Measurements, Hilbert space and quantum logics

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We consider single and multiple measurements on a quantum logic (P,S) as well as states and propositions conditioned by a measurement. We show that corresponding to any measurement A, there is a canonically associated Hilbert space H_A. Algebraic and statistical properties of (P,S) that are preserved in H_A are found. We then study the problem of embedding a quantum logic in Hilbert space.

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1. INTRODUCTION

According to one interpretation of the quantum logic approach, if $\mathscr{S}$ is a physical system, the set of states S of $\mathscr{S}$ represents the set of preparation procedures for $\mathscr{S}$ and the set of propositions P represents the set of physical yes-no experiments for $\mathscr{S}$. Mathematically P is a σ-orthocomplete orthomodular poset and S is a set of countably additive probability measures on P.[1]

We define a measurement (or operation[2]) to be a maximal orthogonal subset of P. For example, suppose the measuring apparatus consists of a finite sequence of n counters with mutually disjoint volumes of sensitivity. Then the measurement consists of mutually orthogonal yes-no experiments $a_1,a_2,...,a_n,b$ [the ith counter clicks (a_i), no counter clicks (b)]. Or suppose the output of the measuring apparatus is a dial reading. A corresponding measurement would be a partition of the scale on the dial face. In any case, the result of a measurement consists of a set of mutually exclusive alternatives one of which always holds.

The physical system $\mathscr{S}$ can be thought of as a black box whose structure we seek to determine. We can prepare the box in various states and we may then subject the box to various measurements $\mathscr{M}$. By repeating preparations of $\mathscr{S}$ in the state $\alpha\in S$ each followed by a performance of the measurement $\{a_i\}\in\mathscr{M}$, we obtain a probability distribution $\{\alpha(a_i)\}$. These probability distributions give the only information about $\mathscr{S}$ available to us. The corresponding sequences $\{\alpha(a_i)^{1/2}\}$ generate a Hilbert space which will play a central role in our study.

In this article we shall consider single and multiple measurements. We shall also study states and propositions conditioned by a measurement. We observe that corresponding to any measurement A, there is a canonically associated Hilbert space H_A. We shall study to what extent the algebraic and statistical properties of (P,S) are preserved in $\{H_A : A\in\mathscr{M}\}$. Finally we consider the problem of embedding a quantum logic in Hilbert space.

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2. MEASUREMENTS

In the sequel P will denote a *σ-orthocomplete orthomodular poset*. From now on we shall make the physically plausible assumption that P is *separable*, that is, every orthogonal set of elements of P is at most countable. Recall that a subset M of P is a *Boolean subσ-algebra* of P in case (i) M, with the ordering the orthocomplementation induced from P, is a Boolean σ-algebra and (ii) the countable joins in M function as joins in P. We say that a set $A\subseteq P$ is *compatible* if A is contained in a Boolean subσ-algebra of P, and in this case we denote the Boolean subσ-algebra generated by A by $\mathscr{B}(A)$. We say that $A,B\subseteq P$ are *compatible* (written $A\leftrightarrow B$) if $A\cup B$ is compatible; and, if $\{a\}\leftrightarrow\{b\}$ we write $a\leftrightarrow b$.

A *measurement* (or *operation*) on P is a maximal orthogonal set in $P\setminus\{0\}$. We denote the set of all measurements on P by $\mathscr{M}=\mathscr{M}(P)$. Note that $\mathscr{M}(P)$ is a covering of $P\setminus\{0\}$. For a general treatment of such structures and their relation to orthomodular posets see Ref. 3. For $A,B\in\mathscr{M}$ we say that B is a *refinement* of A and write $A\leqslant B$ if for every $a\in A$ there exists a subset $B_1\subseteq B$ such that $a=\bigvee B_1$. We call $A\in\mathscr{M}$ *atomic* if every $a\in A$ is an atom of P. The following two lemmas summarize some useful properties of measurements. The proof of the first lemma is a routine verification which we leave to the reader.

Lemma 1: (1) An orthogonal set $A\subseteq P\setminus\{0\}$ is in $\mathscr{M}$ if and only if $\bigvee A=1$.

(2) For each $a\in P\setminus\{0\}$ there exists an $A\in\mathscr{M}$ such that $a\in A$.

(3) If $A\in\mathscr{M}$, then A is compatible.

(4) If $A\leqslant B$, then $a\leftrightarrow b$ for every $a\in A$, $b\in B$.

(5) $a\perp b$ if and only if there is an $A\in\mathscr{M}$ with $a,b\in A$.

(6) $a\leftrightarrow b$ if and only if there is an $A\in\mathscr{M}$ with $a,b\in\mathscr{B}(A)$.

Lemma 2: (1) $(\mathscr{M},\leqslant)$ is an atomistic poset with least element $\{1\}$ and atoms $\{a,a'\}$, $a\neq 0,1$.

(2) A is a maximal element of $(\mathscr{M},\leqslant)$ if and only if A is atomic.

(3) For $A,B\in\mathscr{M}$, $A\leftrightarrow B$ if and only if there exists a $C\in\mathscr{M}$ such that $A,B\leqslant C$.

(4) P is an atomic Boolean σ-algebra if and only if $(\mathscr{M},\leqslant)$ contains a largest element.

Proof: (1) It is clear that $\leqslant$ is reflexive and transitive on $\mathscr{M}$. To show antisymmetry assume that $A,B\in\mathscr{M}$ with $A\leqslant B$

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and $B \leqslant A$. Let $a \in A$. Then there exists an element $b \in B$ and an $a_1 \in A$ with $a_1 \leqslant b \leqslant a$. If $a \neq a_1$, then $a \perp a_1$, hence $a_1 = 0$ which is a contradiction. Thus $A \subseteq B$ and by symmetry $B \subseteq A$ so $A = B$. The other statements are straightforward.

