ANOTHER NONSTANDARD QUANTUM LOGIC (AND HOW I FOUND IT)

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This paper presents an orthomodular lattice with a full set of states which does not admit a strongly order determining set of states. Such a lattice is a quantum logic not embeddable into the standard quantum logic, the lattice $\overline{L}(\mathcal{H})$ of all closed subspaces of a separable complex Hilbert space. In 1969 [1] I gave an example of a poset with this property. Soon after a student, beginning his dissertation, found such a lattice. It was terribly complicated and had about eighty atoms. The student left school and the example was lost. I've been looking for one ever since. Recently I found one. To my surprise the lattice is not much more complicated than the poset. Basically, like its precursor, it's a very symmetric design -- slightly distorted.

The style in which this paper is written is rather unusual. I discuss at length the path that I took in arriving at the final construction (which is given in Figure 6). A few comments follow on intuition and on learning mathematics. The researcher interested only in the technical result can easily skip to Figure 6 and the last section. The reader more interested in the creative process should attempt to read the watered-down section called Preliminaries. If he understands it, fine. If not, he can continue, giving the rest a "once over lightly"; then, if he is so inclined he can reread the whole paper, repeating the process as often as remains profitable.

Preliminaries

In this section I present a discussion of some of the concepts which follow. Most of my readers are already familiar with the notions of a quantum logic, an orthomodular poset, an orthomodular lattice, a state, a full set of states and a strong (sometimes called strongly ordered determining) set of states. Those who are not familiar with these notions and wish to be can find them carefully expounded in [3,4]. Much of this paper can be read without a technical understanding of these terms. Some I won't define here. Others will be glossed over. My aim is to give the non-technical reader some hold on the material, a geometric restatement of the problem which practically anyone should be able to follow.

By a logic I mean either an orthomodular lattice or an orthomodular poset. All of the logics that I shall discuss here will be atomic. This means that they are completely determined by a set X of atoms and a relation $\perp$ called orthogonality. No element of X is orthogonal to itself (in symbols, not $x \perp x$), and if x is orthogonal to y (written $x \perp y$) then $y \perp x$. A subset A of X which consists of mutually orthogonal elements ($x,y \in A$ with $x \neq y$ implies $x \perp y$) is called an orthogonal set. Maximal orthogonal sets (i.e., those contained in no larger orthogonal sets) are called blocks and it is these blocks which we draw as smooth curves (or lines) in each of the figures that follow. The atoms of the logic are the points on the curves; in the figures that follow each block will consist of only three (or four) points. I will refer to these figures depicting atoms of a logic as orthogonality spaces or simply as spaces. If X is the space determined by a logic L then I will say, without explanation, that X generates L and will sometimes write $L = \mathcal{L}(X)$. (Exactly how L is obtained from X may be found in [4].)

Let X be a non-empty set and $\mathcal{E}$ a non-empty collection of subsets of X each of which have at least three elements. Call the elements of X points and the elements of $\mathcal{E}$ lines. Call $(X,\mathcal{E})$ a partial plane in case distinct lines intersect on at most one point, i.e. if $E,F \in \mathcal{E}$ with $E \neq F$ then $^{\#}(E \cap F) \leq 1$. A triangle of $(X,\mathcal{E})$ is a triple E_i, $i = 1,2,3$, of distinct lines each pair of which intersect at a point. A square is a quadruple E_i, $i = 1,2,3,4$, of distinct lines such that each of $E_1 \cap E_2$, $E_2 \cap E_3$, $E_3 \cap E_4$ and $E_4 \cap E_1$ is a point and the other two intersections are empty. It can be shown that (1) every partial plane $(X,\mathcal{E})$ with no triangles generates an orthomodular poset, (2) every partial plane $(X,\mathcal{E})$ with no triangles and no squares generates an orthomodular lattice, and (3) $(X,\mathcal{E})$ is precisely the space of atoms of the logic generated by the partial plane $(X,\mathcal{E})$. This result, especially parts (1) and (2), is known as the Loop Lemma. It is proved, in different terminology, in [2]. Each of the constructions given in Figures 1-6 generates a logic because of part (1) or part (2) of the Loop Lemma.

States on the logic are functions from the logic into the real unit interval [0,1] which are probability measures on each classical sub-logic. The restrictions of these states to the space X of atoms of the logic are called states on X. These are precisely the functions on X which add to 1 over each block. There is a full set of states on a space X provided that, for each pair of elements $x,y \in X$ which are neither equal nor orthogonal, there is a state σ with $\sigma(x) + \sigma(y) > 1$. There is a strong set of states on X provided that for such pair x and y there is a state σ with $\sigma(x) = 1$ and $\sigma(y) > 0$. It should be clear that a space X which has a strong set of states also has a full set of states.

