mu +$ Ker T^*.

We now consider the relationship between generalized weights weights, and perturbations.

<u>Proposition 4.8.</u> Let $(X,\mathcal{E})$ be any finite space, $\Omega \subseteq F(X)^*$ be the weight space of $(X,\mathcal{E})$, and $\Omega_P = \{P_1,P_2,\ldots,P_n\}$ be the set of pure weights. Let $R = \{\nu \in F(X)^* \mid$ There is a number $r > 0$ and weights μ_1,μ_2 such that $r\nu = \mu_1 - \mu_2\}$. Let d be the number of vectors in a maximal linearly independent subset of $\{P_2 - P_1, P_3 - P_1,\ldots P_n - P_1\}$. Then

1) $R \subseteq$ Ker T^* and R is a vector subspace.

2) dimension $R = d$.

3) If $(X,\mathcal{E})$ possesses a positive weight then $R =$ Ker T^*.

<u>Proof.</u> (We assume $\Omega \neq \emptyset$).

1) Choose $\mu \in \Omega$. Then $1\cdot 0 = \mu - \mu$ so that $0 \in R$. Suppose $\nu \in R$ and s is any number. If $s = 0$ then $s\cdot\nu = 0 \in R$. Suppose $s \neq 0$. Choose $r > 0$ and weights μ_1,μ_2 so that $r\nu = \mu_1 - \mu_2$. If $s > 0$ we get $(\frac{r}{s})(s\nu) = \mu_1 - \mu_2$. If $s < 0$ we get $(\frac{r}{-s})(s\nu) = \mu_2 - \mu_1$. Thus $s\nu \in R$. Suppose $0 \neq \nu_1 \in R$ and $0 \neq \nu_2 \in R$. Choose $r_1 > 0, r_2 > 0, \mu_1,\mu_2,\delta_1,\delta_2$ so that $r_1\nu_1 = \mu_1 - \mu_2$ and $r_2\nu_2 = \delta_1 - \delta_2$.

Let $s = \dfrac{r_1 r_2}{r_1+r_2} > 0$. Then $0 < \dfrac{s}{r_1} = \dfrac{r_2}{r_1+r_2} < 1$, $0 < \dfrac{s}{r_2} = \dfrac{r_1}{r_1+r_2} < 1$,

and $\dfrac{s}{r_1} + \dfrac{s}{r_2} = 1$. Now

$$s(\nu_1+\nu_2) = s\left\{\frac{1}{r_1}(\mu_1 - \mu_2) + \frac{1}{r_2}(\delta_1 - \delta_2)\right\}$$

$$= (\frac{s}{r_1}\mu_1 + \frac{s}{r_2}\delta_1) - (\frac{s}{r_1}\mu_2 + \frac{s}{r_2}\delta_2).$$

This proves that $\nu_1 + \nu_2 \in R$ so that R is a vector subspace.

2) Clearly $\{P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1\} \subseteq R$. Let $\nu \in R$. Write $r\nu = \mu_1 - \mu_2$ where $r > 0$ and μ_1, μ_2 are weights. We can write $\mu_1 = \sum_{i=1}^{n} \alpha_i P_i$ and $\mu_2 = \sum_{i=1}^{n} \beta_i P_i$, where $\sum_1^n \alpha_i = 1 = \sum_1^n \beta_i$.

$$\text{Thus } r\nu = \sum_1^n (\alpha_i - \beta_i) P_i = ((1-\sum_2^n \alpha_i)-(1-\sum_2^n \beta_i))P_1 + \sum_2^n (\alpha_i - \beta_i) P_i$$

$$r\nu = (\sum_2^n (\beta_i - \alpha_i))P_1 + \sum_2^n (\alpha_i - \beta_i) P_i = \sum_2^n (\alpha_i - \beta_i)(P_i - P_1).$$

Thus $\nu = \sum_2^n \left(\frac{\alpha_i - \beta_i}{r}\right)(P_i - P_1)$ and we have shown that

$\{P_i - P_1 \mid 2 \leqslant i \leqslant n\}$ generates R. The result now follows.