(2) Assume $A \in \mathcal{M}$ is maximal. If $a \in A$ is not an atom then there exists $a,b \in P$ such that $0 < b < a$. Since $a = b \vee (a \wedge b')$, $(A \setminus \{a\}) \cup \{b, a \wedge b'\}$ is a proper refinement of A contradicting the maximality of A. Thus A is atomic. The converse is clear.

(3) If $A,B \leqslant C$ then $A,B \subseteq \mathscr{B}(C)$ so $A \leftrightarrow B$. Conversely, if $A \leftrightarrow B$, then $C = \{a \wedge b : a \in A, b \in B\} \setminus \{0\} \in \mathcal{M}$ and $A,B \leqslant C$.

(4) If P is an atomic Boolean σ-algebra, then the set of atoms of P is the largest measurement in P. Conversely, assume $\mathcal{M}$ has a largest element A. Then by (2) A is atomic. For $a \in P \setminus \{0,1\}$, $\{a,a'\} \leqslant A$ so P is atomic. Moreover if, $a,b \in P \setminus \{0,1\}$ then $\{a,a'\}, \{b,b'\} \leqslant A$ so, by (3), $a \leftrightarrow b$. Hence P is a Boolean σ-algebra. □

For $a \in P$ and S a set of states (or σ-additive probability measures) on P we define $a^S = \{\alpha \in S \mid \alpha(a) = 1\}$. Throughout this section we assume that P admits a strong set of states S, that is, for $a,b \in P$, $a \leqslant b$ whenever $a^S \subseteq b^S$. We then call the pair (P,S) a *strong quantum logic*. For $A \in \mathcal{M}$, define the Hilbert space

$$\mathscr{H}_A = \left\{ f: A \to \mathbb{C} \,\middle|\, \sum_{a \in A} |f(a)|^2 < \infty \right\}$$

with inner product $\langle f, g \rangle = \Sigma_{a \in A} f(a) \bar{g}(a)$. For $\alpha \in S$, let $\alpha_A \in \mathscr{H}_A$ be the function $\alpha_A(a) = \alpha(a)^{1/2}$, $a \in A$. Notice that α_A is a unit vector in $\mathscr{H}_A$ since $\|\alpha_A\|^2 = \Sigma_{a \in A} \alpha(a) = 1$. We call α_A *the state α conditioned by the measurement A*. Let $P(\mathscr{H}_A)$ be the lattice of all orthogonal projections of $\mathscr{H}_A$. We frequently identify an orthogonal projection with the closed subspace it projects onto. For $a \in P$, define $a_A \in P(\mathscr{H}_A)$ to be the closed span $\overline{\text{sp}}\{\alpha_A \mid \alpha \in a^S\}$. We call a_A the *proposition a conditioned by the measurement A*. Define the maps $J(A): S \to \mathscr{H}_A$ and $K(A): P \to P(\mathscr{H}_A)$ by $J(A)\alpha = \alpha_A$ and $K(A)a = a_A$. The next lemma states that the range of $J(A)$ generates $\mathscr{H}_A$ and that a state $\alpha \in S$, respectively a proposition $a \in P$, is uniquely determined by the maps $J(A)\alpha$, $A \in \mathcal{M}$, respectively, $K(A)a$, $A \in \mathcal{M}$.

Lemma 3: (1) $\mathscr{H}_A = \overline{\text{sp}}\{\alpha_A \mid \alpha \in S\}$.

(2) For $\alpha, \beta \in S$, $\alpha = \beta$ if and only if $\alpha_A = \beta_A$ for all $A \in \mathcal{M}$.

(3) For $a,b \in P$, $a = b$ if and only if $a_A = b_A$ for all $A \in \mathcal{M}$.

Proof: (1) Fix $A \in \mathcal{M}$. For $a \in A$, define $e_a \in \mathscr{H}_A$ by $e_a(b) = \delta_{ab}$ (the Kronecker delta). It is clear that $\{e_a \mid a \in A\}$ is an orthonormal basis for $\mathscr{H}_A$. Moreover, since S is strong, for each $a \in A$ there exists an $\alpha^* \in S$ such that $\alpha^*(a) = 1$. Hence $\alpha^*_A = e_a$ so that $\{e_a : a \in A\} \subseteq \{\alpha_A : A \in \mathcal{M}\}$. The result follows.

(2) Suppose $\alpha_A = \beta_A$ for every $A \in \mathcal{M}$. Let $b \in P \setminus \{0,1\}$ and let $B = \{b,b'\} \in \mathcal{M}$. Then $\alpha(b)^{1/2} = \alpha_B(b) = \beta_B(b) = \beta(b)^{1/2}$, so $\alpha(b) = \beta(b)$; hence $\alpha = \beta$.