The notions of full and strong were originally defined in the logic $\mathcal{L}(X)$ in terms of an ordering $\leq$ on $\mathcal{L}(X)$. In that setting the definitions are perhaps more natural. A set of states $\mathcal{S}$ is full (or order determining) on a logic $\mathcal{L}$ in case the following always holds: $x \leq y$ in $\mathcal{L}$ if and only if $\sigma(x) \leq \sigma(y)$ for all σ in $\mathcal{S}$. A set of states $\mathcal{S}$ is strong (or strongly order determining) on a logic $\mathcal{L}$ in case the following always holds: $x \leq y$ in $\mathcal{L}$ if and only if $\sigma(x) = 1$ implies $\sigma(y) = 1$. There should be no confusion because the notions match up. X has a full set of states if and only if the logic $\mathcal{L}(X)$ also has a full set of states. X has a strong set of states if and only if the logic $\mathcal{L}(X)$ has a strong set of states.

The Construction

The fact that or the proof that a construction works doesn't necessarily give any idea as to how that particular construction came to be discovered. What follows is an attempt to present the somewhat scattered train of thought that led to the construction, called X_{58} and given in Figure 6, which is the main result of this paper. The path that I stumbled along on has little mathematical value. However it may have some psychological value in that unrelated constraints led to a more elegant solution than was anticipated. Here's how it happened.

I felt that an approach similar to the one that I used to generate a lattice with no states would work. In that earlier paper [2] I observed that the logic generated by the 3×4 window, called $W_{3,4}$ and given in Figure 1, admitted no states. (Reason: There are two partitions of the points by blocks, the 3 horizontal blocks and the 4 vertical blocks. Since every state sums to 1 over each block, if there were a state on

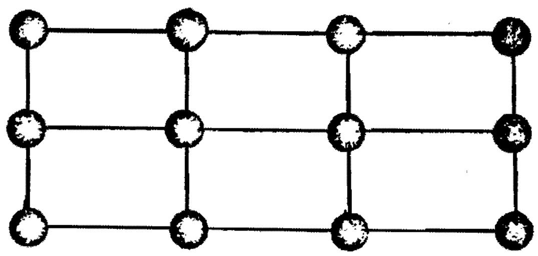

Fig. 1. The space $W_{3,4}$.

$W_{3,4}$ then by summing over all the points in two different ways one could argue that 3 = 4, a contradiction. Thus there are no states on $W_{3,4}$ because $3 \neq 4$.) Because of the existence of squares in $W_{3,4}$ the logic which it generated was a poset.

In order to generate a corresponding lattice, I had to design a space which, while mirroring the same combinatorial restrictions as $W_{3,4}$, had no squares. The example that I arrived at, call it $X_{3,4}$, is given in Figure 2. This 3×4 array of blocks was constructed by replacing each

