Book Review

The Logic of Quantum Mechanics. By Enrico Beltrametti and Gianni Cassinelli. Addison-Wesley, Reading, Massachusetts, 1981, xxi + 305 pp., $31.50 (cloth).

The monograph under review carries the same title as the celebrated 1936 paper by Garrett Birkhoff and John von Neumann. It amplifies the insights of others who have written in this spirit, most notably Mackey, Jauch, and Varadarajan. The volume by Beltrametti and Cassinelli incorporates much from these earlier works, as well as more than a decade of active research which followed the latest of them.

This book is addressed principally to theoretical physicists and to students of elementary nonrelativistic quantum mechanics who want to deepen their understanding of the subject. It is also written with mathematicians in mind, principally, those interested in applications of lattice theory or of the theory of linear operators on a Hilbert space. A knowledge of quantum mechanics is not assumed. Exercises (sometimes with hints) are contained in almost every chapter.

The organizational scheme is tripartite: a rapid presentation of Hilbert space quantum mechanics articulated with rigor and insight, a development of various basic structures in the description of quantum systems, and a reconstruction of Hilbert space quantum mechanics from certain of these basic structures. The three parts of the book constitute, respectively, an excellent overview of elementary quantum theory, a detailed survey of the logic of quantum mechanics, and a description of to what extent the Hilbert space formulation of quantum mechanics is determined.

Part 1 consists of an exposition of the basic formalism of quantum mechanics using the theory of Hilbert space and of linear operators in these spaces. The pace is made quick by assuming a general acquaintance with the elementary theory of operators on a Hilbert space. Appendices on trace-class operators and on the spectral theorem provide detailed accounts of the most important results, with proofs relegated to the references.

States on a physical system are immediately identified with density

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operators in some associated Hilbert space $\mathscr{H}$. Pure states correspond to projections onto one-dimensional subspaces and, therefore, to unit vectors. The stage is quickly set for studying superpositions. Such a study becomes one of the guiding themes of the volume. The fact that pure states can be superposed to get new pure states is deeply intertwined with the nonunique decomposability of quantum mixtures into pure states. This is shown by an explicit calculation and leads directly to the untenability of the ignorance interpretation of quantum mixtures. Indeed, most of the so-called "paradoxes" of quantum mechanics have their root in this nonunique decomposability of mixtures.

Heisenberg's inequality as introduced herein has only a statistical nature. No idea of simultaneous measurement on one and the same physical system in implied. Attempts to do so elsewhere (e.g., in the pioneering works on quantum mechanics) have, in the authors' opinion, "not attained a clear formulation."

Complementarity is treated in as sensitive a manner as Heisenberg's inequality. A lucid argument shows that two complementary physical quantities are always incompatible but that noncompatability does not entail complementarity. It is noted that, in the pedagogical tradition of quantum mechanics, this distinction is often quite blurred. The authors cite the "disturbance theory," which has been used to justify both notions, and point out its naive aspects as well as its nonnecessity for the coherence of quantum mechanics. Indeed the "disturbance theory" is presented as a typical effort at talking about quantum systems with classical concepts. It assumes that—as happens in classical mechanics—every pure state assigns a value, not a probability distribution of values, to every physical quantity. The authors dismiss this theory delicately: "The 'disturbance theory' refers to this exceptional case as if it were the rule." Here, and throughout the book, the authors abandon the tradition of striving to use concepts of classical mechanics to explain quantum facts.