(3) Suppose $a_C = b_C$ for every $C \in \mathcal{M}$. Let $A_0 = \{a,a'\} \in \mathcal{M}$. Then

$$\begin{aligned} \text{sp}\{e_a\} &= \overline{\text{sp}}\{\alpha_{A_0} \mid \alpha \in a^S\} \\ &= a_{A_0} = b_{A_0} = \overline{\text{sp}}\{\alpha_{A_0} \mid \alpha \in b^S\}. \end{aligned}$$

Hence, $\alpha \in b^S$ implies $\alpha_{A_0} \in \text{sp}\{e_a\}$ which implies $\alpha_{A_0} = e_a$ so $\alpha \in a^S$. Thus $b^S \subseteq a^S$ and therefore $b \leqslant a$. By symmetry $a \leqslant b$, so $a = b$. □

For $a \in P$, $\alpha \in S$, $A \in \mathcal{M}$ we define the *probability of a in the state α conditioned by the measurement A* to be $\alpha_A(a_A) = \langle a_A \alpha_A, \alpha_A \rangle = \langle K(A)aJ(A)\alpha, J(A)\alpha \rangle$. In general, $\alpha_A(a_A) \neq \alpha(a)$ (see Sec. 4). This is to be expected since a single measurement on a physical system would not in general determine the statistics of the entire system. Also, in general, $K(A)$ does not preserve all the algebraic properties of P. Again, one would not expect a single measurement to determine the complete internal structure of P. The following result shows that $K(A)$ preserves the order on P and that both $J(A)$ and $K(A)$ preserve the structure and statistics of $\mathscr{B}(A)$.

Theorem 4: (1) For $a,b \in P$ and $A \in \mathcal{M}$, $a \leqslant b$ implies $a_A \leqslant b_A$.

(2) $K(A)$ is an isomorphism on $\mathscr{B}(A)$ and $\alpha_A(a_A) = \alpha(a)$ for every $\alpha \in S$ and $a \in \mathscr{B}(A)$.

Proof: (1) If $a \leqslant b$ then $a^S \subseteq b^S$ so that $\{\alpha_A \mid \alpha \in a^S\} \subseteq \{\alpha_A \mid \alpha \in b^S\}$ and $a_A \leqslant b_A$.

(2) For each $a \in A$, $K(A)a = \text{sp}\{e_a\}$ since $\alpha(a) = 1$ implies $\alpha_A = e_a$. Let $b \in \mathscr{B}(A)$. Then there exists a set $B \subseteq A$ such that $b = VB$. Since $K(A)$ preserves order and $a \leqslant b$ for all $a \in B$ we have $K(A)a \leqslant K(A)b$ and hence $V_{a \in B} K(A)a \leqslant K(A)b$. Let $\psi \in K(A)b = \overline{\text{sp}}\{\alpha_A \mid \alpha \in b^S\}$. If $\alpha(b) = 1$ then $\alpha(a) = 0$ for each $a \in A \setminus B$ so $\alpha_A \in \overline{\text{sp}}\{e_a \mid a \in B\}$ and $\psi \in V_{a \in B} K(A)a$. Hence $K(A)b = V_{a \in B} K(A)a$. It follows that $K(A)$ is an isomorphism from $\mathscr{B}(A)$ to the Boolean σ-algebra generated by $\{\text{sp}\{e_a\} \mid a \in A\}$. To show that $\alpha_A(a_A) = \alpha(a)$ for $\alpha \in S, a \in \mathscr{B}(A)$ let $b \in \mathscr{B}(A)$ with $b = VB$, $B \subseteq A$. For $a \in A$ and $\alpha \in S$ we have

$$\alpha_A(a_A) = \langle \text{sp } e_a(\alpha(b)^{1/2})_{b \in A}, (\alpha(b)^{1/2})_{b \in A} \rangle = \alpha(a).$$

Hence

$$\begin{aligned} \alpha_A(b_A) &= \alpha_A(V_{a \in B} K(A)a) = \sum_{a \in B} \alpha_A(K(A)a) \\ &= \sum_{a \in B} \alpha(a) = \alpha(VB) = a(b). \end{aligned}$$ □

Corollary 5: (1) If $A,B \in \mathcal{M}$ and $A \leftrightarrow B$, then there exists a refinement $C \geqslant A,B$ such that $\alpha_C(a_C) = \alpha(a)$ for all $\alpha \in S$ and $a \in \mathscr{B}(A \cup B)$.

(2) If A is atomic and $a \leftrightarrow A$, then $\alpha_A(a_A) = \alpha(a)$ for all $\alpha \in S$.

Proof: (1) Let C be the refinement in the proof of Lemma (2), part (3). Then $\mathscr{B}(C) = \mathscr{B}(A \cup B)$ and the result follows from Theorem 4.

(2) Let $B = \{a,a'\}$. The result follows from (1) since A has no proper refinement. □

Let $f: \mathcal{M} \to \mathbb{C}$. We say that f has a *limit* $\lambda \in \mathbb{C}$ and write $\lim f(A) = \lambda$ if for any $\epsilon > 0$ there exists an $A(\epsilon) \in \mathcal{M}$ such that $|f(B) - \lambda| < \epsilon$ whenever $B \geqslant A(\epsilon)$. The next corollary shows that statistics is preserved in the limit.

Corollary 6: For every $a \in P$, $\alpha \in S$ we have $\alpha(a) = \lim \alpha_A(a_A)$.

For $\alpha, \beta \in S$, *the transition probability* $T_A(\alpha, \beta)$ of α and β given A is defined by

$$T_A(\alpha, \beta)^{1/2} = \sum_{a \in A} \alpha(a)^{1/2} \beta(a)^{1/2} = \langle \alpha_A, \beta_A \rangle.$$

The *transition probability* $T(\alpha, \beta)$ is defined[4] as $T(\alpha, \beta) = \inf_{A \in \mathcal{M}} T_A(\alpha, \beta)$. It is shown in Ref. 5 that $T(\alpha, \beta)$ possesses the usual properties of a transition probability and reduces to the standard form if P is a Hilbert space logic.

Corollary 7: For any $\alpha, \beta \in S$, $T(\alpha, \beta) = \lim T_A(\alpha, \beta)$.

