SOME EQUATIONS RELATED TO STATES ON ORTHOMODULAR LATTICES

by

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Herein OML denotes orthomodular lattice. Recall that a state on an OML L is a function from L into the real numbers which is a countably additive probability measure on every Boolean subalgebra of L. (What we do here works for finitely additive states.) A set $\mathcal{S}$ of states on L is said to be full (respectively, strong) in case $x \leq y$ in L if and only if $\alpha(x) \leq \alpha(y)$ (respectively, $\alpha(x) = 1$ implies $\alpha(y) = 1$) for all α in $\mathcal{S}$. A state α is dispersion free in case $\alpha(x) \in \{0,1\}$ for all $x \in L$. Let $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{DF}$ denote the classes of OMLs having a full set of states, a strong set of states, and a full set of dispersion free states, respectively. It is easy to see that $\mathcal{DF} \subsetneq \mathcal{S} \subsetneq \mathcal{F}$. Recently one of us [1,2] proved that $\mathcal{DF}$ is a variety while $\mathcal{F}$ and $\mathcal{S}$ are not varieties.

In part 1 of this paper some ortholattice equations are given which are valid in $\mathcal{DF}$ but not in $\mathcal{S}$. These are the only known equations with this property. In part 2, an equation is given which is valid in $\overline{L}(H)$, the lattice of all closed subspaces of a Hilbert space, but not in $\mathcal{DF}$. This allows us to present an example of an OML which is not embeddable in $\overline{L}(H)$, yet belongs to $\mathcal{DF}$. Other such examples [5,6,7] are either more complicated or not in (even) $\mathcal{S}$. Previously the ortho-Arguesian equation [7] of A. Day was the only equation known to be satisfied in $\overline{L}(H)$ but not in some $L \in \mathcal{S}$. This equation involved six variables (and a familiar Gestalt). Our equations involve only 3 variables.

SECTION 1. Some equations valid in $\mathcal{DF}$

Recently one of us [2] proved that $\mathcal{DF}$ is an equational class. This section may be viewed as a beginning of a project to explicitly describe an equational basis for $\mathcal{DF}$. Such a project would be of interest in the light of the association of $\mathcal{DF}$ with certain theories of hidden variables [9] or with recent developments in coarse-grained measure spaces [10].

We shall occasionally refer to "the equation $M \leq N$" meaning "the equation $M \vee N = N$". Recall that, for $x \in L$, the Sasaki projection $\phi_x : L \to L$ is defined by $\phi_x(y) = (y \vee x') \wedge x$. This mapping, being residuated, preserves joins.

Proposition 1: For $L \in \mathcal{DF}$, $n \geq 2$ and $b_i \in L$, $i = 0,1,\ldots,n$, the following equations are satisfied:

$$(E_n) \qquad (b_0 \vee (b_0' \wedge b_1')) \wedge \bigwedge_{i=1}^{n-1} (b_i \vee (\bigwedge_{j=1}^{i+1} b_j')) \leq b_0 \vee (\bigwedge_{i=1}^{n} b_i'),$$

$$(F_n) \qquad b_0' \wedge (\bigvee_{i=1}^{n} b_i) \leq \bigvee_{i=0}^{n-1} \phi_{b_i'}(b_{i+1}) \vee \bigvee_{1 \leq j < i \leq n-1} \phi_{b_i'}(b_j) .$$

Proof. We first prove that every $L \in \mathcal{DF}$ satisfies (E_n). Let α be any dispersion free state on L with $\alpha((b_0 \vee (b_0' \wedge b_1') \wedge \bigwedge_{i=1}^{n-1} (b_i \vee (\bigwedge_{j=1}^{i+1} b_j'))) = 1$. We shall prove that $\alpha(b_0 \vee (\bigwedge_{i=1}^{n} b_i')) = 1$.

