# A NON-STANDARD QUANTUM LOGIC WITH A STRONG SET OF STATES

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Since the discovery of non-Euclidian geometries the academic community has widely recognized the importance of non-standard models of axiomatic systems. This paper presents a non-standard quantum logic, call it ($L_{44}$, $M_{22}$). Previous examples[5,6] of non-standard quantum logics were non-standard by reason of the fact that the states were not strongly order determining. $M_{22}$ is strongly order determining on $L_{44}$. The property violated in $L_{44}$ but satisfied in Hilbert space is a variant of Desargues' Theorem. It is called the ortho-Arguesian law and was first formulated by Alan Day.

Recall that a quantum logic (L, M) is standard in case L is ortho-isomorphic to some sub-orthomodular poset of $\bar{L}(H)$, the ortho-lattice of all closed subspaces of a Hilbert space H. This is, of course, a weak notion of being standard since it puts no condition on the states, but then the corresponding notion of being non-standard is strong. Undefined terms may be found in[6].

$L_{44}$ is presented in Figure 1. The notation is that of[4]. Theorem 3 of[4] shows that $L_{44}$ is an orthomodular lattice. Since it is an ortho-lattice the reader need not agonize over the subtleties of sub-orthomodular posets and can just think sub-ortholattice in the above definition.

Table 1 defines 22 dispersion-free states on the atoms of $L_{44}$, and by extension, on all of $L_{44}$. It does so as follows: let $\sigma_i(x)$ be the $i^{th}$ digit of $\bar{x}$. Thus the $i^{th}$ column of Table 1 gives the state $\sigma_i$ on the atoms of $L_{44}$.

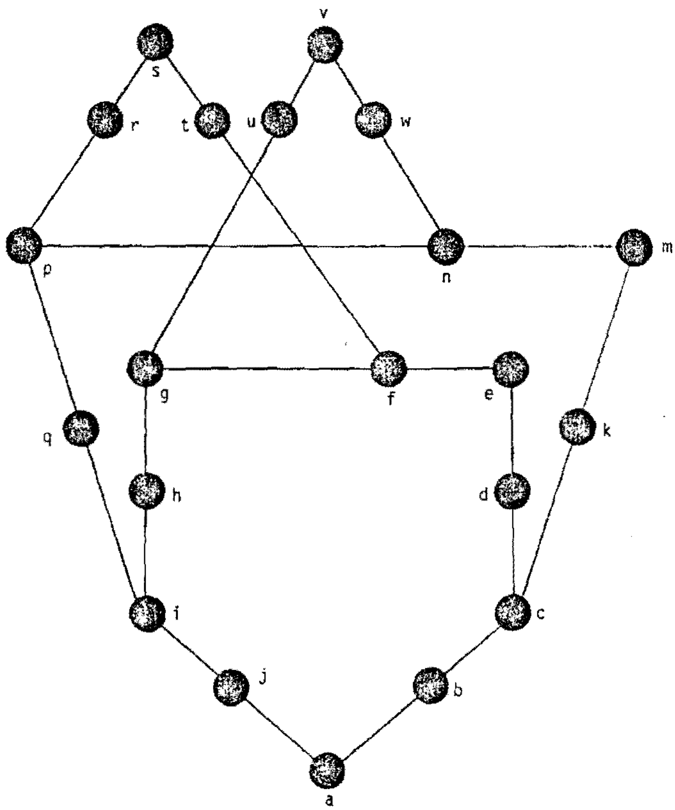


Fig. 1. The Orthomodular Lattice $L_{44}$.

Table 1: $\sigma_i(x)$ = The $i^{th}$ digit in $\bar{x}$.

