ANY COMPLETE ATOMIC ORTHOMODULAR LATTICE WITH COUNTABLY MANY ATOMS IS A SUBLATTICE OF ONE GENERATED BY THREE ELEMENTS

By

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Abstract

The purpose of this paper is to prove that every complete atomic orthomodular lattice with countably many atoms is a suborthomodular lattice of one such lattice generated by three elements. The one point extension of Bruns and Kalmbach is presented as a pasting of orthomodular lattices.

Introduction

In section I we prove a theorem which gives sufficient conditions for when one point closed complete Dacey space may be embedded in another to give a third.

We then exhibit an infinite point closed complete Dacey space $(Y, \mathcal{F})$ such that $F \in \mathcal{F}$ implies $|F| = 3$ and $(Y, \mathcal{F})$ is generated by a three element set. For any complete atomic orthomodular lattice L with countably many atoms X, we embed X in Y and thereby show that L is a sublattice of a complete lattice generated by three elements. From this we derive a corollary which is stronger than the recent result of Bruns and Kalmbach [1] that the orthomodular lattice freely generated by a three element set (two of which are comparable) contains an infinite chain.

In section 2 we present the one point extension of [1] as a "Paste Job" and generalize theorem 3.4 of [3].

Section 1

Let $(X, \perp)$ be an undirected graph with no loops. A subset D of X is called a $\perp$-set if $a \perp b$ for all $a, b \in D$ with $a \neq b$. For

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$M \subset X$ let $M^{\perp} = \{x \in X : x \perp m$ for all $m \in M\}$ and $M^{\perp\perp} = (M^{\perp})^{\perp}$. Let $\mathbf{E}$ be the set of all maximal $\perp$-subsets (or cliques) of X. Then $\{X, \mathbf{E}\}$ is called the orthogonality space corresponding to the orthogonality relation $\perp$. $(X, \mathbf{E})$ is said to be point closed if $\{x\}^{\perp\perp} = \{x\}$ for all $x \in X$. The logic $L(X, \mathbf{E})$ of the orthogonality space $(X, \mathbf{E})$ is defined to be the set $\{D^{\perp\perp} : D \subset X$ and D is a $\perp$-set$\}$ partially ordered by set-theoretic inclusion. $(X, \mathbf{E})$ is called a Dacey space (respectively, a complete Dacey space) whenever $L(X, \mathbf{E})$ is an orthocomplete orthomodular poset (respectively, complete orthomodular lattice) when orthocomplemented by $^{\perp} : L(X, E) \to L(X, E)$. The orthogonality space (X, E) is a complete Dacey space if and only if, for all $M = M^{\perp\perp} \subset X$ and D a maximal $\perp$-subset of M, $D^{\perp\perp} = M$.

Regarding $(X, \mathbf{E})$ as an incidence structure [2] we define, for $x, y \in X$, the distance $d_{\mathbf{E}}(x, y)$ from x to y to be 0 if $x = y$, and the infimum of the lengths of all paths connecting x to y if $x \neq y$. Thus $d_{\mathbf{E}}(x, y) = 1$ if and only if $x \perp y$; $d_{\mathbf{E}}(x, y) = 2$ if and only $x \neq y$, x not $\perp y$ and, for some $z \in X$, $z \in x^{\perp} \cap y^{\perp}$.

An orthogonality space $(X, \mathbf{E})$ is called a *partial plane* in case, for all $E, F \in \mathbf{E}$, $|E| \geq 3$ and $|E \cap F| \leq 1$ whenever $E \neq F$. Thus our notion of a partial plane is slightly more restrictive than Dembowski's [2]. For any set M, we denote the cardinality of M by $|M|$. The following will prove useful.

Lemma 1 1. Under the hypotheses of the following theorem the following statements hold.

(1) If $a, b \in Y$, then $a \# b$ if and only if $a \perp b$ and $\{a, b\} \subseteq i(X)$.