Thus $1 = \alpha(b_0 \vee (b_0' \wedge b_1')) = \alpha(b_0) + \alpha(b_0' \wedge b_1')$ and, for $0 \leq i \leq n-1$, $1 = \alpha(b_i \vee (\bigwedge_{j=1}^{i+1} b_j')) = \alpha(b_i) + \alpha(\bigwedge_{j=1}^{i+1} b_j')$. We may assume that $\alpha(b_0) \neq 1$ so that $\alpha(b_0) = 0$ and $\alpha(b_0' \wedge b_1') = 1$. If $\alpha(b_k) = 0$ then $\alpha(\bigwedge_{j=1}^{k+1} b_j') = 1$ so $\alpha(b_{k+1}') = 1$ and $\alpha(b_{k+1}) = 0$. Thus, by a finite induction, $\alpha(b_{n-1}) = 0$ and $\alpha(\bigwedge_{j=1}^{n} b_j') = 1$. Hence $\alpha(b_0 \vee (\bigwedge_{i=1}^{n} b_i')) = 1$ and (E_n) is established.

We next observe that (E_n) is equivalent to (F_n) in any OML. We do this by taking orthocomplements of both sides of E_n, invoking the deMorgan Laws and using well-known properties of the Sasaki projections. Thus

$$b_0' \wedge (\bigvee_{i=1}^{n} b_i) \leq \phi_{b_0'}(b_1) \vee \bigvee_{i=1}^{n-1} (b_i' \wedge (\bigvee_{j=1}^{i+1} b_j))$$

$$= \phi_{b_0'}(b_1) \vee \bigvee_{i=1}^{n-1} \phi_{b_i'}(\bigvee_{j=1}^{i+1} b_j)$$

$$= \phi_{b_0'}(b_1) \vee \bigvee_{i=1}^{n-1} \bigvee_{j=1}^{i+1} \phi_{b_i'}(b_j)$$

$$= \phi_{b_0'}(b_1) \vee \bigvee_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq i+1}} \phi_{b_i'}(b_j)$$

$$= \phi_{b_0'}(b_1) \vee \bigvee_{i=1}^{n-1} \phi_{b_i'}(b_{i+1}) \vee \bigvee_{1 \leq j < i \leq n-1} \phi_{b_i'}(b_j)$$

$$= \bigvee_{i=0}^{n-1} \phi_{b_i'}(b_{i+1}) \vee \bigvee_{1 \leq j < i \leq n-1} \phi_{b_i'}(b_j).$$

With $x = b_0'$, $y = b_1$, and $z = b_2$, (F_2) reduces to

$$\text{(I)} \qquad x \wedge (y \vee z) \leq \phi_x(y) \vee \phi_{y'}(z).$$

Proposition 2. There exists an $L \in S$ in which equation (I) fails.

Proof. By the Loop Lemma of [4] the diagram given in Figure 1 is the diagram of an orthomodular lattice. We denote this orthomodular lattice by $L_\$$.