| | | | | |
|---|---|---|---|---|
| $\bar{a}$ = | 111100 | 000000 | 001001 | 0000 |
| $\bar{b}$ = | 000011 | 100000 | 110110 | 1110 |
| $\bar{c}$ = | 000000 | 011111 | 000000 | 0001 |
| $\bar{d}$ = | 110110 | 100000 | 100110 | 0100 |
| $\bar{e}$ = | 001001 | 000000 | 011001 | 1010 |
| $\bar{f}$ = | 000110 | 010101 | 100100 | 0100 |
| $\bar{g}$ = | 110000 | 101010 | 000010 | 0001 |
| $\bar{h}$ = | 001110 | 010001 | 011101 | 0110 |
| $\bar{i}$ = | 000001 | 000100 | 100000 | 1000 |
| $\bar{j}$ = | 000010 | 111011 | 010110 | 0111 |
| $\bar{k}$ = | 110010 | 100000 | 011000 | 1110 |
| $\bar{m}$ = | 001101 | 000000 | 100111 | 0000 |
| $\bar{n}$ = | 010010 | 010110 | 010000 | 1001 |
| $\bar{p}$ = | 100000 | 101001 | 001000 | 0110 |
| $\bar{q}$ = | 011110 | 010010 | 010111 | 0001 |
| $\bar{r}$ = | 001110 | 010100 | 110100 | 1001 |
| $\bar{s}$ = | 010001 | 000010 | 000011 | 0000 |
| $\bar{t}$ = | 101000 | 101000 | 011000 | 1011 |
| $\bar{u}$ = | 001010 | 010100 | 110101 | 1100 |
| $\bar{v}$ = | 000101 | 000001 | 001000 | 0010 |
| $\bar{w}$ = | 101000 | 101000 | 100111 | 0100 |

Table 2 exhibits the index of the state showing the non-orthogonality of each non-orthogonal pair of atoms; thus the appearance of i in the x, y position indicates that $\sigma_i(x) = \sigma_i(y) = 1$. By Theorem 1.6 of [3] this is enough to ensure that the set $M_{22}$ of states is full on $L_{44}$.

Since each state in $M_{22}$ is dispersion free, $M_{22}$ is strongly order determining.

To see that L is not a sub-ortholattice of $\bar{L}(H)$ for any Hilbert space H, we show that it violates a condition which is always satisfied in $\bar{L}(H)$ and in each of its sub-ortholattices. The condition is called the ortho-Arguesian law. It is due to Alan Day[2] and is a slight variant of the presentation of Desargues' Theorem as found in, say,[1] page 109.

Table 2.

| | a | b | c | d | e | f | g | h | i | j | k | m | n | p | q | r | s | t | u | v | w |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | . | - | - | 1 | 3 | 4 | 1 | 3 | - | - | 1 | 3 | 2 | 1 | 2 | 3 | 2 | 1 | 3 | 4 | 1 |
| b | | . | - | 5 | 6 | 5 | 7 | 5 | 6 | 5 | 5 | 6 | 5 | 7 | 5 | 5 | 6 | 7 | 5 | 6 | 7 |
| c | | | . | - | - | 8 | 9 | 8 | 10 | 8 | - | - | 8 | 9 | 8 | 8 | 11 | 9 | 8 | 12 | 9 |
| d | | | | . | - | 4 | 1 | 4 | 13 | 5 | 1 | 4 | 2 | 1 | 2 | 4 | 2 | 1 | 5 | 4 | 1 |
| e | | | | | . | - | - | 3 | 6 | 14 | 14 | 3 | 14 | 15 | 3 | 3 | 6 | 3 | 3 | 6 | 3 |
| f | | | | | | . | - | 4 | 10 | 5 | 5 | 4 | 5 | 12 | 4 | 4 | - | - | 5 | 4 | 16 |
| g | | | | | | | . | - | - | 7 | 1 | 17 | 2 | 1 | 2 | 22 | 2 | 1 | - | - | 1 |
| h | | | | | | | | . | - | 5 | 5 | 3 | 5 | 12 | 3 | 3 | 18 | 3 | 3 | 4 | 3 |
| i | | | | | | | | | . | - | 19 | 6 | 19 | - | - | 13 | 6 | 19 | 13 | 6 | 13 |
| j | | | | | | | | | | . | 5 | 17 | 5 | 7 | 5 | 5 | 17 | 7 | 5 | 12 | 7 |
| k | | | | | | | | | | | . | - | 2 | 1 | 2 | 5 | 2 | 1 | 5 | 15 | 1 |
| m | | | | | | | | | | | | . | - | - | 3 | 3 | 6 | 3 | 3 | 4 | 3 |
| n | | | | | | | | | | | | | . | - | 2 | 5 | 2 | 14 | 5 | - | - |
| p | | | | | | | | | | | | | | . | - | - | - | 1 | 20 | 12 | 1 |
| q | | | | | | | | | | | | | | | . | 3 | 2 | 3 | 3 | 4 | 3 |
| r | | | | | | | | | | | | | | | | . | - | 3 | 3 | 4 | 3 |
| s | | | | | | | | | | | | | | | | | . | - | 18 | 6 | 17 |
| t | | | | | | | | | | | | | | | | | | . | 3 | 21 | 1 |
| u | | | | | | | | | | | | | | | | | | | . | - | 3 |
| v | | | | | | | | | | | | | | | | | | | | . | - |
| w | | | | | | | | | | | | | | | | | | | | | . |