Proof: Given $\epsilon > 0$, by definition there exists an $A \in \mathscr{M}$ such that $|T_A(\alpha, \beta) - T(\alpha, \beta)| < \epsilon$. Now let $B \in \mathscr{M}$ with $B > A$. If $a \in A$, then there exists $a_i \in B$, $i = 1,2,\cdots$, such that $a = \bigvee a_i$. By Schwarz's inequality we have

$$\sum \alpha(a_i)^{1/2}\beta(a_i)^{1/2} < \left[\sum \alpha(a_i)\right]^{1/2}\left[\sum \beta(a_i)\right]^{1/2} = \alpha(a)^{1/2}\beta(a)^{1/2}.$$

Hence $T(\alpha, \beta) < T_B(\alpha, \beta) < T_A(\alpha, \beta)$ and $|T_B(\alpha, \beta) - T(\alpha, \beta)| < \epsilon$. □

There is a close relationship between measurements and observables.[6] Let x be an observable with countable spectrum $\sigma(x) = \{\lambda_i | i = 1,2,\cdots\}$. Then corresponding to x we have a measurement $x(\{\lambda_i\}) i = 1,2,\cdots$. Conversely, if $\{a_i | i = 1,2,\cdots\} \in \mathscr{M}$ and $\{\lambda_i | i = 1,2,\cdots\} \subseteq \mathbf{R}$, then there exists an observable x with $\sigma(x) = \{\lambda_i | i = 1,2,\cdots\}$ and $x(\{\lambda_i\}) = a_i$, $i = 1,2,\cdots$. In general, if x is an arbitrary observable, then there exists a sequence of observables x_i with finite spectra such that for any state α for which the expectation $E_\alpha(x)$ exists, we have $\lim E_\alpha(x_i) = E_\alpha(x)$.[7] Moreover, if $m < n$ then the measurements corresponding to x_n is a refinement of the measurement corresponding to x_m. In this way an arbitrary observable can be associated with a sequence of measurements each being a refinement of the previous ones.

We close this section with a brief consideration of multiple measurements. There are three important types of multiple measurements: sequential measurements, simultaneous measurements and independent measurements. If $A_1,\ldots,A_n \in \mathscr{M}(P)$, then a sequential measurement given by $K(A_n)K(A_{n-1})\cdots K(A_1)$ would first apply $K(A_1)$, then on the resulting system $K(A_2)$ would be applied and so forth. In general, there appears to be no reasonable mathematical way to define this if $A_1,\ldots,A_n$ are measurements on the original quantum logic (P,S). Physically, this is because the first measurement may drastically change the original system so that A_2 no longer applies. We can define a sequential measurement if $A_1 \in \mathscr{M}(P)$, $A_2 \in \mathscr{M}[P(H_{A_1})],\cdots$ by the expression $K(A_n)\cdots K(A_1)$.

In case P is a lattice, if $A_1,\ldots,A_n$ are compatible, then there exists a common refinement $B > A_i$, $i = 1,2,\ldots,n$. One can then consider the measurement B as a simultaneous measurement of $A_1,\ldots,A_n$. If the A_i's are not compatible, there appears to be no mathematical or physical sense for their simultaneous measurement. The next result shows that if $A < B$, then in a certain sense $J(A)J(B) = J(A)$ and $K(A)K(B) = K(A)$. If $A < B$ then, by Theorem 4, $\hat{A} = K(B)A$ is a measurement on $P(H_B)$ which is isomorphic to A.

Lemma 8: If $A < B$ then $J(\hat{A})J(B)\alpha = J(A)\alpha$ and $K(\hat{A})K(B)a = K(A)a$ for every $\alpha \in S$ and $a \in P$.

Proof: Since $A < B$, we have $A \subseteq \mathscr{B}(B)$ and by Theorem 4 $\alpha_B(a_B) = \alpha(a)$ for all $a \in A$. Hence

$$J(\hat{A})J(B)\alpha = (\alpha_B(a_B)^{1/2})_{a\in A} = (\alpha(a)^{1/2})_{a\in A} = J(A)\alpha.$$

Also, for any $a \in P$,

$$K(\hat{A})K(B)a = \overline{\mathrm{sp}}\{J(\hat{A})J(B)\alpha : \alpha \in a^S\} = \overline{\mathrm{sp}}\{J(A)\alpha : \alpha \in a^S\} = K(A)a.$$ □

We now consider independent multiple measurements. Let $A_1,\ldots,A_n \in \mathscr{M}$. An independent measurement of $A_1,\ldots,A_n$ may be physically thought of as follows: Prepare a state α, make the measurement A_1, reprepare the state α, make the measurement $A_2,\ldots$, reprepare the state α, make the measurement A_n. The result would be an n-tuple of vectors $(\alpha_{A_1}, \alpha_{A_2}, \ldots, \alpha_{A_n})$. Motivated by this, we define the map $J(A_1,\ldots,A_n): S \to H_{A_1} \otimes \cdots \otimes H_{A_n}$ by

$$J(A_1,\ldots,A_n)\alpha = J(A_1)\alpha \otimes \cdots \otimes J(A_n)\alpha = \alpha_{A_1} \otimes \cdots \otimes \alpha_{A_n}$$

and the map $K(A_1,\ldots,A_n): P \to P(H_{A_1} \otimes \cdots \otimes H_{A_n})$ by

$$K(A_1,\ldots,A_n)a = \overline{\mathrm{sp}}\{J(A_1,\ldots,A_n)\alpha : \alpha \in a^S\} = \overline{\mathrm{sp}}\{\alpha_{A_1} \otimes \cdots \otimes \alpha_{A_n} : \alpha \in a^S\}.$$

The maps $J(A_1,\ldots,A_n)$, $K(A_1,\ldots,A_n)$ correspond to a conditioning by the independent measurements $A_1,\ldots,A_n$. Notice that

$$[J(A_1,\ldots,A_n)\alpha][K(A_1,\ldots,A_n)a] = \alpha_{A_1}(a_{A_1})\alpha_{A_2}(a_{A_2})\cdots\alpha_{A_n}(a_{A_n})$$

which is the correct statistics for independent measurements. The proof of the next lemma follows easily from Theorem 4.