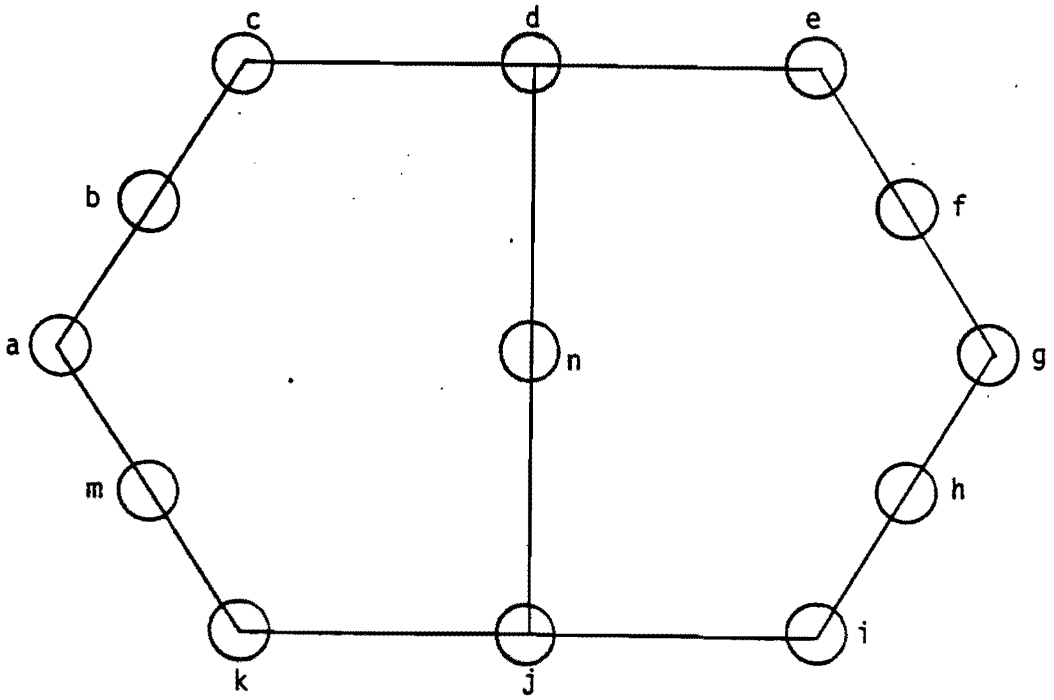

Figure 1 $L_\$$: An OML in S on which equation (I) fails.

With $x = c'$, $y = j$ and $z = i$ we compute in Figure 1 to obtain $x \wedge (y \vee z) = a$, $\phi_x(y) = e$ $\phi_{y'}(z) = i$ and $\phi_x(y) \vee \phi_{y'}(z) = g'$. Clearly $a \not\leq g'$ so that (I) fails in $L_\$$.

Apparently the simplest way to see that $L_\$ \in S$ is to show that for atoms x, y of L $x \not\leq y$ implies there exists a state α with $\alpha(x) = 1$ and $\alpha(y) > 0$ [3]. Here the evident symmetry reduces the verification to listing only a small number of states. (The five given in Table 1 suffice.)

	a	b	c	d	e	f	g	h	i	j	k	m	n
α_a	1	0	0	1/3	2/3	1/6	1/6	1/6	2/3	1/3	0	0	1/3
α_b	0	1	0	1/3	2/3	1/6	1/6	1/2	1/3	1/3	1/3	2/3	1/3
α_c	0	0	1	0	0	2/3	1/3	1/3	1/3	1/3	1/3	2/3	2/3
α_d	1/3	2/3	0	1	0	2/3	1/3	1/6	1/2	0	1/2	1/6	0
α_m	1/3	1/6	1/2	0	1/2	1/6	1/3	1/6	1/2	0	1/2	1/6	1

Table 1

By using the symmetry of $L_\$$ we may extend the five states given in Table 1 to 13 states S which are strong on $L_\$$. The important property of S is that there be a bijection $x \leftrightarrow \alpha_x$ between the atoms of $L_\$$ and S and that $\alpha_x(y) = 1$ iff $x \leq y$. By checking only the original 5 states one may argue that S is strong, indeed that S is a strong set of Jauch-Piron states. (Recall that a state α is called a Jauch-Piron state if $\alpha(x) = \alpha(y) = 1$ implies $\alpha(x \wedge y) = 1$.) This concludes the proof of Proposition 2.

Corollary 1. The equational class generated by $\mathcal{S}$ is strictly contained in $\mathcal{DF}$.

Proof. By Proposition 1 (F_2), i.e. (I), holds in $\mathcal{DF}$. By Proposition 2 it fails in $\mathcal{S}$ and therefore in the equational class generated by $\mathcal{S}$.

