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Commutator-Finite Orthomodular Lattices

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Abstract. The class $\mathfrak{C}$ of orthomodular lattices which have only finitely many commutators is investigated. The following theorems are proved: $\mathfrak{C}$ contains the block-finite orthomodular lattices. Every irreducible element of $\mathfrak{C}$ is simple. Every element of $\mathfrak{C}$ is a direct product of a Boolean algebra and finitely many simple orthomodular lattices. The irreducible elements of $\mathfrak{C}$ which are modular, or are M-symmetric with at least one atom, have height two or less.

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This paper initiates the investigation of orthomodular lattices (OMLs) in which the set of all commutators is finite. We call such an OML *commutator-finite* since the term *block-finite* is used for an OML with finitely many blocks. In Section 1 we prove some facts about commutators. The most notable of these is the fact that the join of the set of all commutators in L, when it exists, is central and the interval up to its orthocomplement is a Boolean factor containing all other Boolean factors of L. The main results of the paper appear in Section 2 where the following are proved. Every block-finite OML is commutator-finite. Every irreducible commutator-finite OML is simple. Every commutator-finite OML decomposes into a direct product of a Boolean algebra and finitely many simple OMLs. Finally, if L is an irreducible commutator-finite OML which is either modular or an AC-lattice, then L has height at most 2.

1. Commutators in an Orthomodular Lattice

Throughout this paper L and each L_i are OMLs. If L_1 is a subalgebra of L and $a, b \in L_1$ then $L_1[a, b]$ is $\{x \in L_1 \mid a \leqslant x \leqslant b\}$. In case $L_1 = L$, the local universe of discourse, we write $[a, b]$ for $L[a, b]$. $|X|$ is the cardinality of the set X. We sometimes write $x \oplus y$ for $x \vee y$ when $x \perp y$. For $M, N \subseteq L$, we write $M \vee N = \{m \vee n \mid m \in M, n \in N\}$ ($= M \oplus N$ when $M \subseteq \{x \in L \mid x \perp n$ for all $n \in N\}$). Readers unfamiliar with the terminology or rudiments of orthomodular lattice theory are referred to [6]. Throughout the paper we invoke, without comment, the distributivity results associated with Foulis–Holland sets [5].

By the *commutator* of two elements a and b of L we mean the element

$$a * b := (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b').$$

For $X \subseteq L$, we define com $X = \{a * b\colon a,\ b \in X\}$. Note that $0 \in \text{com } L$ where L is *commutator-finite* if $|\text{com } L|$ is finite.

The commutator of two projections in a von Neumann algebra is the projection onto the closure of the range of the ring theoretic commutator $PQ - QP$, cf. [9], Theorem 1.

It is not difficult to give an example of an OML having precisely n commutators for any cardinal number n. Let G_{12} be the 12-element OML having two blocks, each with 8 elements. Then G_{12} has precisely 2 commutators. Any Boolean algebra B has just one commutator, in fact L is Boolean if and only if com $L = \{0\}$, and the horizontal sum of B with G_{12} has 3. If n is a cardinal number larger than 3 then the horizontal sum of $n - 2$ copies of G_{12} has precisely n commutators.

LEMMA 1. *Let* $a, b \in L$.

(1) $a * b = a * b' = a' * b = a' * b' \in C(\{a, b\}) \cap CC(\{a, b\})$.

(2) (*R. Wille* [12]) *If* $a, b \in C(x)$ *then* $x \wedge (a * b) = (x \wedge a) * b = (x' \vee a) * b$.

(3) $a * b = a * b_2 = a_2 * b = a_2 * b_2 = a_2 \vee b_2$ *where* $a_2 = a \wedge (a * b)$ *and* $b_2 = b \wedge (a * b)$.

(4) *These are equivalent:* (i) $a * b \leqslant a$, (ii) aCb, (iii) $a * b = 0$.