Lemma 9: Let $K = K(A_1,\ldots,A_n)$.

(1) $K0 = 0$, $K1 = 1$ and $a < b$ implies $Ka < Kb$.

(2) If $a \neq b$ and $a,b \in A_j$ for some $j \in \{1,2,\ldots,n\}$ then $Ka \perp Kb$.

3. HILBERT SPACE EMBEDDINGS

We say that P is *embeddable* in Hilbert space if P is isomorphic to a sub-ortho-modular poset of $P(H)$ for some Hilbert space H. An example is given in Ref. 8 of a finite strong quantum logic (P,S) for which P is not embeddable in Hilbert space. One might ask if there are stronger conditions that can be placed on S which forces P to be embeddable in Hilbert space. One possible such condition is the Jauch–Piron condition.[9] A state α on P is a Jauch–Piron state if $\alpha(a) = \alpha(b) = 1$ implies $\alpha(c) = 1$ for some $c \leqslant a,b$; note that this reduces to the usual definition[10] when P is a lattice. It has been conjectured that if (P,S) is a quantum logic in which S is a strong set of Jauch–Piron states, then P is embeddable in Hilbert space. That this conjecture is false can be seen by combining the example cited above and the following Corollary 12. Another possible such condition is the following.

Call two states α and β of P *mutually singular* and write $\alpha \# \beta$ if $\alpha(c) = \beta(c') = 1$ for some $c \in P$. This relation of mutual singularity has been studied in Ref. 11 in the context of the Jordan–Hahn decomposition of signed states. Now let S be any set of states on P. Write $a \perp_S b$ in case $a^S \times b^S \subseteq \#$, i.e., $\alpha \# \beta$ whenever $\alpha \in a^S$ and $\beta \in b^S$. Clearly, $\perp \subseteq \perp_S$. Call S ultrastrong in case $\perp_S \subseteq \perp$.

Remark 10: (1) If S is ultrastrong then S is strong.

(2) Ultrastrong (like strong) is "ascendingly hereditary," i.e., if $S_1 \subseteq S_2$ and S_1 is ultrastrong then so is S_2.

Proof: (1) Assume S is ultrastrong and $a^S \subseteq b^S$. If $\alpha \in a^S$ and $\beta \in (b')^S$ then $\alpha(b) = \beta(b') = 1$. Hence $a \perp_S b'$ so $a \perp b'$ and $a < b$. Hence S is strong.

(2) This follows immediately from the fact $S_1 \subseteq S_2$ implies $\perp_{S_2} \subseteq \perp_{S_1}$.

We give an example in the next section which shows that strong need not imply ultrastrong. An order determining set of dispersion-free states S on P is ultrastrong. Indeed, suppose $a\perp_S b$. If $\alpha\in a^S$ then $\alpha(b)=0$ since otherwise $\alpha\in a^S\cap b^S$ implying $\alpha\#\alpha$ which is impossible. Hence $\alpha(b')=1$ which implies $a\leqslant b'$ or $a\perp b$. In particular any σ-class[12] or σ-algebra of subsets of a set admits an ultrastrong set of states. Also, for any Hilbert space logic $P(H)$, the set $S(H)$ of all pure states is ultrastrong. To see this, let $a,b\in P(H)$ with $a\perp_S b$. If a or b equals 0 we are finished. Otherwise, let ϕ and ψ be unit vectors with $\phi\in a$ and $\psi\in b$. Then the corresponding pure states α_ϕ, α_ψ satisfy $\alpha_\phi\in a^S$ and $\alpha_\psi\in b^S$. Hence there exists $c\in P(H)$ such that $\alpha_\phi(c)=\alpha_\psi(c')=1$. This implies that $\phi\in c$ and $\psi\in c'$ so $\phi\perp\psi$. Hence $a\perp b$. □

Theorem 11: If P is atomic and every atom $a\in P$ admits a Jauch–Piron state μ_a with $\mu_a(a)=1$, then P admits an ultrastrong set S of Jauch–Piron states.

Proof: Let A be the atoms of P and let $S_0=\{\mu_a \mid a\in A\}$ be a set of Jauch–Piron states indexed by A and satisfying $\mu_a(a)=1$. For each $b\in P\setminus\{0\}$ let A_b be a maximal orthogonal set of atoms under b. Since P is separable A_b is countable so we may write $A_b=\{a_1,a_2,\cdots\}$ and, since P is σ-orthocomplete, $b=\vee A_b$. For $c\in P$ define

$$\mu_{A_b}(c)=\begin{cases}\dfrac{1}{n}\sum\limits_{a_i\in A_b}\mu_{a_i}(c) & \text{if } A \text{ is finite with } n \text{ elements,}\\ \sum\limits_{i=1}^{\infty}\dfrac{1}{2^i}\mu_{a_i}(c) & \text{if } A \text{ is infinite.}\end{cases}$$

Then μ_{A_b} is a state on P.