3) Suppose there is a weight μ such that $\mu(x) > 0$ for each $x \in X$. Let $\nu \in \operatorname{Ker} T^*$. Let $m = \min\{\mu(x) \mid x \in X\}$. Choose $r > 0$ so that $|r\nu(x)| < m$ for all $x \in X$. Let $\mu_1 = \mu + r\nu$. Then $\mu_1 \in \Omega$ and since $r\nu = \mu_1 - \mu$ we have that $\nu \in R$. Thus $\operatorname{Ker} T^* = R$.

Let $(X, \mathcal{E})$ be any finite space and let $\Omega_p \subseteq \Omega$ be the set of pure weights. We known by Corollary 4.6 that Ω_p is a finite set. Let $\Omega_p = \{\mu_0, \mu_1, \mu_2, \ldots, \mu_n\}$. It is a well known fact that Ω is a simplex if and only if $\{\mu_k - \mu_0 \mid 1 \leqslant k \leqslant n\}$ is a linearly independent subset of $F(X)^*$. $(X, \mathcal{E})$ is said to be a <u>classical space</u> if $\mathcal{E} = \{X\}$. In this case Ω is a simplex and $|\Omega_p| = |X|$. We now obtain a converse to this result.

Let $(X, \mathcal{E})$ be any finite space. Let $E \subseteq F(X)^*$ be a complement of $\operatorname{Ker} T^*$. That is $F(X)^* = \operatorname{Ker} T^* \oplus E$. Then $F(X) = F(X)^{**} = (\operatorname{Ker} T^*)^* \oplus E^*$ so that $(\operatorname{Ker} T^*)^{\perp} = E^*$. Now $\operatorname{Ker} T^* = (\operatorname{im} T)^{\perp}$ so that we get $\operatorname{im} T = (\operatorname{im} T)^{\perp\perp} = (\operatorname{Ker} T^*)^{\perp} = E^*$. Now $|X| = \dim \operatorname{Ker} T^* + \dim E = \dim \operatorname{Ker} T^* + \dim E^*$ so that $|X| = \dim \operatorname{Ker} T^* + \dim \operatorname{im} T$.

<u>Theorem 4.9.</u> Let $(X, \mathcal{E})$ be any finite space which has a positive weight. Then Ω is a simplex if and only if $|X| + 1 = \dim(\operatorname{im} T) + |\Omega_p|$.

<u>Proof.</u> Suppose $|X| + 1 = \dim(\operatorname{im} T) + |\Omega_p|$. Then $|\Omega_p| - 1 = \dim \operatorname{Ker} T^*$ and by Proposition 4.8 we can conclude that Ω is a

simplex.

Suppose Ω is a simplex. Then dim Ker $T^* = |\Omega_p| - 1$ and the sult follows from $|X|$ = dim Ker T^* + dim (im T).

Corollary. 4.10. Suppose $(X,\mathcal{E})$ is a finite space, Ω is a simplex, and there is a positive weight. If $|\Omega_p| \geqslant |X|$ then $(X,\mathcal{E})$ is a classical space.

Corollary 4.11. Let $(X,\mathcal{E})$ be a finite space such that $x \neq y$ implies the existence of weight ω so that $\omega(x) = m_x$ and $\omega(y) > m_y$. If Ω is a simplex then $(X,\mathcal{E})$ is a classical space.

Proof. By theorem 3.5 there is a bijection between the points of X and the facets of Ω. Since Ω is a simplex the number of facets of Ω is the same as the number of pure weights. Thus $|\Omega_p|=|X|$ and the result follows from corollary 4.10.

Recall that $\Omega(X,\mathcal{E})$ satisfies the projection postulate in case for all $x \in X$ there exists $\omega \in \Omega$ with $\omega(x) = 1$.

Remark 4.12. Let $(X,\mathcal{E})$ be a finite point closed space such that Ω is full and satisfies the projection postulate. If Ω is a simplex then $(X,\mathcal{E})$ is a classical space.

