PERSPECTIVITY IN SEMIMODULAR ORTHOMODULAR POSETS

L. HASKINS, S. GUDDER AND R. GREECHIE

1. *Introduction*

Perspectivity is an extremely fruitful concept in lattice theory [**2**; p. 74, 91, 269]. In fact, von Neumann's result that perspectivity is an equivalence relation in a continuous geometry (von Neumann lattice) [7] is one of the deepest and most celebrated theorems in this field. Using this concept, von Neumann was able to show, among other things, that an element is in the centre of a continuous geometry if and only if it has a unique complement. Perspectivity can also be used to define the superposition principle for quantum mechanical systems, and thus theorems in this area become important for studies in axiomatic quantum mechanics [**5**; p. 106]. In this note we show that some of the classical lattice theorems concerning perspectivity can be proved in semimodular orthomodular posets of finite length. In particular, we show that perspectivity is an equivalence relation on atoms in such posets; that the centre of such posets is generated by the suprema of equivalence classes of perspective atoms; and that an element is in the centre if and only if it has a unique complement.

2. *The u and l Operations*

Let P be a poset. We would like to generalize the lattice operations $\vee$ and $\wedge$ to P. It will turn out that our generalization provides a useful tool for working in posets which enjoys many of the properties of $\vee$ and $\wedge$. If $A \subseteq P$ define

$$U(A) = \{x \in P : x \geqslant y \quad \text{for some} \quad y \in A\}$$

$$L(A) = \{x \in P : x \leqslant y \quad \text{for some} \quad y \in A\}$$

$$M(A) = \{x \in A : y > x \quad \text{for no} \quad y \in A\}$$

$$m(A) = \{x \in A : y < x \quad \text{for no} \quad y \in A\}$$

If R is a collection of subsets A of P we define

$$\underset{R}{l}(A) = M \bigcap_R L(A) \quad \text{and} \quad \underset{R}{u}(A) = m \bigcap_R U(A).$$

If $R = \{A, B\}$ we write $A\,l\,B$ and $A\,u\,B$ for $l\{A, B\}$ and $u\{A, B\}$ respectively, and we denote a singleton set $\{x\}$ by x. Notice that if P is a lattice then $x\,u\,y = x \vee y$ and $x\,l\,y = x \wedge y$, so our operations generalize the lattice operations. In general, however, $x\,u\,y$ and $x\,l\,y$ need not be singleton sets (they may even be empty). Our first lemma, whose simple proof we omit, gives an associative law.

LEMMA 2.1. *Let R_q be a collection of subsets A of P for each $q \in Q$. If P has no infinite chains then*

$$\underset{Q}{l}\left[\underset{R_q}{l}(A)\right] = \underset{\cup R_q}{l}(A), \quad \underset{Q}{u}\left[\underset{R_q}{u}(A)\right] = \underset{\cup R_q}{u}(A).$$

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As a special case of Lemma 2.1 we have $A\,l\,(B\,l\,C) = (A\,l\,B)\,l\,C$. One can give examples which show that this associative law need not hold if P has infinite chains even if A, B, C are singleton sets. One can also show by examples that if $U(A)$ and $L(A)$ are defined in the traditional way (i.e., $U(A) = \{x \in P : x \geqslant y \text{ for all } y \in A\}$ and $L(A) = \{x \in P : x \leqslant y \text{ for all } y \in A\}$) then the associative law need not hold even for singleton sets when P has no infinite chains. This is one reason for our present definition of U and L.

The proof of the next lemma is straightforward and will be omitted.

LEMMA 2.2. *Let R be a collection of subsets A of P and let S_A be a collection of subsets of A such that $A = \bigcup\{B \in S_A\}$. If P has no infinite chains then*

$$\underset{R}{l}(A) = M\cup\{l(x) : x \in \underset{R}{\Pi} S_A\}.$$

As a special case of Lemma 2.2 we see that

$$A\,l\,B = M \cup\{a\,l\,b : a \in A, b \in B\}.$$

Now let P be a poset with a universal lower bound 0. We define the *height* $h(x)$ of $x \in P$ as the supremum of the number of elements in the maximal chains from 0 to x minus one. For $x, y \in P$ we define $x\,u^*\,y = \{z \in x\,u\,y : h(z) \text{ is minimal}\}$ and $x\,l^*\,y = \{z \in x\,l\,y : h(z) \text{ is maximal}\}$. A poset P is *upper semimodular* if whenever distinct elements $x, y \in P$ both cover $z \in P$ there exists $s \in P$ such that s covers both x and y. P is *lower semimodular* if whenever distinct elements $x, y \in P$ are covered by $z \in P$, there exists $t \in P$ such that both x and y cover t.