We claim that, for any $b\in P\setminus\{0\}$, $\mu_{A_b}(c)=1$ if and only if $b<c$. For, if $\mu_{A_b}(c)=1$ then, for each $a_i\in A_b$, $\mu_{a_i}(c)=1$ and hence there exists $d_i\in P$ with $d_i<a_i$ and $d_i<c$ and $\mu_{a_i}(d_i)=1$. Now $d_i\neq 0$ and $a_i\in A$ imply $d_i=a_i$ so that $a_i<c$ for each i. Hence $b=\vee A_b<c$. The converse is clear.

Let $S=\{\mu_{A_b} \mid b\in P\setminus\{0\}\}$. If $\mu_{A_b}(c_1)=\mu_{A_b}(c_2)=1$ then $b<c_1$, $b<c_2$ and $\mu_{A_b}(b)=1$ so each μ_{A_b} is a Jauch–Piron state. To see that S is ultrastrong on P assume that $a\perp_S b$. We may assume that $a,b\neq 0$. Then, in particular, $\mu_{A_a}\#\mu_{A_b}$ so there exists $c\in P$ with $\mu_{A_a}(c)=\mu_{A_b}(c')=1$. Hence $a<c$ and $b<c'$ so that $a\perp b$. □

Corollary 12: Every finite P which admits a strong set of states also admits an ultrastrong set of Jauch–Piron states.

Proof: Let $S(P)$ denote the set of all countably additive states on P and let S be any strong set of states on P. For each atom a of P $a^{S(P)}$ is a nonempty polytope and therefore has finitely many extreme points $\mu_1,\mu_2,\ldots,\mu_n$. The state $\mu_a:=(1/n)\Sigma_{i=1}^n \mu_i$ satisfies the following:

$\mu_a(b)=1$ if and only if $a<b$.

For, if $\mu_a(b)=1$ then $\mu_i(b)=1$ for each i. Hence $\mathrm{ext}(a^{S(P)})\subseteq b^{S(P)}$ and, since $a^{S(P)}$ is the convex hull of $\mathrm{ext}(a^{S(P)})$ and $b^{S(P)}$ is convex, if follows that $a^{S(P)}\subseteq b^{S(P)}$ so that $a^S=a^{S(P)}\cap S\subseteq b^{S(P)}\cap S=b^S$. Since S is strong, $a<b$. The converse is easy. As in the proof of the preceding theorem μ_a is a Jauch–Piron state so that the hypotheses of that theorem are satisfied and the assertion follows. □

Corollary 13: If (P,S) is a strong quantum logic in which P is a finite lattice, then P admits a convex and ultrastrong set of Jauch–Piron states.

Proof: Review of the foregoing result and Remark 10, if P is a lattice then the set of Jauch–Piron states is a convex subset of $S(P)$. □

We note in passing that this result fails for finite orthomodular posets. The smallest orthomodular poset which is not a lattice, J_{18} the 18-element orthomodular poset due to M. F. Janowitz,[13] provides a counterexample. In this poset the convex combination of Jauch–Piron states may not be Jauch–Piron.

We now use our previous work to prove a weak embedding theorem for a certain type of quantum logic. We say that P is *measurement finite* if there exists a finite collection of measurements $A_1,\ldots,A_n$ such that $a\perp b$ implies $a,b\in\mathscr{B}(A_j)$ for some $j\in\{1,\ldots,n\}$. Orthomodular lattices L which are block-finite in the sense that there are only finitely many maximal Boolean sub-algebras were studied in Ref. 14. An immediate corollary of the main result of Ref. 15 is that an orthomodular lattice L is block finite if and only if it is measurement finite. The corresponding result for posets is still open.

We say that (P,S) is *weakly embeddable* in a Hilbert space H if there exist injective maps $J:S\to H$ and $K:P\to P(H)$ such that

(1) $K0=0$, $K1=1$, and $a\leqslant b$ if and only if $Ka\leqslant Kb$.

(2) $a\perp b$ if and only if $Ka\perp Kb$.

(3) $\alpha\#\beta$ if and only if $J_\alpha\perp J_\beta$.

(4) $Ka=\overline{\mathrm{sp}}J(a^S)$.

Theorem 14: A separable, measurement finite strong quantum logic (P,S) is weakly embeddable in a Hilbert space if and only if S is ultrastrong.

Proof: Since P is measurement finite, there exists $A_1,\ldots,A_n\in\mathscr{M}$ such that $a\perp b$ implies, $a,b\in\mathscr{B}(A_j)$ for some $j\in\{1,\ldots,n\}$. Let $J=J(A_1,\ldots,A_n)$ and $K=K(A_1,\ldots,A_n)$. It follows from Lemma 9 that $K0=0$, $K1=1$, and $a<b$ implies $Ka<Kb$. If $a\perp b$ then $a,b\in\mathscr{B}(A_j)$ for some j so $K(A_j)a\perp K(A_j)b$ by Theorem 4. It follows that $Ka\perp Kb$. Suppose that $Ka\perp Kb$. Now there exists an A_j such that $b\in B(A_j)$. Then $b=\vee B$ for some $B\subseteq A_j$. If $\alpha\in a^S$, then

$$\alpha_{A_1}\otimes\cdots\otimes\alpha_{A_n}\in Ka\subseteq Kb\subseteq K(A_1)b\otimes\cdots\otimes K(a_n)b.$$