(5) $\alpha \in \text{com } L$ *iff there exists* $b \leqslant \alpha$ *with* $b \sim_{sp} \alpha \wedge b'$, i.e., *with b perspective to* $\alpha \wedge b'$ *in* $[0, \alpha]$.

(6) *No atom is a commutator.*

Proof. The proof of (1) is immediate.

(2) We may assume that $L = C(x) \simeq [0, x] \times [0, x']$. The first equality holds in each factor and therefore in L. The second follows from the first by (1) above and the De Morgan Law.

(3) $a_2 \vee b_2 = (a \wedge (a * b)) \vee (b \wedge (a * b)) = (a * b) \wedge (a \vee b) = a * b$ by (2). Similarly $a_2 * b = (a \wedge (a * b)) * b = (a * b) \wedge (a * b) = a * b$.

(4) This follows either by a straightforward calculation or by observing that it works in any Boolean algebra as well as in MO(2), the six element OML, and therefore in the free orthomodular lattice on two generators [6], p. 27, and, therefore, in any orthomodular lattice.

(5) If $\alpha = b * c = b \vee c$, then an elementary computation shows that c is a common complement for b and $\alpha \wedge b'$ in $[0, \alpha]$. Conversely, if $c\colon b \sim_{sp} (\alpha \wedge c')$ for some $b \leqslant \alpha$, then $b * c = (b \vee c) \wedge (b \vee c') \wedge (b' \vee c) \wedge (b' \vee c') = \alpha$ since $b \vee c = b \vee (\alpha \wedge b') = \alpha$, $0 = c \wedge (\alpha \wedge b') = c \wedge b'$, $\alpha = c \vee (\alpha \wedge b') \leqslant c \vee b'$, and $b \wedge c = 0$.

(6) This follows from (5).

Part (5) of the above Lemma yields the following observation in $\bar{L}(\mathscr{H})$, the lattice of all closed subspaces of a Hilbert space. Since, for orthogonal closed subspaces M and N, $M \sim_{sp} N$ if and only if $\dim M = \dim N$ (cf. [8], Section 1) the only noncommutators in $\bar{L}(\mathscr{H})$ are the finite-dimensional subspaces of an odd dimension. Notice that, by part (3) of the above Lemma, for each commutator α there exist $x, y \in L$ with $\alpha = x * y =$

$x \vee y$. Moreover, for such a pair $x, y, x \wedge y = x \wedge y' = x' \wedge y = 0$. Part (4) of the above lemma yields the following important result.

COROLLARY 2. *Every upper bound of* com L *is central in* L.

REMARK 3. For $X_i \subseteq L_i$, com $(X_1 \times X_2) = (\text{com } X_1) \times (\text{com } X_2)$; equivalently, $e \in C(L)$ implies com $L =$ com $[0, e] \oplus$ com $[0, e']$. If $a, b \leqslant c$, then the commutator of a and b is the same whether computed in L or in the OML $[0, c]$.

The proof of the above Remark is straightforward and left to the reader. In the sequel, we shall use the following well-known interplay between internal and external direct sums: $L \simeq L_1 \times L_2$ iff there exists $e \in L$ with $[0, e] \simeq L_1$, $[0, e'] \simeq L_2$ and $L = [0, e] \oplus [0, e']$.

As in the theory of von Neumann algebras, we shall say that an element $e \in L$ is *Abelian* if $[0, e]$ is a Boolean algebra. This concept is basic to the type decomposition of projection geometries and will be used here similarly. Since a nonzero commutator must dominate nonzero noncommuting elements, no nonzero Abelian element is a commutator. We shall write CA(L) for the set of Abelian elements of L which are also central Clearly an element e is in CA(L) if and only if $[0, e] \subseteq C(L)$.