(2) If $a, b \in Y$, then $a * b$ if and only if $a \perp b$ and $\{a, b\} \not\subseteq i(X)$.

(3) If $A \subset i(X)$ and $|A| > 1$ then $A^{\perp} = A^{\#}$

(4) If $A \subset Y/i(X)$ than $A^{\perp} = A^{*}$.

(5) If D is a $\perp$-set with $D \not\subset i(X)$ then $D^{\perp} = D^{*}$.

(6) If $M = M^{\perp\perp} \neq M^{**}$ and $M \cap (Y \setminus i(X)) \neq \phi$, then $M \cap i(X) \neq \phi$, $M^{*} = \phi$ and $|M^{\perp}| = 1$.

Proof. We leave the verification of (1)—(4) to the reader.

(5) Let D be a $\perp$-set with $D \not\subset i(X)$. Then $D^{*} \subset D^{\perp}$. Snppose there exists an $e \in D^{\perp} \setminus D^{*}$ then there exists a $d_o \in D$ with $e \perp d_o$ and not $e * d_o$. Thus $e \# d_o$ and $e, d_o \in i(X)$. Let $d_1 \in D \setminus i(X)$. Then $d_o * d_1 * e$ so that $d_F(d_o, e) \leq 2$, a contradiction. Hence $D^{*} = D^{\perp}$.

(6) Let $M = M^{\perp\perp} \neq M^{**}$ and $M \not\subset i(X)$. Then there exists $a \in M^{**} \setminus M^{\perp\perp}$ and $n \in M^{\perp}$ that $a \not\perp n$. It follows that $n \in M^{\perp} \setminus M^{*}$. Hence, for some $m \in M$, not $n * m$ so that $n \# m$ and $n, m \in i(X)$. Thus $M \cap i(X) \neq \phi$. By hypothesis there exists an $m_1 \in (Y \setminus i(X)) \cap M$. If $|M^{\perp}| > 1$ then there exists $n_1 \in M^{\perp} \cap (Y \setminus i(X))$; hence $m^{*} n_1 * m_1 * n$ and $d_F(m, u) \leq 3$, a contradiction. Therefore $|M^{\perp}| = 1$. Now $M^{*} \subset M^{\perp} = \{n\}$ and $n \notin M^{*}$ so that $M^{*} = \phi$. This completes the proof of the Lemma.

Theorem 1.2. Let $(X, \mathbf{E})$ and (Y, F) be disjoint point closed complete Dacey spaces such that (Y, F) is a partial plane and $(X, \mathbf{E})$ satisfies the condition $|E| \geq 3$ for all $E \in E$. Let $i : X \| \longrightarrow Y$ be an injection of X into Y satisfying $d_F(i(x), i(y)) \geq 4$ for all $x, y \in X$ with $x \neq y$. For each $E \in E$ let $i(E) = \{i(x) : x \in E\}$, let $i(\mathbf{E}) = \{i(E) : E \in E\}$ and let $G = i(E) \cup F$. Then (Y, G) is a point closed complete Dacey space.

Proof. Let $\#$ and $*$ denote the othogonality relations on $i(X)$ and Y corresponding to $i(E)$ and $\mathbf{F}$, respectively. Let $\perp = \# \cup *$. Then it is easy to see that (Y, G) is the orthogonality space corresponding to the orthogonality relation $\perp$ on Y.

To see that (Y, G) is point closed let $y \varepsilon Y$ and $F \varepsilon \boldsymbol{F}$ with $y \varepsilon F$. Since $F \setminus \{y\} \not\subset i(X)$, $(F\setminus\{y\})^{\perp} = (F\setminus\{y\})^{*}$ by (5) above. But $(F\setminus\{y\})^{*} = \{y\}$ since (Y, F) is a point closed complete Dacey space. Thus $\{y\} = (F\setminus\{y\})^{\perp}$ and $\{y\}^{\perp\perp} = \{y\}$.