Now suppose P has no infinite chains and is both upper and lower semimodular. One can then show that each connected component of P has universal bounds 0,1 [**4**]. The next theorem, which will be quite useful in the sequel, is proved in [**4**].

THEOREM 2.3. *For all $x, y \in P$, $x\,u^*\,y$ and $x\,l^*y$ are non-empty and*

$$h(x)+h(y) = h(u)+h(l)$$

for all $u \in x\,u^*\,y$ and all $l \in x\,l^*\,y$.

3. *Perspectivity*

Let P be a poset with 0,1. If $x \in P$, an element $y \in P$ is a *complement of* x if $x\,u\,y = \{1\}$ and $x\,l\,y = \{0\}$. Two elements $x, y \in P$ are *perspective* (written $x \sim y$) if they have a common complement. We say that P is *orthocomplemented* if there exists a map $x \to x'$ on P where x' is a complement of x satisfying: $x'' = x$, $x \leqslant y$ implies $y' \leqslant x'$. If $x \leqslant y'$, we say x and y are *orthogonal* and write $x \perp y$. An orthocomplemented poset is *orthomodular* if: $x \vee y$ exists whenever $x \perp y$ and $x \leqslant y$ implies $y = x \vee (y \wedge x')$.

LEMMA 3.1. *Let P be an orthomodular poset.* (1) *If $x, y \in P$ are atoms and $x \not\perp y$ then $x \sim y$.* (2) *If x, y, z are distinct atoms and $z \leqslant u$ for some $u \in x\,u\,y$ then $x \sim y$.*

Proof. (1) If $x \not\perp y$ then $y \wedge x' = 0$ and $y \vee x' = 1$ since y is an atom and x' is a dual atom; so x' is a complement of both x, y and hence $x \sim y$. (2) If $x \not\perp y$ then $x \sim y$; so assume $x \perp y$. If $x \perp z$ then $y = x' \wedge (x \vee y) \geqslant z$; so $y = z$, which is a contradiction. Hence $x \not\perp z$ and similarly $y \not\perp z$. As in the proof of (1) z' is a common complement of x and y, so $x \sim y$.

LEMMA 3.2. *Let P be an upper and lower semimodular poset with 0,1 of finite length (i.e., $h(1) < \infty$) and let x, y be distinct atoms in P. If $x \sim y$ and $u \in x\,u^*\,y$, then there exists an atom $z \neq x, y$ such that u covers z.*

Proof. Let e be a common complement for x, y and let $u \in x\,u^*\,y$ and $z \in u\,l^*\,e$. Then $h(e)+h(x) = h(1)$ and $u\,u^*\,e = \{1\}$ since $u \geqslant x, y$. Now $h(u) = 2$ since x, y cover 0. Since $h(u)+h(e) = h(1)+h(z)$ we have $h(z) = 1$. Then z is an atom and $z \neq x, y$, since $z \leqslant e$. Clearly u covers z.

Simple examples show that the hypotheses of Lemma 3.1 and 3.2 cannot be interchanged. In the sequel we shall assume that P is a semimodular (i.e., upper and lower semimodular) orthomodular poset of finite length.

LEMMA 3.3. *If $a \in P-\{0\}$, then $[0, a]$ is an orthomodular semimodular poset under the induced order and orthocomplement $\#$ defined by: $b^\# = b' \wedge a$, $b \in [0, a]$.*

Proof. The lower semimodularity of $[0, a]$ is direct in any lower semimodular poset and that $[0, a]$ is orthomodular is similarly straightforward. To show $[0, a]$ upper semimodular, let $p, q, r \in [0, a]$ such that p, q cover r in $[0, a]$. Then $r^\#$ covers $p^\#, q^\#$; so there exists $n \in [0, a]$ such that $p^\#, q^\#$ cover n. Then $n^\#$ covers p, q.