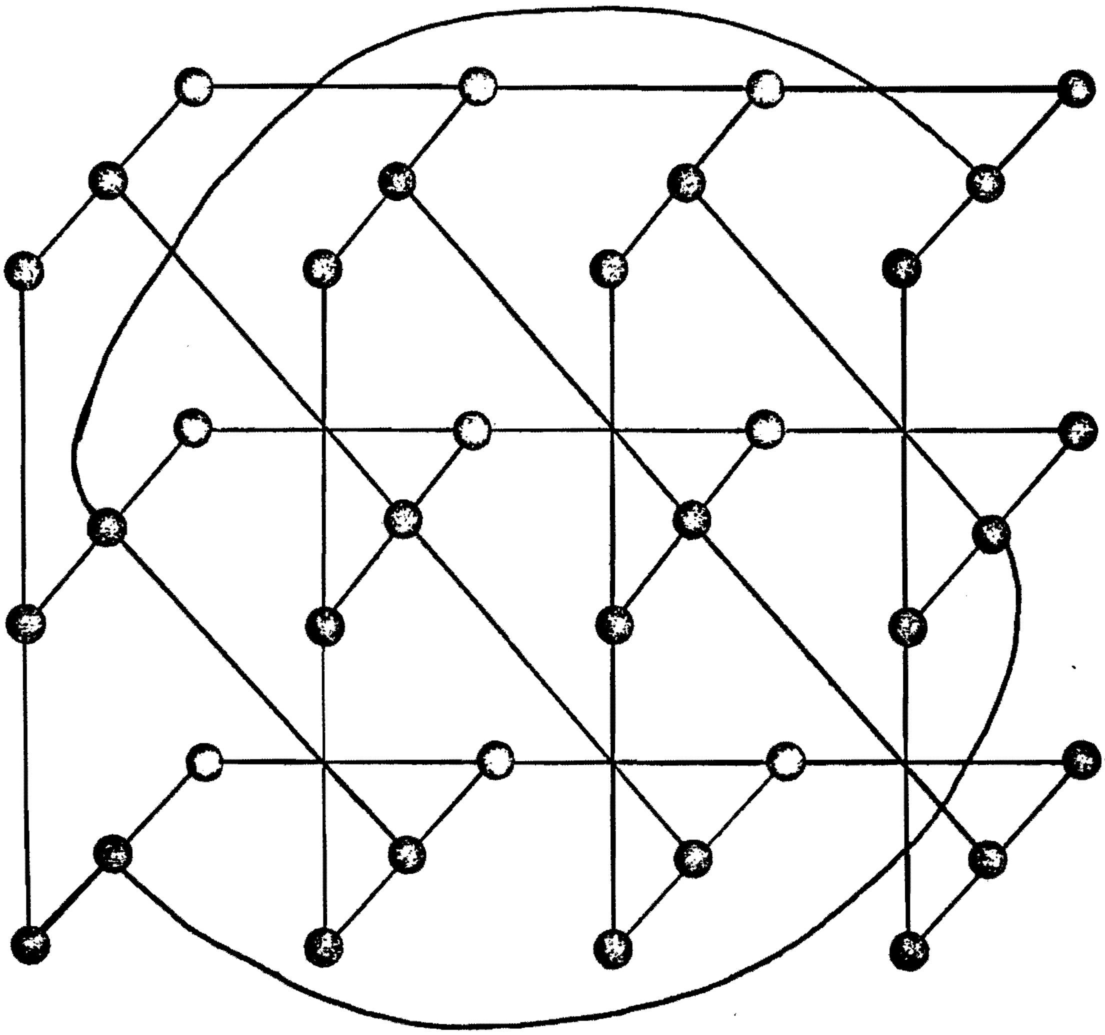

Fig. 2. The space $X_{3,4}$

point of $W_{3,4}$ by a block consisting of 3 points, by maintaining 3 horizontal and 4 vertical blocks and then by intertwining 4 diagonal blocks. I again had two partitions of points by blocks. One partition consisted of the 12 replacement blocks; the other consisted of 11 blocks: 3 horizontal, 4 vertical and 4 diagonal. The combinatorial restrictions mirrored from $W_{3,4}$ involved the existence of these two partitions having different cardinalities. By making an argument similar to that made above I conclude that $X_{3,4}$ admits no states because $11 \neq 12$.

I figured that, somehow, the same trick would work again. So I considered the example given in Figure 3, which I call X_{52} because its logic has 52 elements (25 atoms, the points of the space, their primes or "negations", zero and one). This space has a full but not strong set of states. But because of the many squares which make up its design it generates a poset and not a lattice. It is the space which I presented in 1969 [1] and from which I hoped to evolve a space having no squares yet having a full but not strong set of states.