To show that (Y, G) is a complete Dacey space let D be a maximal $\perp$-subset of $M = M^{\perp\perp} \subset Y$. We must show that $D^{\perp\perp} = M$. We may assume that $|M| > 1$ and $M \neq Y$. We consider three cases.

Case (*i*) $M \subset i(X)$. By (3) above $M^{\perp} = M^{\#} \subset i(X)$ so that $M = M^{\#\perp} \subset i(X)$. It follows that $|M^{\#}| > 1$ and $M^{\perp\perp} = M^{\#\perp} = M^{\#\#}$. Since D is a maximal $\perp$-subset of $M \subset i(X)$, D is a maximal $\#$-subset of $M = M^{\#\#}$. Since $(X, \boldsymbol{E})$ is a complete Dacey space $D^{\#\#} = M$. Hence $|D| > 1$ and $D^{\perp} = D^{\#} \subset i(X)$. If $D^{\#} = \{a\}$, then $D^{\perp\perp} = a^{\perp} \not\subset i(X)$; but $D^{\perp\perp} \subset M \subset i(X)$, a contradiction. Thus $|D^{\#}| > 1$ and $D^{\perp\perp} = D^{\#\perp} = D^{\#\#} = M$, again by (3) above.

Case (*ii*) $M \subset Y \setminus i(X)$. The proof, similar to that of case (*i*), is left to the reader.

Case (*iii*) $M \not\subset i(X)$ and $M \not\subset Y\setminus i(X)$. If $M \neq M^{**}$ then $M^{\perp} = \{n\} \subset i(X)$ by (6) of the above Lemma. It follows that $n^{*} = M \cap (Y\setminus i(X))$ and $n^{\#} = M \cap i(X)$. Suppose there exist $d \in D \cap i(X)$ and $d_1 \in D \cap (Y\setminus i(X))$; then $n * d_1 * d$ and $d_{F}(n, d) \leq 2$, a contradiction. Thus either $D \subset i(X)$ or $D \subset Y\setminus i(X)$

If $D \subset i(X)$ then D is a maximal $\#$-subset of $M \cap i(X) = n^{\#}$, $D^{\#\#} = n^{\#}$ and $D^{\#} = \{n\}$; thus $|D| > 1$, $D^{\#} = D^{\perp} = \{n\}$ and $D^{\perp\perp'} = n^{\perp} = M$. If $D \subset Y\setminus i(X)$ then D is a maximal $*$-subset of $M \cap (Y\setminus i(X))\, n^{*}$ so that $D^{**} = n^{*}$, $D^{\perp} = D^{*} = \{n\}$ and $D^{\perp\perp} = n^{\perp} = M$.

We may therefore assume that $M = M^{**}$. If $|M \cap i(X)| = 1$

then D is a maximal *-subset of $M = M^{**}$ so $D^{**} = M$; thus $|D| > 1$ and $D^{\perp} = D^{*}$; it follows that $D^{\perp\perp} = D^{*\perp} \supseteq D^{**} = M$, *i.e.* $D^{\perp\perp} = M$. If $|M \cap i(X)| > 1$ then there exist $m, m_1 \in M \cap i(X)$ with $m \neq m_1$ and, by hypothesis, there exists $m_2 \in M \cap (Y \setminus i(X))$; it follows that $M^{\perp} \subset i(X)$ and hence that $M^{\perp} = \{n\} \subset i(X)$. The remainder of the proof is the same as in the preceding paragraph.

Theorem 1.3. Every complete atomic orthomodular lattice with countably many atoms is a suborthomodular lattice of a complete atomic orthomodular lattice generated by three elements.