LEMMA 3.4. *For all $x, y \in P$, $x\,u^*\,y = x\,u\,y$ and $x\,l\,y = x\,l^*\,y$.*

Proof. Let $x, y \in P$ and let $a \in x\,u\,y$. Consider the induced poset $[0, a]$. If $b \in x\,l^*\,y$ w.r.t. $[0, a]$, then $b \in x\,l^*\,y$ w.r.t. P. If $c \in x\,u^*\,y$ w.r.t. $[0, a]$, then

$$h(b)+h(c) = h(x)+h(y)$$

by Theorem 2.4. But this implies that $c \in x\,u^*\,y$ w.r.t. P. Then $c \leqslant a$ and $a \in x\,u\,y$ implies $c = a \in x\,u^*\,y$ w.r.t. P. Since $x\,u^*\,y \subseteq x\,u\,y$, the proof is complete.

THEOREM 3.5. *Perspectivity is an equivalence relation on the atoms of P.*

Proof. Perspectivity is clearly symmetric and reflexive. To show transitivity, let x, y, z be distinct atoms of P such that $x \sim y$ and $y \sim z$. If $x \not\perp z$ then by Lemma 3.1 (1) $x \sim z$. Suppose $x \perp z$ and both $x, z \not\perp y$. Let $w = x \vee z$ and let $a \in y'\,l\,w$. Since $1 \in y'\,u\,w$ we have $h(a)+h(1) = h(y')+h(w) = h(1)+1$, and $h(a) = 1$. Hence a is an atom and since $a \perp y$, $a \neq x, z$. We now apply Lemma 3.2 (2) to obtain $x \sim z$. Otherwise there are two possibilities. (1) x, y, z are pairwise orthogonal, or (2) $x \perp z$ and y is orthogonal to exactly one of x, z.

Case (1). Now $x \vee y$, $y \vee z$, $x \vee z$ exist and by Lemma 3.2 $x \vee y$ covers an atom $s \neq x, y$ and $y \vee z$ covers an atom $t \neq y, z$. We shall show the existence of an atom $w \neq x, z$ which is covered by $x \vee z$, so that $x \sim z$ by Lemma 3.1. Letting $r = (x \vee y) \wedge s'$, $v = (z \vee y) \wedge t'$, we see that r, v are atoms and $r \perp s$, $v \perp t$. If $r = x$, then $x \perp s$ implies that $s \vee x$ exists and $x \vee y = s \vee x$. Then

$$s = (s \vee x) \wedge x' = (x \vee y) \wedge x' = y,$$

a contradiction. Similarly we can show that $r \neq y$ and $v \neq y, z$. By the preceding contradiction we see that $s, r \not\perp x$, and since $(y \vee z) \perp x$ we have $t, v \perp x$; so s, r, t, v are distinct. If s or $r = z$ the proof is finished. Therefore we may assumed the seven atoms to be distinct. The orthogonality relations ensure the existence of the following elements: $z\vee s$, $z\vee r$, $x\vee t$, $x\vee v$, $r\vee s$, $t\vee v$. By semimodularity, all these elements have height two. Suppose $z < a\in s\,u^*\,v$. Then since $h(a)=2$ and $a\geqslant z, s$ we have $a=z\vee s$. Since $z, s\perp r$, we have $a\perp r$. Hence $v\perp r$ and so $r\vee v$ exists. Suppose $x<r\vee v$, so that $r\vee v=x\vee v$. Then $x=v'\wedge(x\vee v)=v'\wedge(r\vee v)=r$ is a contradiction. Suppose $z<r\vee v$, so that $r\vee z=r\vee v$. Then