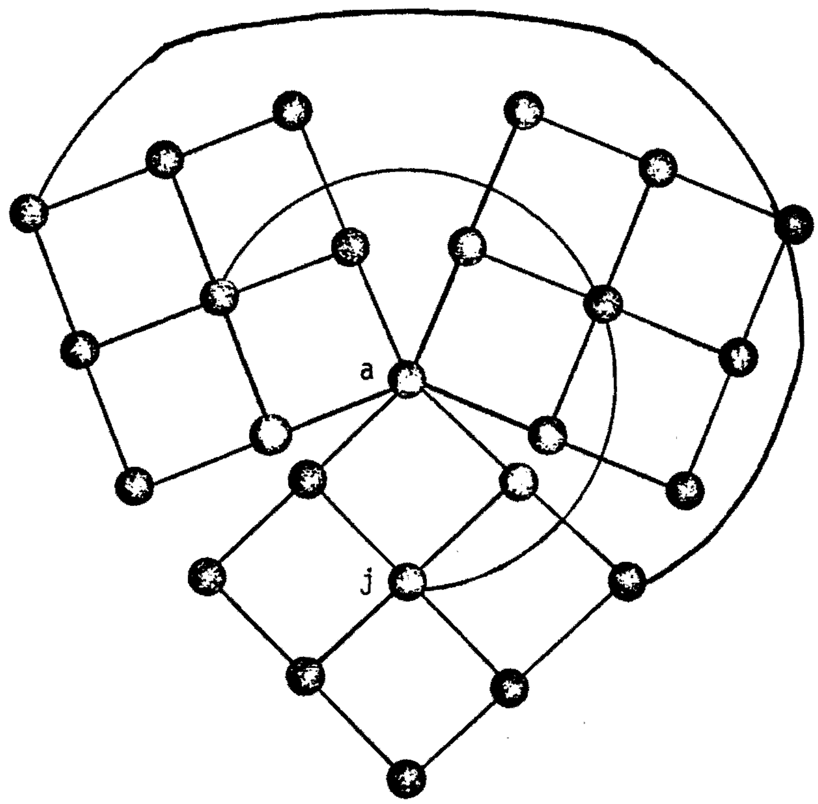

Fig. 3. The space X_{52} .

X_{52} is composed of three 3 × 3 windows. The window $W_{3,3}$ is given in Figure 4.

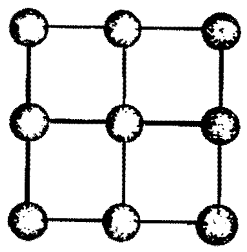

Fig. 4. The 3 × 3 window $W_{3,3}$.

The corresponding latticial design $X_{3,3}$ is given in Figure 5. (The correspondence is the same as that between $W_{3,4}$ and $X_{3,4}$.) The states on $W_{3,3}$, as well as on $X_{3,3}$, are both full and strong.

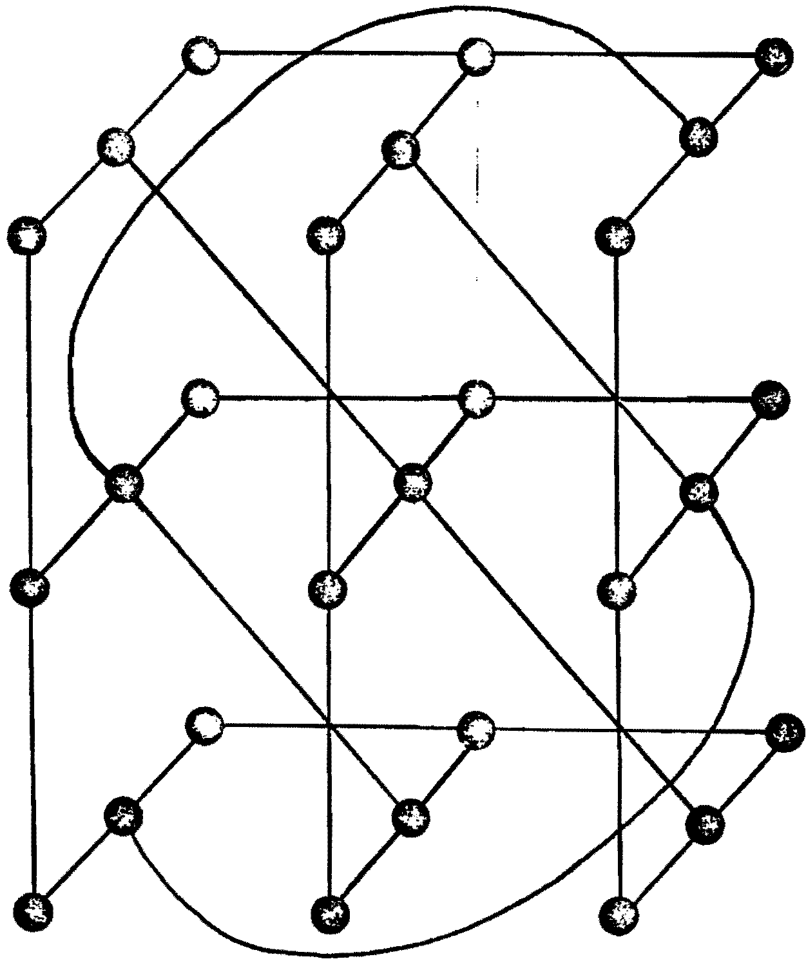

Fig. 5. The space $X_{3,3}$.