Observe in passing that (I) holds in any OML provided that $xCy \vee z$. For

$$\phi_x(y) \vee \phi_{y'}(z) = ((y \vee x') \wedge x) \vee ((z \vee y) \wedge y')$$

$$= ((y \vee x') \vee (z \vee y)) \wedge ((y \vee x') \vee y') \wedge (x \vee (z \vee y)) \wedge (x \vee y')$$

by the Marsden-Herman lemma [8] since $xC(y \vee z)Cy' C (y \vee x')Cx$

$$= (x' \vee (y \vee z)) \wedge (x \vee (y \vee z)) \wedge (x \vee y')$$

$$= (y \vee z) \wedge (x \vee y') \geq x \wedge (y \vee z)$$

since $xC(y \vee z)$.

The following proposition affords alternative generalizations of (E_2) .

Proposition 3. Let $L \in \mathcal{DF}$ and let $a, b, c, d \in L$ with $a \perp b \perp c \perp d$. Then $(a \vee b) \wedge (c \vee d) \leq a \vee d$.

Proof. Assume α is a dispersion free state on L with $\alpha((a \vee b) \wedge (c \vee d)) = 1$.

If $\alpha(a) = 0$ and $\alpha(d) = 0$ then $\alpha(b) = \alpha(a \vee b) = 1$ and $\alpha(c) = \alpha(c \vee d) = 1$ contradicting $b \perp c$. Therefore $\alpha(a) = 1$ or $\alpha(d) = 1$. In any case $\alpha(a \vee d) = 1$. $L \in \mathcal{DF}$ implies $(a \vee b) \wedge (c \vee d) \leq a \vee d$.

Corollary 2. For $L \in \mathcal{DF}$ the following equations hold.

(1) $((y \wedge x') \vee x) \wedge ((z \wedge y) \vee y') \leq x \vee (y \wedge z)$ (This is (E_2).),

(2) $(x \vee (y \wedge x')) \wedge (((y' \vee x) \wedge w) \vee ((w' \vee (y \wedge x')) \wedge u))$
$\leq x \vee ((w' \vee (y \wedge x')) \wedge u)$,

(3) $((x \wedge y') \vee (y \wedge z')) \wedge ((z \wedge w') \vee w) \leq (x \wedge y') \vee w$,

(4) $(x \vee (y \wedge x')) \wedge ((z' \wedge (y' \vee x)) \vee z) \leq x \vee z$.

Proof. (1) Let $a = x$, $b = y \wedge x'$, $c = y'$, $d = y \wedge z$ in the lemma.

(2) $a = x$, $b = y \wedge x'$, $c = (y' \vee x) \wedge w$, $d = (w' \vee (y \wedge x')) \wedge u$.

(3) Apply Proposition 3.

(4) Apply Proposition 3.

SECTION 2

Interest in which orthomodular lattices are sublattices of projection lattices reaches back to the preverbal regions of Kaplanski's, Loomis' and F. Maeda's minds as they first began tasting the abstract algebraic flavor of projection lattices. There have been several papers investigating which OMLs or which quantum logics are embeddable in Hilbert space [5,6,7,11,13]. One of the most recent [7] presents an example L^* to show that even if an OML has a strong set of states it need not be embeddable in $\overline{L}(H)$, the OML of all closed subspaces on the Hilbert space H. It was shown that L^* failed to satisfy an equation in six variables satisfied by all subOMLs of $\overline{L}(H)$.

Here we present an equation in three variables satisfied by all subOMLs of $\overline{L}(H)$; we also present an OML $\hat{L}$ in $\mathcal{DF}$ on which this equation fails. We note in passing that this equation also fails in $L_\$$ of Section 1. (Recall that $L_\$ \in \mathcal{S} \setminus \mathcal{DF}$.) These OMLs have a simpler design and a smaller cardinality than any of the preceding examples of OMLs having at least a strong set of states and known not to be in $\overline{L}(H)$.