Here is the definition. Given $a_i$, $b_i \in L$, $i = 0,1,2$, let

$$x = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2), \quad c_i = (a_j \vee a_k) \wedge (b_j \vee b_k)$$

where $\{i, j, k\} = \{0, 1, 2\}$, $y = c_2 \wedge (c_0 \vee c_1)$ and $z = (a_0 \wedge (a_1 \vee y)) \vee b_0$; then $L$ is said to satisfy the <u>ortho-Arguesian Law</u> in case $a_i \perp b_i$, $i = 0, 1, 2$, implies $x \leq z$.

To see that $L_{44}$ does not satisfy the ortho-Arguesian Law, let $a_0 = n$, $b_0 = p$, $a_1 = j$, $b_1 = i$, $a_2 = g$, and $b_2 = f$. Then $a_i \perp b_i$, $i = 0, 1, 2$, $x = c$, $c_0 = h$, $c_1 = 0$, $c_2 = q'$, $y = 0$ and $z = p$. But $x \nleq z$, i.e., $c \nleq p$, so the ortho-Arguesian Law fails in $L_{44}$.

To see that $\bar{L}(H)$ satisfies the ortho-Arguesian Law let $a_i, b_i \in \bar{L}(H)$ with $a_i \perp b_i$, $i = 0,1,2$. Since the orthogonal sum of closed subspaces is closed

$$x = \bigcap_{i=0}^{2} (a_i + b_i)$$

where + indicates the sum of the vector subspaces. Let

$$z^* = b_0 + [a_0 \cap (a_1 + (c_2^* \cap (c_0^* + c_1^*)))] \quad \text{where}$$

$$c_i^* = (a_j + a_k) \cap (b_j + b_k), \ \{i,j,k\} = \{0,1,2\}$$

In order to prove that $x \leq z$ we need only prove that $x \subseteq z^*$ since evidently $z^* \subseteq z$. Let

$$\vec{x} \in x \quad \text{so that} \quad \vec{x} \in a_i + b_i, \quad i = 0,1,2, \text{ and } \vec{x} = \vec{x}_i + \vec{y}_i$$

where $\vec{x}_i \in a_i$ and $\vec{y}_i \in b_i$.

In order to show that $\vec{x} \in z^*$ we need only show that

$$\vec{x}_0 \in a_1 + (c_2^* \cap (c_0^* + c_1^*)). \quad \text{But } \vec{x}_0 = \vec{x} - \vec{y}_0 = \vec{x}_1 + \vec{y}_1 - \vec{y}_0$$

so the proof reduces to showing that

$$\vec{y}_1 - \vec{y}_0 \in c_2^* \cap (c_0^* + c_1^*).$$

But $\vec{y}_1 - \vec{y}_0 = \vec{x}_0 - \vec{x}_1 \in (a_0 + a_1) \cap (b_1 + b_0) = c_2^*$.

Moreover $\vec{y}_1 - \vec{y}_0 = (\vec{y}_1 - \vec{y}_2) + (\vec{y}_2 - \vec{y}_0)$, $\vec{y}_1 - \vec{y}_2 = \vec{x}_2 - \vec{x}_1 \in c_0^*$

and $\vec{y}_2 - \vec{y}_0 = \vec{x}_0 - \vec{x}_2 \in c_1^*$ so that $\vec{y}_1 - \vec{y}_0 \in c_0^* + c_1^*$

completing the proof.

Essentially the same argument proves that any sub-ortholattice of $\bar{L}(H)$ satisfies the ortho-Arguesian Law. Alan Day was the first to realize that the pre-condition, $a_i \perp b_i$ $(i = 0,1,2)$, allowed the above simple vector space argument to go through. Without the pre-condition, the law would have implied modularity and thus not be satisfied in $\bar{L}(H)$ for an infinite dimensional Hilbert space H.

The ortho-Arguesian Law may be stated equationally. Indeed, Day's original reason for inventing it was to exhibit an equation valid in $\bar{L}(H)$ but not in every orthomodular lattice. G. Kalmbach was the first to find an orthomodular lattice which violated the law. $L_{44}$, I am told, is also due to her. There may be an interesting characterization of the class of all orthomodular lattices satisfying the ortho-Arguesian Law. This, however, a coordinatization theorem and a physical interpretation are at present beyond the research horizon.

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