The simplest way to see this is to look at the dispersion free states on both of these spaces. (These are states which take on only the values 0 and 1. They are completely determined by where they are 1 and they are 1 exactly once on each block.) The dispersion free states on $W_{3,3}$ are in one-to-one correspondence with the 3×3 permutation matrices, i.e. those matrices of 0's and 1's which have exactly one 1 in each row and in each column; there are six of them. The dispersion free states on $X_{3,3}$ are in one-to-one correspondence with the 3×3 matrices over the set $\{r,c,d\}$ having one r in each row, one d in each diagonal and one c in each column. There are 42 of them. To see that a set of dispersion free states is both full and strong one needs only exhibit for each pair of distinct non-orthogonal elements x,y a state σ with $\sigma(x) = \sigma(y) = 1$. Knowing what the dispersion free states are makes this easy to do.

By intertwining the three windows which make up X_{52} in just the way I did, I had obtained a space with a full but not strong set of states. Now I replaced each window $W_{3,3}$ by the corresponding latticial design $X_{3,3}$ and made a variety of designs by intertwining other blocks through the structure. None of them worked. I soon gave up, at least temporarily. I still had a strong intuition that an example could be found that way. (This must have been the approach taken by the student mentioned earlier. What else could I have suggested to him?) Yet I didn't feel compelled to find it. I also had no specific ideas as to what to do next. As my energy evaporated, a new thought hit me.

I vaguely remembered that a colleague had once told me that, by a very minor adjustment (something like deleting a block or adding a new point or two) of the space $X_{3,4}$ (which admits no states), he could produce a space with a full set of states. (He said that this insight was the key which eventually led to the main results of [6].) Since I needed a structure with a full but not strong set of states (full but with control) I started playing with variations on the theme of a triplication of a modification of $X_{3,4}$, with some identifications and a few extra blocks intertwined, always, of course, respecting the Loop Lemma. What a mess. I got nowhere with it. Triplications of things like $X_{3,4}$ left me just as cold as triplications of things like $X_{3,3}$ which I'd already given up on.

I started modifying the design, simplifying it. I rapidly came back to modifications of $X_{3,4}$ itself, then to modifications in $X_{3,3}$. Having studied the states on $X_{3,3}$ I had a good idea how much play I had while maintaining a full set of states. I also knew that I had to rig two (non-orthogonal) elements, call them a and j, so that $\sigma(a) = 1$ entailed $\sigma(j) = 0$. That's a direct violation of having a strong set of states. (I labelled a and j in Figure 3. The calculation is easy.)

From previous work [4] on spectra of elements in a logic I knew that one way to effect that was to force for some fixed a, f_1 and f_2, something like the equation

$$1 + \sigma(a) = \sigma(f_1) + \sigma(f_2)$$

Then $\sigma(a) = 1$ would entail $\sigma(f_1) = \sigma(f_2) = 1$. After some playing around I found Figure 6, which I call X_{58} because its logic has 58 elements, which provides the desired result. The key fact is that, in X_{58},

$$1 + \sigma(a) = \sigma(b) + \sigma(c) + \sigma(f) + \sigma(g)$$

so that $\sigma(a) = 1$ entails $\sigma(g) = 1$ and therefore $\sigma(j) = 0$. The details are presented in Lemma 1. The proof that X_{58} admits a full set of states is given in Lemma 2.