The equation with which we are concerned is given by the following inequality:

$$\text{(II)} \qquad x \leq y \vee [\phi_{y'}(x) \wedge [\phi_{z'}(x) \vee ((y \vee z) \wedge (\phi_{y'}(x) \vee \phi_{z'}(x)))]]$$

That (II) holds in $\overline{L}(H)$ follows from the following Remark since $x \leq (x \vee y) \wedge (x \vee z) = (y \vee \phi_{y'}(x)) \wedge (z \vee \phi_{z'}(x))$.

Remark 1. For $M, N, P \in \overline{L}(H)$,

$$(N + \phi_{N^\perp}(M)) \cap (P + \phi_{P^\perp}(M)) \subseteq N + [\phi_{N^\perp}(M) \cap [\phi_{P^\perp}(M)$$

$$+ ((N + P) \cap (\phi_{N^\perp}(M) + \phi_{P^\perp}(M)))]]$$

Proof. The easy proof is omitted.

To see that (II) fails in the lattice $L_\$$ of Figure 1, let $x = a$, $y = d$ and $z = j$; then $\phi_{y'}(x) = e$ and $\phi_{z'}(x) = i$ so $(y \vee z) \wedge (\phi_{y'}(x) \vee \phi_{z'}(x)) = n' \wedge g' = 0$ and $y \vee [\phi_{y'}(x) \wedge (\phi_{z'}(x) \vee 0)] = d \vee [e \wedge i] = d \vee 0 = d$. But $a \not\leq d$ so (II) fails.

Figure (II) is the diagram of an orthomodular lattice $\hat{L}$ (see [4]) which is in $\mathcal{DF}$ and for which Equation (II) fails. It is easy to see

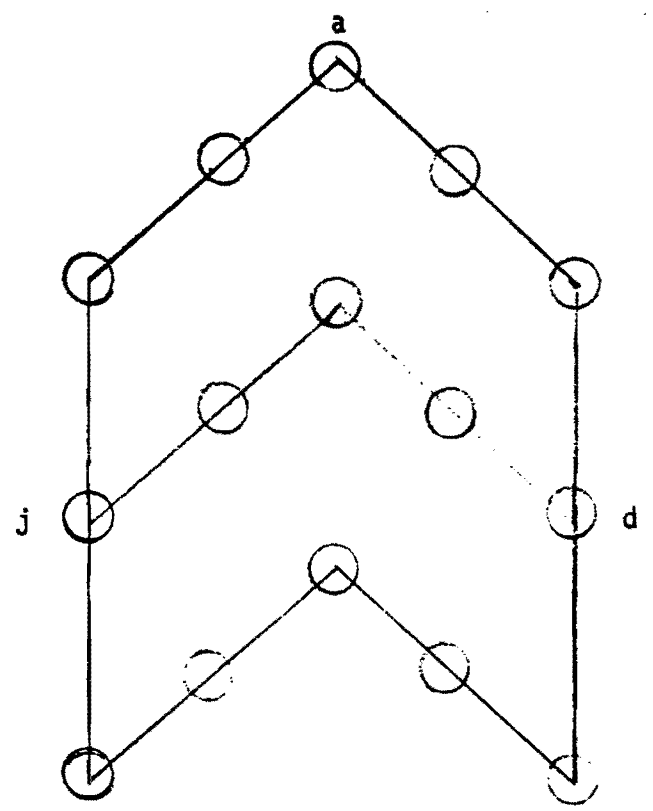

Figure (II): An OML in $\mathcal{DF}$ on which Equation (II) fails.

that $L \in \mathcal{DF}$ and we omit the argument. Equation (II) with $x = a$, $y = d$, $z = j$ readily reduces (as in $L_{\$}$) to $a \leq d$ which is false. Thus Equation (II) fails in $\underset{-}{L}$.

Equation (II) admits many interesting variants. For example it may be shown that, if $\alpha(x,y,z) = (y \vee z) \wedge (\phi_{y'}(x) \vee \phi_{z'}(x))$ then equation (II) is equivalent to

$$\text{(III)} \qquad \phi_{y'}(x) \vee \alpha(x,y,z) = \phi_{z'}(x) \vee \alpha(x,y,z).$$

Professor R. Mayet [14] of the Université Claude-Bernard in Lyon, France has written up several very interesting equivalent forms of equation (II), some closely related to our (III).