$$z=r'\wedge(r\vee z)=r'\wedge(r\vee v)=v$$

is a contradiction. Hence $x,z\nless r\vee v$. In this case we let $w\in(r\vee v)\,l^*\,(x\vee z)$ and let $u\in(r\vee v)\,u^*\,(x\vee z)$. Since $x\vee y\vee z>r\vee v, x\vee z$ and $r\vee v\neq x\vee z$ we must have $h(u)=h(x\vee y\vee z)=3$. Hence $h(x\vee z)+h(r\vee v)=4=h(u)+h(w)$; so $h(w)=1$. Now w is the atom we seek since $w<x\vee z$ and $w\neq x,z$ since $x,z\nless r\vee v$. A similar result follows if $x<a\in s\,u^*\,v$; so we may assume neither x nor $z<a\in s\,y^*\,v$. By Lemmas 3.3 and 3.4 there must exist $a\in s\,u^*\,v$ such that $a\in[0,x\vee y\vee z]$. But then $x\vee y\vee z$ covers a and $x\vee z$; so there exists $w\neq x,z$ covered by a and $x\vee z$. Again w is the atom we seek.

Case (2). Suppose $x\perp z$, $y\perp z$ but $x\not\perp y$. As above, $y\vee z=t\vee v$. If $x\perp t,v$, then $x'\geqslant t\vee v=y\vee z>y$, which is a contradiction. Therefore $x\not\perp t$ or $x\not\perp v$, say $x\not\perp t$. If $z\perp t$ then $z\vee t=z\vee y$, so $y=z'\wedge(z\vee y)=z'\wedge(z\vee t)=t$, a contradiction. Hence $z\not\perp t$ and t' is a common complement for x and z. If $x\perp z$, $x\perp y$, and $y\not\perp z$, the proof is similar.

LEMMA 3.6. (1) *If $x\in P$ then x is the supremum of $h(x)$ pairwise orthogonal atoms.* (2) *If $x\in P$ then $x=\bigvee\{a\leqslant x: a$ an atom$\}$.*

Proof. (1) Let Q_n be the proposition that the result holds if $h(x)=n$. Now Q_1 is trivial. Suppose Q_{k-1} holds, $k\geqslant 2$, and $h(x)=k$. Then $x>a$ for some atom a. Then $x=a\vee(x\wedge a')$ and $h(x\wedge a')=h(x)-h(a)=k-1$; so $x\wedge a'$ is the supremum of $k-1$ pairwise orthogonal atoms. Since every atom $\leqslant x\wedge a'$ is orthogonal to a, Q_k holds. (2) Let $A=\{a\leqslant x: a$ an atom$\}$. Now $x\geqslant A$ and if $y\geqslant A$ then $y\geqslant x$, since x is the supremum of an orthogonal collection of atoms in A.

A *basis* for $x\in P$ is a set of pairwise orthogonal atoms a_i, $i=1,\dots,n$, such that $x=\bigvee a_i$. It is clear that any $x\neq 0$ has a basis and that every basis for x has $h(x)$ elements.

LEMMA 3.7. *If E is an equivalence class of perspective atoms in P and B is a basis for* 1 *then $\bigvee\{a\in E\}=\bigvee\{b\in E\cap B\}$.*

Proof. It is clear that $\bigvee\{b\in B_0\}$ exists for any $B_0\subseteq B$. Let $B_2=E\cap B$ and $B_1=B-B_2$. If $a\in E$, $b\in B_1$ then $a\perp b$ since otherwise $a\sim b$ and $b\in E$, a contradiction. Hence if $a\in E$ then $a\perp f\equiv\bigvee\{b\in B_1\}$. Now $g\equiv\bigvee\{b\in B_2\}\leqslant f'$ and if $g<f'$ then $f'\wedge g'\neq 0$, which contradicts $g\vee f=1$; so $g=f'$. Hence $E\leqslant g$. If $h\geqslant E$ then $h\geqslant B_2$; so $h\geqslant g$ and hence $\bigvee\{a\in E\}=g$.

It follows from Lemma 3.7 that there are a finite number of equivalence classes E_i, $i=1,\dots,n$, of perspective atoms. Let $e_i=\bigvee\{a\in E_i\}$. It is clear that $e_i\perp e_j$, $i\neq j$, and $\bigvee e_i=1$.

An element $e\in P$ is in the centre of P if P is isomorphic to a direct product $X\cdot Y$ of posets X, Y so that e corresponds to $(1,0)$. We denote the centre of P by $Z(P)$.