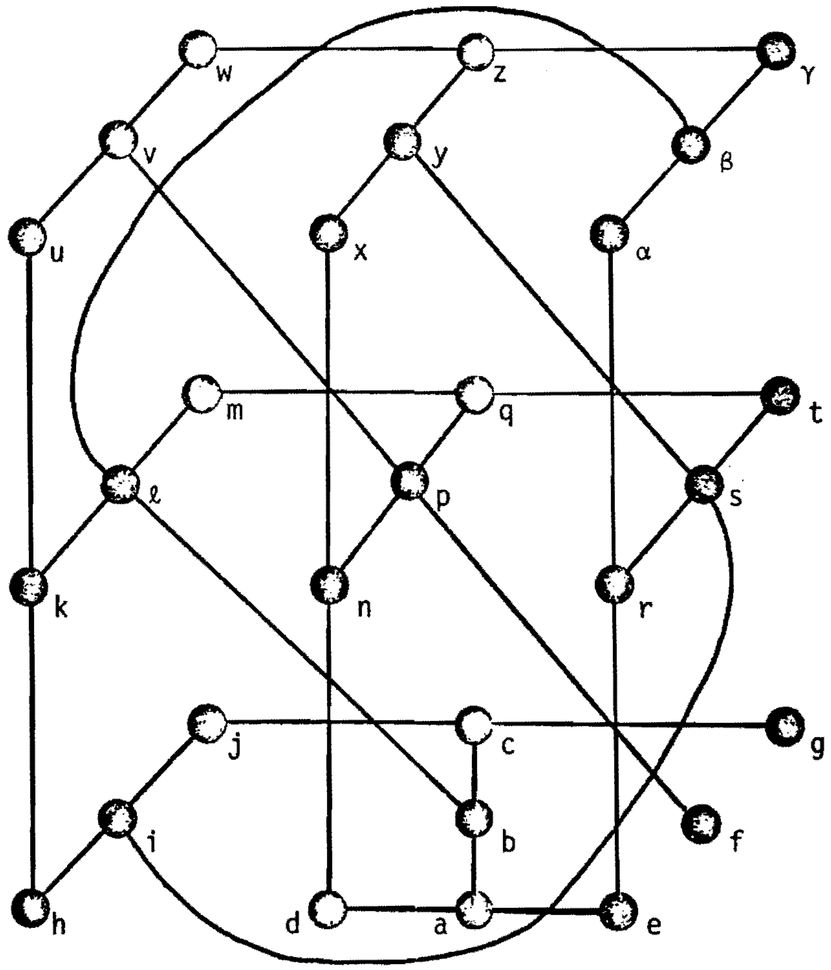

Fig. 6. The space X_{58} .

The above exposition attempted to indicated how X_{58} was discovered. Somehow the circumambulation was important. It provided, together with all the dead ends and minor insights which fit together to yield a major insight, a structure with just the right amount of non-symmetry. As a lesson in the psychology of discovery (or the psychology of invention - depending on your point of view) the route was rather prototypical. Many of the steps had little to do with the cold mathematics of the problem. They were fictions of my mind beyond my capability to verify or refute, or they were vaguely remembered truisms, too loose or trivial to be considered mathematics. Yet these fictions and truisms led me to the final result. In the above case the final result was an example; more frequently in mathematics, it's a theorem. But the general process of fitting it all together frequently consists of many extraneous dimensions, unmentionable dead ends and, at a deeper level, restrictions in patterns of thinking. These facets of research account for much of what is called intuition.

If it wasn't fashionable to sweep such data under the rug and present only finished products, folks would have a better appreciation of mathematical research. There would be fewer students with mental blocks working against their understanding of abstract ideas. There would be more mathematics majors and less fear of mathematical notation.

If more people could witness the transitions from specific examples to abstract concepts, from nebulous intuitions to exact analyses, and from difficult specifics to simple and elegant generalizations, then our culture might begin to assimilate the state of consciousness called mathematics.

Aftermath

In this section we provide the technical proofs that X_{58} has a full but not strong set of states. This easily entails that the logic $\mathcal{L}(X_{58})$, which is a lattice by the Loop Lemma, also enjoys these properties.

Lemma 1. If σ is a state on the space X_{58} then $1 + \sigma(a) = \sigma(b) + \sigma(c) + \sigma(f) + \sigma(g)$. Hence $\sigma(a) = 1$ implies $\sigma(j) = 0$ so that X_{58} does not admit a strong set of states.

Proof. Let $\mathcal{A} = \{\{u,v,w\}, \{x,y,z\}, \{\alpha,\beta,\gamma\}, \{k,\ell,m\}, \{n,p,q\}, \{r,s,t\}, \{h,i,j\}, \{d,a,e\}, \{b,c,f,g\}\}$, let $\mathcal{B} = \{\{w,z,\gamma\}, \{m,q,t\}, \{j,c,g\}, \{u,k,h\}, \{x,n,d\}, \{\alpha,r,e\}, \{v,p,f\}, \{y,s,i\}, \{\beta,\ell,b\}, \{a\}\}$, and let $N_\sigma = \sum_{x \in X_{58}} \sigma(x)$, where σ is any state on X_{58}. Since $\mathcal{A}$ and $\mathcal{B}$

both partition X_{58} and the members of $\mathcal{A} \cup \mathcal{B}$ are blocks of X_{58} except for {b,c,f,g} in $\mathcal{A}$ and {a} in $\mathcal{B}$ we have, by calculating N_σ in two different ways,

$$8 + \sigma(b) + \sigma(c) + \sigma(f) + \sigma(g) = 9 + \sigma(a)$$

and therefore $1 + \sigma(a) = \sigma(b) + \sigma(c) + \sigma(f) + \sigma(g)$. If $\sigma(a) = 1$ then $\sigma(g) = 1$ and $\sigma(j) = 0$. Since a is not orthogonal to j, X_{58} does not admit a strong set of states.