Proof. Let $\{(Y'_i, F'_i) : i = 1,2,3, \ldots\}$ be countably many disjoint copies of the orthogonality space given in figure 1 and let (Y', F')

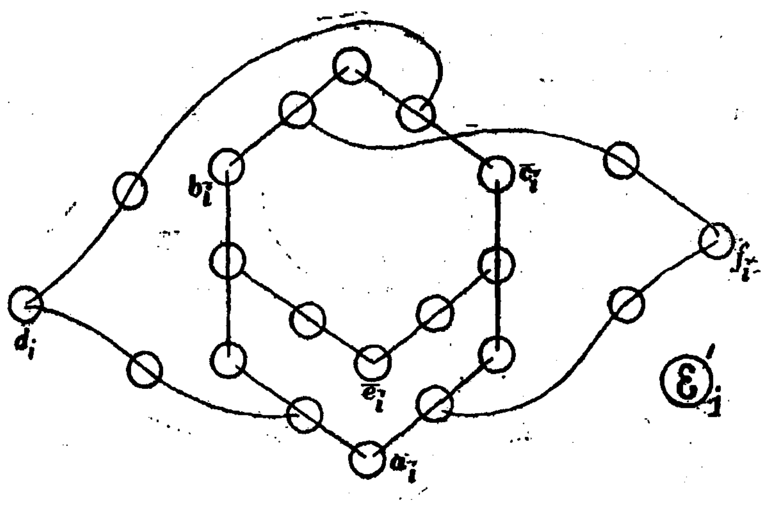

Figure 1. The orthogonality space (Y'_i, E'_i)

denote the (disjoint) union of these spaces. For $i = 1, 2, 3, \ldots$ identify $a_{i+1} = d_i$, $b_{i+1} = e_i$, $c_{i+1} = f_i$ to obtain an orthogonality space (Y, F). In view of [4] Theorem 3 (Y, F) is easily seen to be a point closed complete Dacey Space which is a partial plane. Moreover the set $\{a_1, b_1, c_1\}$ generates $L(Y, F)$.

Let $X = \{x_j\}$ be the set of atoms of L and E the family of all maximal orthogonal sets in X. Then (X, E) is a point closed complete Dacey space since $L(X, E) \cong L$. If $E \in E$ implies $|E| \geq 3$, then we may apply theorem 1.1 to the mapping $i(x_j) = a_j$ to obtain a point closed complete Dacey space (Y, G). $L(X, E)$ is a sub-complete suborthomodular lattice of the complete orthomodular lattice $L(Y, G)$. Moreover $L(Y, G)$ is generated, as a complete lattice, by the three elements $\{a_1, b_1, c_1\}$.

Now, since we may assume that $|X| > 1$, we need only consider the case in which there exists $E \in E$ with $|E| = 2$. Each such set corresponds to a copy of 2^2 as a horizontal summand of L. There are countably many such summands. Select one atom, say y_j, from each such summand. Define $i(x_j) = a_{2j}$ and $i(y_j) = a_{2j+1}$ and apply theorem 1.1. This completes the proof.

Note that the three generating elements above are in fact atoms of the resulting lattice. By utilizing a more complicated combinatorial design for the space (Y, F) - "based" on a pentagon rather than a hexagon - one can strengthen theorem 1.3 to the case in which two of the generating atoms are orthogonal (cf. [1]).

Section 2.

In this section we discuss the one point extension of Bruns and Kalmbach [1]. We give a theorem which generalizes both this construction and "The Paste Job" of [3].

Following Bruns and Kalmbach define a **quasi-ideal** in an orthocomlemented lattice L to be a subset A of L which satisfies the following conditions

1. $0 \in A$,
2. if $a \in A$ and $b \leq a$ then $b \in A$,
3. if $a \in A$ then $a' \notin A$,
4. If $M \subset A$, if $\bigvee M$ exists and if $\bigvee M \notin A$ then $(\bigvee M)' \in A$,
5. for every $x \in L$: $\bigvee([0, x] \cap A)$ exists.