THEOREM 4. *CA(L) is the set of orthocomplements of the upper bounds for the set* com L, *and* $\bigvee$CA(L) *exists if and only if* $\bigvee$com L *exists.*

If $h = \bigvee$com L *exists, then* CA(L) $= [0, h']$ *and* $[0, h]$ *contains no nonzero elements which are central Abelian elements of* $[0, h]$ *(and, therefore, of* L*).*

Proof. Let U(com L) be the set of upper bounds for com L and put $I = \{e' : e \in U(\text{com } L)\}$. I is an order ideal. Also by part (4) of Lemma 1, U(com $L) \subseteq C(L)$. It follows that $I \subseteq$ CA(L). Conversely suppose that $e \in$ CA(L). Let $x, y \in L$. Then by part (2) of Lemma 1, $e \wedge (x * y) = (e \wedge x) * (e \wedge y) = 0$. Therefore, $x * y \leqslant e'$. So $e \in I$. Thus $I =$ CA(L). It follows by the De Morgan Law that $\bigvee$com $L = \bigwedge U(\text{com } L) = (\bigvee I)' = (\bigvee \text{CA}(L))'$, where the equation is to be interpreted in the sense that if any join or meet exists, then the others do as well and equality obtains.

Let us now assume that $h = \bigvee$com L exists. Then h is the smallest element of U(com L) and so h' is the largest element of $I =$ CA(L). Finally CA$[0, h] = [0, h] \cap \text{CA}(L) \subseteq [0, h] \cap [0, h'] = \{0\}$.

This result shows that in an arbitrary OML L, whenever one can establish that $\bigvee$CA(L) exists, for example by some finiteness condition (as in Section 2) or from the completeness of L itself (as in projection geometries), then CA(L) must be a principal ideal.

PROPOSITION 5. *If* $a \vee c \perp b \vee d$ *then* $(a \vee b) * (c \vee d) = (a * c) \oplus (b * d)$.

Proof. Let $e = a \vee c$. Then e is central in the subalgebra generated by $\{a, b, c, d\}$. Using the Foulis–Holland Theorem and part (2) of Lemma 1, we compute

$$(a \oplus b) * (c \oplus d)$$
$$= \{[(a \oplus b) * (c \oplus d)] \wedge e\} \oplus \{[(a \oplus b) * (c \oplus d)] \wedge e'\}$$

$$= \{[(a \oplus b) \wedge e] * [(c \oplus d) \wedge e]\} \oplus \{[(a \oplus b) \wedge e'] * [(c \oplus d) \wedge e']\}$$
$$= \{[(a \wedge e) \oplus (b \wedge e)] * [(c \wedge e) \oplus (d \wedge e)]\} \oplus$$
$$\oplus \{[(a \wedge e') \oplus (b \wedge e')] * [(c \wedge e') \oplus (d \wedge e')]\}$$
$$= \{a * c\} \oplus \{b * d\}.$$

COROLLARY 6. (1) *If* $\alpha, \beta \in \operatorname{com} L$ *and* $\alpha \perp \beta$, *then* $\alpha \vee \beta \in \operatorname{com} L$.

(2) *If* $\alpha, \beta \in \operatorname{com} L$ *with* $\beta = z * w$ *and* $\alpha C z, w$, *then* $\alpha \vee \beta \in \operatorname{com} L$ *and* $\alpha \wedge \beta \in \operatorname{com} L$.

(3) $C(L) \cap \operatorname{com} L$ *is a sublattice of* L *as well as an ideal of* $C(L)$.

Proof. (1) Let $\alpha = a * c = a \vee c$ and $\beta = b * d = b \vee d$. Then $a \vee c \perp b \vee d$ and Proposition 5 applies.

(2) $\alpha \wedge \beta = \alpha \wedge (z * w) = (\alpha \wedge z) * w$ by part (2) of Lemma 1. Since $\alpha' \wedge \beta = \alpha' \wedge (z * w) = (\alpha' \wedge z) * w$, $\alpha \vee \beta = \alpha \oplus (\alpha' \wedge \beta) \in \operatorname{com} L$ by (1).

(3) Apply (2).

It follows immediately from (1) of the above Corollary that α is a maximal commutator of L only if $\operatorname{com}[0, \alpha'] = \{0\}$. We conclude this section with the following characterization of the minimal nonzero commutators of L.