Two elements $x, y \in P$ are compatible (written $x \mathrm{C} y$) if there exist mutually orthogonal elements x_0, y_0, z such that $x = x_0 \vee z$, $y = y_0 \vee z$. The following lemma holds on any orthomodular poset.

LEMMA 3.8. *$e \in Z(P)$ if and only if $e \mathrm{C} x$ for all $x \in P$.*

Proof. If $e \in Z(P)$, $x \in P$, then $e = (1, 0)$ and $x = (x_1, x_2)$; so $x = (x_1, 0) \vee (0, x_2)$, $e = (x_1, 0) \vee (x_1', 0)$. Conversely, if $e \mathrm{C} x$ for every $x \in P$ then $x = (x \wedge e) \vee (x \wedge e')$ [**3**, **6**]. Define $f\colon P \to [0, e] \cdot [0, e']$ by $f(x) = (x \wedge e, x \wedge e')$. Clearly f is injective and order-preserving. If $(a, b) \in [0, e] \cdot [0, e']$ then $a \perp b$; so $a \vee b$ exists. Now by [**3**; p. 75] or [**6**; p. 200] $(a \vee b) \wedge e = (a \wedge e) \vee (b \wedge e) = a$ and similarly $(a \vee b) \wedge e' = b$; so $f(a \vee b) = (a, b)$ and f is bijective. Since

$$f^{-1}[(a, b)] = a \vee b,$$

f^{-1} is order preserving; so $e \in Z(P)$.

THEOREM 3.9. *The centre of P is the Boolean algebra generated by $e_1, \ldots, e_n$.*

Proof. Let $x \in P$ and let $a_1, \ldots, a_m$ be a basis for x. Extend $a_1, \ldots, a_m$ to a basis $A = \{a_1, \ldots, a_r\}$, $r \geqslant m$, for 1. Now by Lemma 3.5 $e \equiv e_i = \bigvee\{a_j \leqslant e : a_j \in A\}$. Let

$$c = \bigvee\{a_j \leqslant e : j = 1, \ldots, m\},$$
$$x_0 = \bigvee\{a_j \nleqslant e : j = 1, \ldots, m\},$$
$$e_0 = \bigvee\{a_j \leqslant e : j = m+1, \ldots, r\}.$$

Then c, x_0, e_0 are mutually orthogonal and $x = x_0 \vee c$, $e = x_0 \vee e_0$; so $e \mathrm{C} x$. Hence $e \in Z(P)$. Now suppose $x \in Z(P)$ and hence $x = (1, 0)$ for some direct product decomposition. Suppose $a \leqslant x$ is an atom. Then $a = (a_1, 0)$. Suppose b is an atom and $a \sim b$. Then there is a common complement $\bar{b}$. Hence $a \vee \bar{b} = (1, 1)$, $a \wedge \bar{b} = (0, 0)$; so $\bar{b} = (\bar{a}_1, 1)$. Since $0 = \bar{b} \wedge b = (\bar{a}_1, 1) \wedge (b_1, b_2)$ we must have $b_2 = 0$. Hence $b = (b_1, 0) \leqslant x$ and $e_i \leqslant x$, where $a \in E_i$. Thus $x = \bigvee\{e_i : e_i \leqslant x\}$. Since $Z(P)$ is a Boolean algebra [**2**; p. 67] the proof is complete.

COROLLARY 3.10. *$P = \prod[0, e_i]$ is the complete factorization of P.*

COROLLARY 3.11. *P is indecomposable if and only if the atoms of P are all mutually perspective.*