Let $A' = \{a' : a \in A\}$ and assume that L, $A \times \{s'\}$, $A' \times \{s\}$ are pairwise disjoint where s, s' are arbitrary elements. Then the extension E_A^L of L by A is defined to be the set

$$E_A^L = L \cup (A \times \{s'\}) \cup (A' \times \{s\})$$

with the following partial ordering and orthocomplementation. We write E for E_A^L and, for $a \in E$, we write $\pi_1(a)$ only when $a \in E \setminus L$ and mean $\pi_1(a) = x$ if $a = (x, s)$ or $a = (x, s')$. Then, for $a, b \in L$, define $a \leq b$ to mean that one of the following hold :

1. $a \leq_L b$ ($\leq_L$ is the partial ordering in L)
2. $a \in A$ and $a \leq_L \pi_1(b)$
3. $b \in A'$ and $\pi_1(a) \leq_L b$
4. $\pi_1(a) \leq_L \pi_1(b)$ and $\pi_1(a), \pi_1(b)$ are both in A or both in A′.

Define $^\#$: $E \to E$ by

$$a^\# = \begin{cases} a' & \text{if } a \in A \cup A' \\ (x', s) & \text{if } a = (x, s') \in A \times \{s'\} \\ (x, s') & \text{if } a = (x', s) \in A' \times \{s\} \end{cases}$$

Theorem. (Bruns and Kalmboch [1]) : If L is an orthocomplemented (respectively, orthomodular) lattice and A is a quasi-ideal in L, then E_A^L is an orthocompleted (respectively, orthomodular) lattice.

Henceforth we restrict our attention to orthomodular lattices L.

Let $S_A = A \cup A'$ whenever A is a quasi-ideal of L. Then S_A is a **section** of L - as defined in [3] - and S_A is a sub-othomodular lattice of L.

Let $S_A^o = A \cup A' \times \{s\}$ with the ordering induced from $E_A^{S_A}$ and the orthocomplementation induced from $a \to (a', s)$. Then $S_A^o \simeq S_A$ and $E_A^{S_A} \simeq S_A^o \times 2^1$

Let $L_o = E_A^{S_A}$, let $P(L, L_o ; S_A , S_A)$ denote the pasting of L and L_o along S_A and S_A as described in [3], Definition 3.2. Then

$$E_A^L = P(L, L_o : S_A , S_A)$$

Thus if L is complete then L_o is complete and S_A is subcomplete in both ; hence, we may apply theorem 3.4 of [3] and obtain the one point extention as a special case of that result. It follows that if L is complete so is the one point extension of L. By inspection of the proof of that theorem we see that the stronger hypothesis on S_A (namely, for all $x \in L$. $\bigvee([0, x] \cap A)$ exists) allows us to restate theorem 3.4. of [3] to obtain a general theorem which contains as a special case the one point extension.

Theorem 2.1. Let S_1 and S_2 be corresponding sections of the disjoint orthomodular lattices L_1 and L_2. Let $S_i = A_i \cup A'_i$ where A_i is a quasi-ideal in L_i. Then $P(L_1 , L_2 ; S_1 , S_2)$ is an orthomodular lattice.

Proof. The proof follows that of [3], theorem 3.4. except for the discussion of the existence of $e \vee f$ when $e \in L_1 \setminus S_1$ and $f \in L_2 \setminus S_2$. We now discuss this case utilizing the notation of [3]. Let $e_1 \in L_1 \setminus S_1, f_2 \in L_2 \setminus S_2$, $a_1 = \inf_{L_1} \{x \in A'_1 : x \geq_{L_1} e\}$ and $b_2 = \inf_{L_2} \{x \in A'_2 : x \geq_{L_2} f\}$. Then $a_1 \in S_1$ and $b_2 \in S_2$, let $b_1 = \theta^{-1}(b_2)$. Then $[a_1 \vee_{L_1} b_1] = [e_1] \vee [f_2]$ is easily verified. This completes the proof.

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