Proof. We may assume $|X| > 1$. For $\omega \in \Omega_p$ let F_ω denote the facet of Ω opposite ω. As in the first part of the proof of Theorem 3.5 there exists $x_\omega \in X$ such that $F_\omega \subseteq S_o(x_\omega)$ and, since $M_{x_\omega} > 0$, $F_\omega = S_o(x_\omega)$. The mapping $\phi : \omega \to x_\omega$ is an injection of Ω_p into X. Let $\omega_1,\dots,\omega_n$ be the pure weights. Then $\omega_i(x_{\omega_j}) = \delta_{ij}$, the Kronecker delta. Fullness implies that image $(\phi) = E$ is an (in fact, maximal) orthogonal set. If $y \in X \setminus E$ then there exists $\omega \in \Omega_p$ with $\omega(y) = 1$ and there exists $e \in E$ with $\omega(e) = 1$. It follows that $y^\perp \subseteq e^\perp$ and hence $\{e\} = e^{\perp\perp} \subseteq y^{\perp\perp} = \{y\}$, i.e. e=y a contradiction. Thus $X = E$ and $\mathcal{E} = \{E\}$.

5. SYMMETRIES AND DUALITY

Let $(X,\mathcal{E})$ be any space with $X = \bigcup \mathcal{E}$. We define the dual space $(X^*,\mathcal{E}^*)$ by $X^* = \mathcal{E}$ and $\mathcal{E}^* = \{\mathcal{E}_x | \ x \in X\}$. We say that $(X,\mathcal{E})$ is separating if $\mathcal{E}_x \nsubseteq \mathcal{E}_y$ or $\mathcal{E}_x = \mathcal{E}_y$ whenever $x \neq y$. We say that $(X,\mathcal{E})$ is distinguishing if $\mathcal{E}_x \neq \mathcal{E}_y$ whenever $x \neq y$. Note that the logic of a separating and distinguishing Dacey-space need not be hyperirreducible [6].

Proposition 5.1.

1) If $(X,\mathcal{E})$ is any space then $(X^*,\mathcal{E}^*)$ is a distinguishing space.

2) If $E\in\mathcal{E}$, $F\in\mathcal{E}$, $E\neq F$ implies $E\not\subset F$ then $(X^*,\mathcal{E}^*)$ is also a separ ting space.

3) $(X,\mathcal{E})$ is distinguishing if and only if $(X^{**},\mathcal{E}^{**})\simeq(X,\mathcal{E})$.

4) Suppose that X is finite and $\Omega_{\mathcal{E}}\neq\emptyset$. Then $\sum_{x\in X}\mu_1(x)=\sum_{x\in X}\mu_2(x$ for all μ_1, μ_2 in $\Omega_{\mathcal{E}}$ if and only if there exists a generalized weight on $(X^*,\mathcal{E}^*)$.

Proof. For the proof of 5.1 and more results on dual spaces sec [9].

Let $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ be spaces, then every bijection $\phi : X \to$ induces a bijection $\tilde{\phi} : 2^X \to 2^Y$ defined by $\tilde{\phi}(E)=\phi(E)$ for all $E\in$ If ϕ is such that $\tilde{\phi}(\mathcal{E})=\mathcal{F}$, then we say that ϕ is an isomorphism from $(X,\mathcal{E})$ to $(Y,\mathcal{F})$ and write $(X,\mathcal{E})\simeq(Y,\mathcal{F})$. If $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ are orthogonality spaces then the isomorphisms from $(X,\mathcal{E})$ to $(Y,\mathcal{F}$ are precisely the ortho-bijections (c.f. 3.11) of the induced graphs $(X,\perp_{\mathcal{E}})$ and $(Y,\perp_{\mathcal{F}})$.

An isomorphism from $(X,\mathcal{E})$ to itself is called an automorphis the set of all automorphisms on $(X,\mathcal{E})$ forms a group $\mathrm{Aut}(X,\mathcal{E})$. One can show [9] that if $(X,\mathcal{E})$ is a distinguishing space then $\mathrm{Aut}(X,\mathcal{E}$ is isomorphic to $\mathrm{Aut}(X^*,\mathcal{E}^*)$. Gerald Schrag [18] has combined this observation and some of our earlier results with some known resul of graph theory to prove the following theorem.

Theorem (Schrag) : Every finite group is the automorphism group of some orthomodular lattice.