We turn now to a short discussion of where equation (II) came from. In 1958 Bjarni Jónsson [12] introduced an equational formulation of Desargues condition. A. Day restricted the condition in ortholattices to the case in which the corresponding vertices of the two triangles involved were pairwise orthogonal, obtaining his ortho-Arguesian property which yielded an equation in six variables valid in $\overline{L}(H)$, whether or not H was finite dimensional, namely:

$a_0 \perp b_0,\ a_1 \perp b_1,\ a_2 \perp b_2 \Rightarrow (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2)$
$\leq (a_0 \wedge (a_1 \vee y)) \vee b_0$ where $y = c_2 \wedge (c_0 \vee c_1)$ and $c_i = (a_j \vee a_k) \wedge (b_j \vee b_k)$, $\{i,j,k\} = \{0,1,2\}$. G. Kalmbach then found an OML in which this equation failed. Eventually that example was simplified to the one given in [7]. Working with the equation directly we observed that it's essential feature could be captured in an equation with 4 variables - essentially by ignoring one pair of corresponding vertices. The new equation was given by

$a_0 \perp b_0$ and $a_1 \perp b_1 \Rightarrow (a_0 \vee b_0) \wedge (a_1 \vee b_1) \leq b_0 \vee [a_0 \wedge [a_1 \vee ((a_0 \vee a_1) \wedge (b_0 \vee b_1))]]$. We then found a simpler lattice violating the simpler equation. This lattice, $\hat{L}$, was 3-generated. Picking generators led to the more restrictive equation

$$\text{(IV)} \quad (y \vee \phi_{y'}(x)) \wedge (z \vee \phi_{z'}(x))$$

$$\leq y \vee [\phi_{y'}(x) \wedge [\phi_{z'}(x) \vee ((y \vee z) \wedge (\phi_{y'}(x) \vee \phi_{z'}(x)))]].$$

The left hand side of this equals $(x \vee y) \wedge (x \vee z)$ which dominates x. The inequality goes in the right direction so we replaced the left hand side of equation (IV) by x arriving at equation (II).

Note that the only orthomodular structures "simpler" than $L_\$$ and $\hat{L}$ known not to be embeddable in $\overline{L}(H)$ are the orthomodular posets given in Figures 3 and 4.

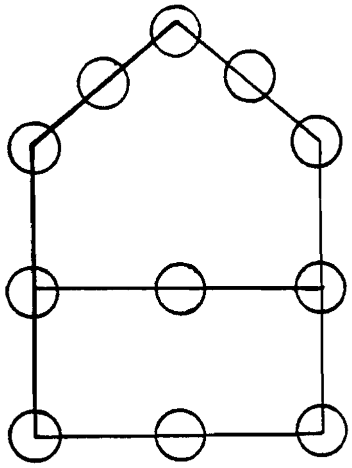

Figure 3

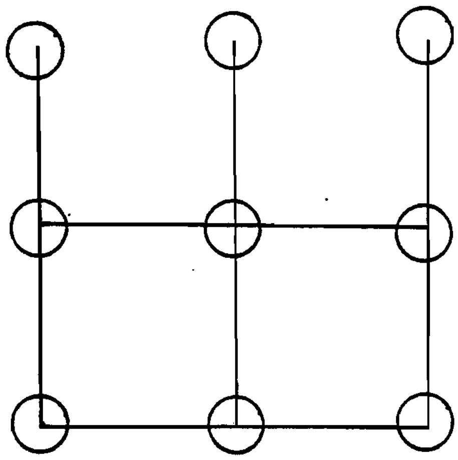

Figure 4

Figure 3 has a full but not a strong set of states. Figure 4 does not have a full set of states.

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