PROPOSITION 7. *Let* α *be a nonzero commutator of* L. *The following statements are equivalent:*

(1) α *is a minimal nonzero commutator of* L.

(2) $x < \alpha$ *implies* $[0, x]$ *is a Boolean algebra.*

(3) $[0, \alpha]$ *is a horizontal sum of Boolean algebras.*

Proof. Assume (1) and let $x < \alpha$. Then $\{0\} = \operatorname{com} L \cap [0, x] = \operatorname{com}[0, x]$ by part (3) of Lemma 1 so that $[0, x]$ is a Boolean algebra. Now assume (2). Fix $x < \alpha$ with $0 < x$. We need only show that $C(x)$ is a block of L. Now there exists a block B of L containing x. Since $B \subseteq C(x)$ we need only show $C(x) \subseteq B$. Since $C(x) = [0, x] \oplus [0, x']$ and B is a subalgebra of L we need only argue that $[0, x] \cup [0, x'] \subseteq B$. We shall prove that $[0, x] \subseteq B$. A similar argument shows that $[0, x'] \subseteq B$. To this end let $z \leq x$ and let $w \in B$. Since $[0, x]$ is a Boolean algebra by (2), $zCw \wedge x$; since $z \perp w \wedge x'$, $zCw \wedge x'$; hence zCw since w, $x \in B$ implies $w = (w \wedge x) \vee (w \wedge x')$. Hence, $[0, x] \subseteq C(B) = B$ so that $[0, \alpha]$ is the horizontal sum of Boolean algebras. (3) implies (1) is trivial and the proof is complete.

COROLLARY 8. *If* $\operatorname{com} L$, *regarded as a poset with the ordering inherited from* L, *is atomic then the set of abelian elements is join dense in* L.

Proof. Let $x \in L$. Either $[0, x]$ is Boolean or there exists a minimal nonzero commutator α in $[0, x]$. We may assume the latter. By Lemma 1 α is not an atom of L so there exists an element a with $0 < a < \alpha$. By Proposition 7, a is abelian. Thus every nonzero element of L dominates a nonzero abelian element. A standard argument [7], Lemma 7.2, shows that the set of abelian elements is join dense.

2. Commutator-Finite Orthomodular Lattices

We now consider the principal subject of our paper. Note that in a commutator-finite orthomodular lattice $\bigvee \mathrm{com}\, L$ exists so, by Theorem 4, the central abelian elements of L form a principal ideal. Also, by Corollary 8, the set of abelian elements in a commutator-finite OML is join dense. In the theory of projection geometries of von Neumann algebras, lattices with this latter property would be called type I or discrete. Note that the class of commutator-finite OMLs is closed under horizontal sums, finite direct products, subalgebras and homomorphic images. The following proposition shows that this class contains all block-finite OMLs.

Let $\mathfrak{A}_L$ denote the set of all blocks, i.e., maximal Boolean subalgebras, of L and let $a^0 = a$ and $a^1 = a'$.

PROPOSITION 9. *Every block-finite orthomodular lattice is commutator-finite.*

Proof. We proceed by induction on the number of blocks. OMLs with one block are Boolean and therefore have only one commutator. Assume that the statement holds for all OMLs having fewer than k blocks. Let L be an OML having k blocks. By [2], Theorem 1, we may assume, without loss of generality, that L has no Boolean factor and, since the blocks of a product are the product of the blocks in the factors, that L is irreducible. Since $\mathfrak{A}_{C(x)} = \{B \in \mathfrak{A}_L \mid x \in B\}$, $0 < x < 1$ forces $\mathfrak{A}_{C(x)} \not\subseteq \mathfrak{A}_L$ so, by the induction hypothesis, $C(x)$ is commutator-finite. Since $a, b \in C(a * b)$, $(\mathrm{com}\, L)\backslash\{1\} \subseteq \cup\{\mathrm{com}\; C(x) : 0 < x < 1\}$. Moreover since, by [4, Theorem 1], there are only finitely many $C(x)$'s, it follows that L is commutator-finite.