If $(X,\mathcal{E})$ is a homogeneous space of dimension n then the set of all bijective structure maps of (Ω,e) onto itself is a group. Denote it by $\mathrm{Aut}(\Omega,e)$. We conclude with the following theorem.

Theorem 5.2. For a homogeneous point closed $S_{\mathcal{E}}$ compact spac of dimension n with a strong set of weights Ω,

$$\mathrm{Aut}(\Omega,e)\simeq\mathrm{Aut}(X,\mathcal{E})\simeq\mathrm{Aut}(X^*,\mathcal{E}^*).$$

Proof. Define ϕ: $\mathrm{Aut}(X,\mathcal{E})\to\mathrm{Aut}(\Omega,e)$ by the following : for $g\in\mathrm{Aut}(X,\mathcal{E})$, $[\phi(g)]\,\omega(x)=\omega(g^{-1}(x))$. It is straight forward to see that ϕ is an injective homomorphism. That it is surjective fo lows from the proof of 3.13.

BIBLIOGRAPHY

1. Berge, C., Graphs and Hypergraphs, North Holland Publishing Co., Amsterdam-London, 1973.

2. Dacey, J.C., "Orthomodular Spaces and additive measurement", Caribbean Journal of Sciences and Mathematics, Vol. I, No II, July, 1969.

3. Dembowski, P., Finite Geometries, Springer-Verlag, New York, 1968.

4. Foulis, D.J. and Randall, C.H., "Operational Statistics I Basic Concepts", J.Math Physics, Vol. 13, No. 11, November 1972.

5. Gerelle, E.R., "A Preliminary Report on the Affine Geometry of Quantum Logics", Kansas State University, Technical Report No. 34, July 1973.

6. Greechie, R.J., "Hyper-irreducibility in an orthomodular lattice", J.Nat.Sc. and Math Vol. VIII, No. 1, April 1968.

7. Greechie, R.J., "Orthomodular lattices admitting no states", Journal of Combinatorial Theory, Vol. X, No. 2, March 1971.

8. Greechie, R.J. and Miller, F.R., "On Structures Related to States on an Empirical Logic I. Weights on Finite Spaces", K.S.U. Technical Report No. 14, April 1970.

9. Greechie, R.J. and Miller, F.R., "On Structures Related to States on an Empirical Logic II. Weights and Duality on Finite Spaces", K.S.U. Technical Report No. 26, August 1972.

10. Greechie, R.J. and Gudder, S.P., "Quantum Logics" in Contemporary Research in The Foundation and Philosophy of Quantum Theory, D.Reidel Publ. Co., Dordrecht-Holland, Boston-U.S.A., (C.A.Hooker ed.), pp. 143-173.

11. Grünbaum, B., Convex Polytopes, Interscience, New York-London, 1967.

12. Gudder, S.P., "Hilbert Space, independence, and generalized probability", J.Mathematical Analysis and Applications, Vol. 20, No. 1, Oct. 1967.

13. Holland, S.S., Jr., "The current interest in orthomodular lattices", Trends in Lattice Theory, (J.C.Abbott, ed.), Van Nostrand, Princeton, 1970.

14. Jauch, J.M., Foundations of Quantum Mechanics, Addison Wesley, Reading, London, 1968.

15. Kolmogorov, A.N., Foundations of the Theory of Probability, 2nd edition, Chelsea, New York, 1956.

16. Randall, C.H., "A mathematical foundation for empirical science with special reference to quantum theory. Part I. A calculus of experimental propositions", Knolls Atomic Power Lab. Report, KAPL-3147, June 1966.

17. Randall, C.H. and Foulis, D.J., "States and the free orthogonality monoid", Math Systems Theory 6, No. 3 (1972) 268-276.

18. Schrag, G., "Combinatories and Graph Techniques in Orthomodular Theory", Ph.D.Dissertation, Kansas State Univ., 1971.

19. Shultz, F., "A representation of the state spaces of orthomodular lattices", (to appear).

20. Varadarajan, V.S., "Probability in physics and a theorem on simultaneous observability", Comm. Pure and Applied Math., Vol. 15 (1962), 189-217.