THEOREM 3.12. *$x \in Z(P)$ if and only if x has a unique complement.*

Proof. We know that an element in $Z(P)$ has a unique complement. Now suppose x has a unique complement. Suppose $a \leqslant x$, $b_1 \leqslant x'$ are atoms and $a \sim b_1$. Since $a \perp b_1$, $a \vee b_1$ exists and covers an atom $c \neq a, b_1$. Extend b_1 to a basis $b_1, \ldots, b_n$ for x' and let $z = b_2 \vee b_3 \vee \ldots \vee b_n$ (let $z = 0$ if $b_1 = x'$). Now $c \perp z$, so $c \vee z$ exists. However $c \vee z \neq x'$, since $b_1 = z' \wedge (b_1 \vee z) = z' \wedge (c \vee z) = c$ is a contradiction. Suppose there is an atom $d \leqslant c \vee z, x$. Then $d \perp z$ and $d \vee z = c \vee z$. Hence $d = z' \wedge (d \vee z) = z' \wedge (c \vee z) = c$. It follows that $c \perp b_1$; so $c \vee b_1 = a \vee b_1$ and $c = b_1' \wedge (c \vee b_1) = b_1' \wedge (a \vee b_1) = a$, which is a contradiction. Hence $(c \vee z) \wedge x = 0$. If $u \in (c \vee z) u^* x$ then $h(u) = h(x) + h(c \vee z) = h(x) + h(x') = h(1)$ so $u = 1$. It follows that $c \vee z \neq x'$ is a complement of x, which is a contradiction. We have thus shown that no atom contained in x is perspective to any atom contained in x'. If P

is indecomposable, applying Corollary 3.11 either x or x' must not contain an atom; so $x = 0$ or 1 and hence $x \in Z(P)$. Otherwise $P = \Pi[0, e_i]$. Since direct products preserve complements, $x = (x_1, \ldots, x_m)$ implies that each x_i has a unique complement in $[0, e_i]$. Then as above, each $x_i = 0$ or 1; so $x \in Z(P)$.

Dilworth's lattice, D16 [**1**], is a poset which is orthomodular but not semimodular. Perspectivity is not an equivalence relation on the atoms of D16 since in Figure 2, p. 93 of [**1**] we see that $a \sim c$, $a \sim e$ but $c \nsim e$. Thus semimodularity cannot be deleted from the hypothesis of Theorem 3.5.

To indicate the need for orthomodularity in the hypothesis, consider the 14-element semimodular orthocomplemented poset whose atoms are u, v, w, x, y, z and whose dual atoms are $u^\perp, \ldots, z^\perp$. Let $U(u) = \{1, u, w^\perp, z^\perp\}$, $U(v) = \{1, v, w^\perp, x^\perp\}$,

$$U(w) = \{1, w, x^\perp, y^\perp, z^\perp, u^\perp, v^\perp\},\ U(x) = \{1, x, w^\perp, y^\perp, z^\perp, v^\perp\},$$

$U(y) = \{1, y, w^\perp, x^\perp, z^\perp\}$ and $U(z) = \{1, z, w^\perp, x^\perp, y^\perp, u^\perp\}$. Perspectivity is not an equivalence relation in the atoms, since $x \sim y$, $y \sim z$ but $x \nsim z$. If we omit $u, v, u^\perp, v^\perp$, we have a counter-example to the conclusion of Lemma 3.1, in that every element covering x, y covers a distinct third atom z, but $x \nsim y$. Note that neither poset is orthomodular, since $x \perp w$ but $\{y^\perp, z^\perp\} \leqslant x\,u\,w \neq x\,v\,w$.

References

1. J. C. Abbott, ed., *Trends in lattice theory* (Van Nostrand Reinhold, New York, 1970).
2. G. Birkhoff, *Lattice theory*, 3rd edn. (Amer. Math. Soc., Providence, R.I., 1967).
3. S. Gudder, "Axiomatic quantum mechanics and generalized probability theory", in *Probabilistic methods in applied mathematics*, ed. A. Bharucha-Reid (Academic Press, Inc., New York, 1970).
4. L. Haskins and S. Gudder, "Height on posets and graphs", *Discrete Math.*, 2 (1972), 357–382.
5. J. Jauch, *Foundations of quantum mechanics* (Addison-Wesley, Reading, Mass, 1968).
6. V. Varadarajan, "Probability in physics and a theorem on simultaneous observability", *Commun. Pure Appl. Math.*, 15 (1962), 189–217.
7. J. von Neumann, *Continuous geometry* (Princeton, N.J., 1960).

Carleton College,
Northfield, Minnesota 55057.

University of Denver,
Denver, Colorado 80210.

Kansas State University,
Manhattan, Kansas 66502.

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