Lemma 2. X_{58} admits a full set of states.

Proof: It suffices to exhibit enough states to exhibit all non-orthogonalities, i.e. for each x,y with $x \neq y$ and not $x \perp y$ we must exhibit a state σ with $\sigma(x) + \sigma(y) > 1$. This is done in Tables 1 and 2. The states σ_i are obtained by multiplying the τ_i given in Table 1 by 1/6. For example, in Table 1 the entry 4 in the row labelled τ_9 and in the column labelled k indicates that $\tau_9(k) = 4$ so that $\sigma_9(k) =$ 2/3. Table 2 gives, for each non-orthogonal pair x and y, the subscript i which determines the state $\sigma_i = \tau_i/6$ for which $\sigma_i(x) + \sigma_i(y) > 1$. For example, the entry 3 in row i column k indicates that $\sigma_3(i) + \sigma_3(k) > 1$ where $\sigma_3 = \tau_3/6$. Since no element is orthogonal to itself the diagonal is left blank. Since the matrix is symmetric the lower half is omitted. Two dashes indicate an orthogonality; for example, there are two dashes in the a,b position of the Table because $a \perp b$ (and therefore $\sigma(a) + \sigma(b) \leq 1$ for all states σ).

Perhaps the main importance of the lattice $\mathcal{L}(X_{58})$ is that it has a full set of states so that it determines a quantum logic and yet is not embeddable in the logic of Hilbert space, the standard quantum logic. There are two relevent notions of embedding. Both involve a function $\phi: L_1 \to L_2$ which is one-to-one and preserves ordering in both directions.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q	r	s	t	u	v	w	x	y	z	α	β	γ
τ_1	6	0	0	0	0	6	6	3	3	0	0	3	3	6	0	0	3	0	3	3	0	3	0	3	3	3	3	0
τ_2	6	0	0	0	0	6	6	3	3	0	0	6	0	0	0	6	3	3	0	3	0	3	6	0	0	3	0	3
τ_3	6	0	0	0	0	6	6	0	6	0	3	3	0	0	0	6	6	0	0	3	0	3	6	0	0	0	3	3
τ_4	4	2	0	2	0	2	6	2	4	0	1	4	1	0	1	5	6	0	0	3	3	0	4	2	0	0	0	6
τ_5	4	2	0	1	1	5	3	0	3	3	0	3	3	5	1	0	3	0	3	6	0	0	0	3	3	2	1	3
τ_6	4	0	2	2	0	4	4	3	3	0	3	3	0	1	2	3	3	0	3	0	0	6	3	3	0	3	3	0
τ_7	4	2	0	2	0	2	6	0	6	0	2	1	3	0	3	3	6	0	0	4	1	1	4	0	2	0	3	3
τ_8	0	6	0	4	2	0	0	0	0	6	4	0	2	2	2	2	0	4	2	2	4	0	0	2	4	4	0	2
τ_9	0	6	0	4	2	0	0	0	0	6	4	0	2	2	2	2	4	0	2	2	4	0	0	6	0	0	0	6
τ_{10}	0	0	6	6	0	0	0	6	0	0	0	6	0	0	0	6	0	6	0	0	6	0	0	0	6	6	0	0
τ_{11}	0	0	6	0	6	0	0	0	6	0	6	0	0	6	0	0	0	0	6	0	6	0	0	0	6	0	6	0
τ_{12}	0	0	6	3	3	0	0	3	3	0	0	0	6	0	6	0	3	3	0	3	0	3	3	0	3	0	6	0
τ_{13}	2	0	4	4	0	4	0	4	0	2	2	4	0	2	2	2	2	0	4	0	0	6	0	6	0	4	2	0
τ_{14}	2	2	2	4	0	0	4	6	0	0	0	0	6	2	4	0	6	0	0	0	2	4	0	6	0	0	4	2
τ_{15}	2	2	2	4	0	0	4	2	4	0	4	0	2	2	0	4	4	2	0	0	6	0	0	0	6	2	4	0
τ_{16}	2	0	4	0	4	4	0	2	2	2	4	2	0	0	2	4	0	4	2	0	0	6	6	0	0	2	4	0
τ_{17}	2	4	0	0	4	0	4	4	0	2	2	2	2	6	0	0	2	0	4	0	6	0	0	6	0	0	0	6
τ_{18}	2	2	2	0	4	4	0	2	0	4	2	4	0	0	0	6	0	6	0	2	2	2	6	0	0	2	0	4
τ_{19}	2	2	2	0	4	4	0	2	0	4	0	0	6	4	2	0	0	6	0	4	0	2	2	0	4	2	4	0
τ_{20}	4	2	0	0	2	2	6	4	2	0	2	4	0	4	2	0	0	0	6	0	2	4	2	4	0	4	0	2
τ_{21}	2	2	2	2	2	4	0	0	2	4	2	4	0	4	2	0	0	0	6	4	0	2	0	4	2	4	0	2
τ_{22}	4	2	0	0	2	2	6	2	4	0	0	4	2	0	4	2	2	2	2	4	0	2	6	0	0	2	0	4
τ_{23}	4	2	0	0	2	2	6	2	4	0	4	2	0	2	4	0	0	0	6	0	0	6	4	2	0	4	2	0
τ_{24}	0	0	6	2	4	0	0	6	0	0	0	4	2	4	0	2	2	2	2	0	6	0	0	4	2	0	2	4