In the discussion of the completely unpolarized state of a spin-1/2 system, the authors again focus on the nonunique decomposability of quantum mixtures which makes it impossible to deduce, from the knowledge of a nonpure state, the preparation procedure actually used to prepare the state. Consider a spin-1/2 system, and write P_z^{up} and P_z^{down} for the spin-up and spin-down states relative to some axis z. Then the mixture

$$1/2P_z^{\text{up}} + 1/2P_z^{\text{down}} \qquad (*)$$

represents the completely unpolarized state and is independent of the choice of the z axis. It has spherical symmetry. However, an obvious preparation procedure, consisting of a random choice out of an ensemble of identical replicas of the physical system, where half the replicas pass the polarizer that filters out P_z^{up} and half pass the polarizer that filters out P_z^{down}, yields, by inspection of the instruments used, only a cylindrical symmetry around z. Only our trust in quantum mchanics guarantees that the mixture (*) is indeed spherically symmetric. The authors propose tests of such spherical symmetry which evidently have not yet been carried out. The section is tantalizing in its import.

Part 1 continues with penetrating discussions of dynamical evolution, with and without superselection rules, compound systems, the EPR problem and the measurement process—a fast-moving, lucid exposition of the highlights of elementary quantum mechanics. Rounding off the first part is a pivotal chapter (Chapter 9) in which several mathematical substructures associated with Hilbert space and important for quantum mechanics are isolated and later abstracted. Readers new to the logic of quantum mechanics should pay special attention to this chapter.

In it, the projection operators, viewed as representing physical quantities which take on only the values 0 and 1, are treated as propositions. The set of all projection operators $\mathscr{P}(\mathscr{H})$ is viewed as a partially ordered set by defining $P \leqslant Q$ to mean $PQ = P$. States are regarded as probability measures on the propositions, and physical quantities are viewed (via the projection-valued measure form of the spectral theorem) as measures on the real line taking values in $\mathscr{P}(\mathscr{H})$.

The abstraction of this, which constitutes a large part of the rest of the volume, is roughly this: Let L be any partially ordered set having enough structure so as to support a definition of generalized probability measures (i.e., countably orthogonally additive functions into the real unit interval) and of spectral measures (functions from the real Borel sets into L which mimic projection-valued measures). Let $\mathscr{S}$ denote all such probability measures on L and $\mathscr{O}$ all such spectral measures on L. Question: When is it possible to identify L with the set $\mathscr{P}(\mathscr{H})$ of all projection operators on some Hilbert space $\mathscr{H}$, such that $\mathscr{S}$ corresponds with the set of all density operators on $\mathscr{H}$ and $\mathscr{O}$ with the set of all self-adjoint operators on $\mathscr{H}$?

Partly to address this question and partly to provide a survey of the field, Part 2 decomposes quantum theory into its conceptual constituents. It singles out the basic mathematical structures and isolates what may be found on direct empirical evidence, pointing out how individual assumptions contribute to shape the theory. (Note: The definitions of terms not defined here may be found in the book itself by consulting, for example, its subject index.)

The most prominent of these basic structures is the partially ordered set (poset) of projections. In its abstracted form, these are the propositions whose minimal structure is taken to be orthomodular. Results on orthomodular lattices and orthomodular posets are interlaced. (As the reader is warned, he or she must be aware that completeness is implicitly assumed—as in the material concerning the central cover operator.) A variety of examples are given and the basic results on commutativity (i.e., compatability) are developed.

The main distinction between orthomodular lattices and orthomodular posets (other than the possible nonexistence of nonothogonal joins) is mentioned: namely, that in an orthomodular poset which is not a lattice there may exist mutually commuting elements x, y, z with $x \vee y$ existing but with z not commuting with $x \vee y$. Such elements do not lie in a common Boolean subalgebra. It is precisely for posets which contain no such triples that Vardarajan's theorem on compatible observables works. This is the result which generalizes von Newmann's theorem on commuting self-adjoint operators in Hilbert space. It states that a set Y of observables consists of pairwise compatible observables if and only if there exists a single observable x such that each observable y in Y is a Borel function of the observable x. While a simple example of an orthomodular poset which admits such triples can be found in the volume (viz., the set of all subsets of an eight-element set having an even number of elements, partially ordered and orthocomplemented set-theoretically), attention is not called to the fact. Perhaps it should be, since the example is in a sense prototypical of badly behaved orthomodular posets.