It follows that

$$\alpha_{A_j}\in K(A_j)b=\overline{\mathrm{sp}}\{e_a: a\in B\}.$$

Hence $\alpha(a)=0$ for $a\in A_j\setminus B$. Therefore $\alpha(b)=\Sigma_{a\in B}\,\alpha(a)=1$. Thus $a^S\subseteq b^S$ and, since S is strong, $a<b$. It follows that K is injective and (1) holds. To complete the proof of (2) assume $Ka\perp Kb$. If $\alpha\in a^S$ and $\beta\in b^S$ then there exists an A_j such that $J(A_j)\alpha\perp J(A_j)\beta$. Hence

$$0=\langle J(A_j)\alpha, J(A_j)\beta\rangle=\sum_{a\in A_j}\alpha(a)^{1/2}\beta(a)^{1/2}.$$

Let $B=\{a\in A_j:\beta(a)=0\}$ and let $b=\vee B$. Since $\alpha(a)\beta(a)=0$ for all $a\in A_j$ we have $\alpha(b)=\beta(b')=1$. Hence $\alpha\#\beta$ and $a\perp_S b$. Assuming that S is ultrastrong, we conclude that $a\perp b$.

To show that J is injective, assume $\alpha,\beta\in S$ with $\alpha\neq\beta$. Then there exist $b\in P$ such that $\alpha(b)\neq\beta(b)$. Now $b\in\mathscr{B}(A_j)$ for some $j\in\{1,\ldots,m\}$ so $\alpha(a)\neq\beta(a)$ for some $a\in A_j$. Hence $J(A_j)\alpha\neq J(A_j)\beta$ and $J\alpha\neq J\beta$. To prove (3) assume $\alpha\#\beta$. Then $\alpha(c)=\beta(c')=1$ for some $c\in P$. Again $c\in\mathscr{B}(A_j)$ for some A_j and $c=\vee B$ for some $B\subseteq A_j$. Hence $\beta(a)=0$ for all $a\in B$ and $\alpha(a)=0$ for all $a\in A_j\setminus B$. Thus $\alpha(a)\beta(a)=0$ for all $a\in A_j$ and

$J(A_j)\alpha \sqcup J(A_j)\beta$. It follows that $J\alpha \sqcup J\beta$. Conversely assume $J\alpha \sqcup J\beta$. Then $J(A_j)\alpha \sqcup J(A_j)\beta$ for some A_j. As in the previous paragraph $\alpha \# \beta$. Condition (4) follows by definition.

Conversely, assume (P,S) is weakly embeddable in a Hilbert space and assume $a \perp_S b$. Let $\alpha \in a^S$ and $\beta \in b^S$. Then $\alpha \# \beta$, so $J\alpha \sqcup J\beta$. Hence,

$$Ka = \overline{\text{sp}} J(a^S) \perp \overline{\text{sp}} J(b^S) = Kb.$$

It follows that $a \perp b$ and S is ultrastrong. □

Corollary 15: If (P,S) is a finite strong quantum logic, then $(P,S(P))$ is weakly embeddable in a Hilbert space.

Proof: Since S is strong, P admits an ultrastrong set of states by Corollary 12. Hence, by Remark 10 part (2) $S(P)$ is ultrastrong. □

The example (P_0,S_0) [16] cited earlier of a strong quantum logic not embeddable in Hilbert space yields an example, namely $(P_0,S(P_0))$, of a strong quantum logic weakly embeddable in a Hilbert space but not embeddable in a Hilbert space.

4. EXAMPLES

We first give an example of a quantum logic (P,S) in which P, given in Fig. 1, is a finite orthomodular lattice and S is a strong set of pure (extremal) states which is not ultrastrong. Simpler examples can be constructed in which the states are not pure. In our notations (see Ref. 17), the vertices represent atoms of P and the straight line segments group these atoms into (3 element) maximal orthogonal sets. Table I lists a set S of 22 states and the values that these states attain on each atom. One can check[18] that each state in S is pure and that S is strong on P. However S is not ultrastrong since $a \perp_S k$ while $a \not\perp k$.

We now present some examples of measurement conditioning maps on quantum logics. Let P be a separable Boolean σ-algebra of subsets of a set X and let S be the set of Dirac measures on P. That is, every state in S has the form $\alpha_x, x \in X$, where $\alpha_x(a) = 1$ if $x \in a$ and $\alpha_x(a) = 0$ if $x \notin a$ for each a in P. Let $A \in \mathscr{M}(P)$. Then $J(A)\alpha_x = e_a$, where a is the unique element of A containing x. It follows that

$$K(A)b = \overline{\text{sp}}\{e_a : a \cap b \neq \phi, a \in A\}.$$

Hence, $[J(A)\alpha_x][K(A)b] = \alpha_x(b)$ for all $x \in X$, $b \in P$.

Let $(P(H),S(H))$ be the Hilbertian logic in which $P(H)$ is the lattice of all closed subspaces of a separable Hilbert space H and $S(H)$ is the set of all pure states on $P(H)$. Let $\{e_i : i = 1,2,\cdots\}$ be an orthonormal basis for H. Then $A = \{\text{sp } e_i : i = 1,2,\cdots\}$ is a measurement. If α_ϕ is a pure state corresponding to the unit vector $\phi \in H$, then

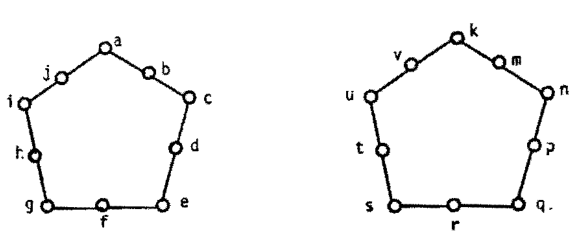

FIG. 1. An orthomodular lattice.