Table 1. The set of states $\sigma_i = \tau_i/6$ are full on X_{58} .

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q	r	s	t	u	v	w	x	y	z	α	β	γ
a		--	--	--	--	1	1	1	1	5	6	1	1	1	7	2	1	2	1	1	4	1	2	1	1	1	1	2
b			--	9	9	5	4	14	7	9	9	--	9	5	9	9	7	19	9	9	9	23	22	9	15	8	--	4
c				10	11	13	--	10	11	--	11	10	12	11	12	10	12	10	11	12	10	6	12	13	10	10	12	24
d					--	13	14	10	15	9	9	10	12	--	14	10	7	10	13	5	9	6	--	9	10	10	14	9
e						16	17	17	11	9	11	18	12	11	12	16	--	16	11	5	17	16	16	17	19	--	12	9
f							1	1	1	5	3	1	1	1	--	2	1	2	1	1	--	1	2	1	1	1	1	2
g								1	1	--	3	1	1	1	7	2	1	2	1	1	7	1	2	1	1	1	1	2
h									--	--	--	10	12	17	14	2	14	10	17	--	10	6	2	14	10	10	14	14
i										--	3	3	12	11	22	3	3	--	20	3	15	3	3	--	11	20	3	3
j											9	18	9	9	9	9	9	18	9	5	9	13	16	9	19	21	19	9
k												--	--	11	23	3	3	16	11	--	9	6	3	9	11	23	11	4
l													--	17	22	3	3	10	13	4	4	6	3	13	10	10	--	4
m														14	12	--	7	12	--	5	14	12	12	14	12	19	12	4
n															--	--	5	19	5	1	15	1	--	1	1	1	1	5
p																--	7	12	20	5	--	6	12	14	12	23	12	4
q																	4	10	--	4	4	6	4	4	10	2	3	2
r																		--	--	4	4	6	3	4	7	--	7	4
s																			--	18	10	16	16	--	10	10	12	18
t																				5	17	6	16	13	11	13	11	9
u																					--	--	2	5	5	5	5	4
v																						--	4	14	10	10	11	4
w																							2	13	--	13	12	--
x																								--	--	2	3	2
y																									--	20	13	4
z																										10	11	--
α																											--	--
β																												--
γ																												

Table 2. The entry 3, for example, in the i,k position indicates that $\sigma_3(i) + \sigma_3(k) > 1$ where $\sigma_3 = \tau_3/6$.

We may insist that ϕ preserve orthogonal suprema or just that it preserve suprema; if ϕ satisfies the former condition call it an ordinary embedding, or just an embedding, if it satisfies the latter call it a strict embedding. Thus strict embeddings are also ordinary embeddings. The point is that $\mathcal{L}(X_{58})$ cannot be embedded in $\overline{L}(\mathcal{H})$ and therefore not strictly embedded. If it were embedded then maximal orthogonal sets (i.e. blocks) in X_{58} would be mapped into maximal orthogonal sets in $\overline{L}(\mathcal{H})$. Since $\overline{L}(\mathcal{H})$ admits a strong (i.e., strongly order determining) set of states it would induce such a set of states on the image of X_{58} which we've shown to be impossible. Therefore $\mathcal{L}(X_{58})$ can be embedded in neither sense of embedding. This answers conjecture 1 of [1]. The question concerning whether a logic with a strong set of states is embeddable in $\overline{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, remains open.

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