The heart of the book is found in the middle of Part 2. After the appropriate mathematical material on orthomodular structures is presented, the authors make different choices about which ingredients (from among states, propositions, physical quantities, and operations) are primitive and which are derived. A major strength of the book lies in the fact that, although the authors may favor one approach over another, the reader never has the feeling that one is right whereas the other is wrong. Instead, something is gleaned from each approach. This generous disposition allows the reader some latitude in deciding, if he must, which ingredient is the more fundamental.

Unlike Piron's approach, the authors favor the probabilistic standpoint in which states are probability measures on the propositions. The set of states are studied from several perspectives: as a closure space, as a convex structure, and as a transition-probability structure. Particular attention is paid to superpositions of such abstract states. This leads to insight into the "superposition principle" of quantum mechanics and its relation to the semimodularity (or the covering property) of the lattice of propositions.

While the authors present the connection between superposition and irreducibility, no mention is made of hyperirreducibility (when not only the lattice but each of its intervals is irreducible). This could be viewed as unfortunate if one tried to conceive of a development which avoided the covering property. For example, Theorem 14.8.9 states that, when the lattice of propositions L has the covering property, any two pure states admit quantum superpositions if and only if L is irreducible. A more focused result is this: When the lattice of propositions L is atomic, any two pure states admit quantum superpositions if and only if L is hyperirreducible.

There are several orderings which the states $\mathscr{S}$ may determine on the propositions L. One could declare that propositions a, b in L satisfy $a \leqslant_1 b$ case $\alpha(a) \leqslant \alpha(b)$ for all α in $\mathscr{S}$. Alternatively, $a \leqslant_2 b$ could be defined by the condition: For all α in $\mathscr{S}$, $\alpha(a) = 1$ implies $\alpha(b) = 1$. The authors present a simple example to show that only a restricted class of propositions can be taken if one would require these two orderings to be equivalent. They show that other natural conditions on the ordering of L (e.g., that it be orthocomplemented) entail restrictions on the set of propositions.

Part 3 deals with the attempt to amalgamate much of Part 2 to achieve the standard quantum mechanics of Part 1. The results of Baer, Birkhoff and von Neumann, and Piron are carefully interlaced to describe which lattices are isomorphic to lattices of closed subspaces of some Hilbert space over some number field K. Restrictions on K as well as the special cases in which K is the reals, the complexes, or the quaternions are discussed. The volume closes with discussions of compound systems, noncontextual hidden-variable theories in Hilbert spaces, and quantum probability theory.

While there are some minor typographical errors, there is only one worth reporting here. An important reference fortunately changed title between the time of the preliminary version which the authors had access to and the time of its publication. Reference 5 on page 190 is to G. Kalmbach's *Orthomodular Lattices*, Academic Press, London, 1983. This reference could be regarded as a companion volume to the volume under review.

My final point also involves a title. The title of the book under review might well have been "The Mathematical Foundations of Quantum Mechanics." The term "logic" in the actual title appears perhaps out of respect for the direction of thought created by Birkhoff and von Neumann. It need not infer an analysis of a propositional calculus, as one might imagine finding in a technical treatise written by a logician. We find "logic" used for "structure" frequently elsewhere, but here there is an ambiguity caused by the dominant presence of elements of the theory called propositions, and it is not clear just how much technical logic they support. The one chapter which approaches a truly logical development, entitled "An Introduction to Quantum Logic," accounts for just over 3 percent of the text and is marginal to the content of the volume. A lasting connection between some formal language which might be quantum logic and the mass of mathematical physics glossed "the logic of quantum mechanics" is yet to be made. The authors provide "a loose hint on how a discussion on quantum logic might start." By mentioning and skirting the most sensitive offshoot of Birkhoff's and von Neumann's proposal, the authors have created a volume that, while daringly conceived, is appropriately conservative.

This volume constitutes an important contribution to the understanding of quantum mechanics. Because of the tripartite organization this contribution can be appreciated by a large and varied audience.

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