TABLE I. Strong but not ultrastrong states.

	a	b	c	d	e	f	g	h	i	j	k	m	n	p	q	r	s	t	u	v
1	1	0	0	1	0	1	0	1	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	0	1	0	1	0	1	0	0
3	1	0	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1
4	0	1	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	1	0	0
5	1	0	0	0	1	0	0	1	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
6	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	0	0	1	0	0	1	0	0
7	1	0	0	1	0	0	1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
8	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	0	1	0	0	1	0	0	0
9	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1
10	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0	0	1	0	0	1
11	0	0	1	0	0	1	0	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
12	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1	0	0	1	0	1	0	1
13	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1
14	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	1
15	0	1	0	1	0	1	0	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
16	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	0	1	0	1	0	0	1	0
17	0	0	1	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1	0	1
18	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0
19	0	1	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	1	0	1
20	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	0	1	0
21	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1
22	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0

$$J(A)\alpha_\phi = (|\langle \phi, e_i \rangle| : i = 1,2,\cdots) \in l_2.$$

For $a \in P(H)$ we have

$$K(A)a = \overline{\text{sp}}\{(|\langle \phi, e_i \rangle| : i = 1,2,\cdots) : \phi \in a\}.$$

In particular, $K(A)(\text{sp } e_i) = \text{sp } \delta_i$, where $\delta_i(j) = \delta_{ij}$, $i,j = 1,2,\cdots$. If a is one-dimensional and $\psi \in a$ with $\|\psi\| = 1$, then

$$[J(A)\alpha_\phi][K(A)a] = \left[\sum |\langle \psi, e_i \rangle \langle e_i, \phi \rangle|\right]^2.$$

In particular, if $\langle \psi, e_i \rangle$, $\langle e_i, \phi \rangle \geqslant 0$ for each $i = 1,2,\cdots$, then

$$[J(A)\alpha_\phi][K(A)a] = |\langle \psi, \phi \rangle|^2 = \alpha_\phi(a).$$

Finally, let P be the six element orthomodular lattice, i.e., $P = \{0,1,a,a',b,b'\}$ is the horizontal sum of two copies of 2^2, and let S be the set of all states on P. Then $\mathscr{M}(P)$ contains two nontrivial measurements, $A = \{a,a'\}$ and $B = \{b,b'\}$. The Hilbert spaces H_A and H_B both are $\mathbb{C}^2$. Let $e_1 = (1,0)$ $e_2 = (0,1)$ be the natural orthonormal basis for $\mathbb{C}^2$. Then for every $\alpha \in S$ we have

$$J(A)\alpha = \alpha(a)^{1/2} e_1 + (1 - \alpha(a))^{1/2} e_2$$

and

$$J(B)\alpha = \alpha(b)^{1/2} e_1 + (1 - \alpha(b))^{1/2} e_2.$$

Now $K(A)a = \text{sp } e_1$, $K(A)a' = \text{sp}(e_2)$, $K(A)b = K(A)b' = \mathbb{C}^2$ and $K(B)b = \text{sp } e_1$, $K(B)b' = \text{sp } e_2$, $K(B)a = K(B)a' = \mathbb{C}^2$. Hence $\alpha_A(a_A) = \alpha(a)$, $\alpha_A(a'_A) = \alpha(a')$, $\alpha_A(b_A) = \alpha_A(b'_A) = 1$ for every $\alpha \in S$ with similar equations holding for α_B. Also $H_A \otimes H_B = \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\begin{aligned} J(A,B)\alpha &= J(A)\alpha \otimes J(B)\alpha \\ &= \alpha(a)^{1/2}\alpha(b)^{1/2} e_1 \otimes e_1 + \alpha(a)^{1/2}[1 - \alpha(b)]^{1/2} e_1 \otimes e_2 \\ &\quad + [1 - \alpha(a)]^{1/2}\alpha(b)^{1/2} e_2 \otimes e_1 \\ &\quad + [1 - \alpha(a)]^{1/2}[1 - \alpha(b)]^{1/2} e_2 \otimes e_2, \end{aligned}$$

$$
\begin{aligned}
K(A,B)a &= \mathrm{sp}\{J(A,B)\alpha:\alpha(a)=1\} \\
&= \mathrm{sp}\{e_1 \otimes [\alpha(b)^{1/2}e_1 + (1-\alpha(b))^{1/2}e_2]:\alpha(a)=1\} \\
&= \mathrm{sp}\, e_1 \otimes \mathbb{C}^2 \\
&= K(A)a \otimes K(B)a.
\end{aligned}
$$

Similarly $K(A,B)a' = \mathrm{sp}\, e_2 \otimes \mathbb{C}^2$, $K(A,B)b = \mathbb{C}^2 \otimes \mathrm{sp}\, e_1$, and $K(A,B)b' = \mathbb{C}^2 \otimes \mathrm{sp}\, e_2$. Moreover, if P_{e_1} denotes the projection onto sp e_1, we have

$$
\begin{aligned}
&[J(A,B)\alpha][K(A,B)a] \\
&\quad = \langle P_{e_1} \otimes IJ(A)\alpha \otimes J(B)\alpha, J(A)\alpha \otimes J(B)\alpha\rangle \\
&\quad = \langle \alpha(a)^{1/2}e_1 \otimes J(B)\alpha, J(A)\alpha \otimes J(B)\alpha\rangle \\
&\quad = \alpha(a)^{1/2}\langle e_1, J(A)\alpha\rangle \, \langle J(B)\alpha, J(B)\alpha\rangle = \alpha(a).
\end{aligned}
$$

In a similar way, the statistics is preserved for a',b,b',0, and 1. This example can be generalized to any finite horizontal sum of separable Boolean σ-algebras.

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