The class of commutator-finite OMLs strictly contains the class of block-finite OMLs. In fact, the horizontal sum of infinitely many copies of the four element Boolean algebra is the simplest example of a commutator-finite OML which is not block-finite.

We now turn our attention to proving the following theorem: a commutator-finite OML is irreducible iff it is simple. Michael Roddy recently proved this for block-finite OMLs using a substantially different technique. He presented it as an orthomodular analogue of the Birkhoff–Menger Theorem [10]. Hereafter we use the notation $a\phi_x$ for $(a \vee x') \wedge x$; $a\phi_x$ is the *Sasaki projection* of a onto x.

LEMMA 10. (1) *If I is an ideal of L, then* $\mathrm{com}\, I = I \cap \mathrm{com}\, L$. (2) *If I is a p-ideal of L, then $I \cap C(I) = I \cap C(L)$.*

Proof. (1) If $\alpha \in \mathrm{com}\, I$, then $\alpha = a * b$ for some $a, b \in I$ and $\alpha \leqslant a \vee b \in I$ so that $\alpha \in I$. Conversely, if $\alpha \in I \cap \mathrm{com}\, L$, then $\alpha = a * b = a \vee b$ for some $a, b \in L$ by part (3) of Lemma 1. However, $\alpha \in I$ implies $a, b \in I$ and, hence, $\alpha \in \mathrm{com}\, I$.

(2) Let I be a p-ideal of L. We need prove only that $I \cap C(I) \subseteq I \cap C(L)$. Let $a \in I \cap C(I)$ and $x \in L$. Then $a\phi_x \in I \subseteq C(a)$ so $x = (x \wedge a') \vee a\phi_x \in C(a)$. Thus $a \in C(L)$.

LEMMA 11. *Let I be a nonzero ideal of the OML L with* $|\mathrm{com}\, I|$ *finite. Then* (1) $\bigvee \mathrm{com}\, I \in I \cap C(I)$ *and* (2) $I \cap C(I) \neq \{0\}$.

Proof. (1) Let $i \in I$ and let $h = \bigvee \mathrm{com}\, I$. By Lemma 10, part (1), we know that $\mathrm{com}\, I \subseteq I$. Then, since $\mathrm{com}\, I$ is finite, it follows that $h \in I$. Hence $h * i \leqslant h \vee i \in I$ so that

$h * i \in I$. Therefore, $h * i \leq h$ so that hCi by part (4) of Lemma 1. Hence, $h \in I \cap C(I)$.

(2) If $I \subseteq C(I)$ this is trivial. If $I \nsubseteq C(I)$ then $0 < \vee\text{com } I \in I \cap C(I)$.

THEOREM 12. *If L is an irreducible orthomodular lattice such that no proper p-ideal of L contains infinitely many commutators, then L is simple.*

Proof. Let I be a nonzero p-ideal of L. Then, by Lemmata 10 and 11,

$$\{0\} \neq I \cap C(I) = I \cap C(L) = I \cap \{0, 1\}$$

so that $1 \in I$ and $I = L$. Hence, L is simple.

COROLLARY 13. *If L is a commutator-finite orthomodular lattice, then L is irreducible if and only if L is simple.*

The Theorem is, in fact, stronger than the Corollary. For example, if L is the horizontal sum of infinitely many copies of any commutator-finite OML having at least three commutators, then L satisfies the hypothesis of the Theorem but is not commutator-finite.

As we observed just before Proposition 5, a largest Boolean summand may be found in a projection geometry. Günter Bruns [2], Theorem 1, has proved that this phenomenon occurs in block-finite OMLs. The following Theorem, which is a reflection of Theorem 4, is the corresponding result for commutator-finite OMLs.

THEOREM 14. *If L is a nonBoolean commutator-finite OML, then L has a unique orthogonal decomposition* $L = [0, e_0] \oplus [0, e_1] \oplus \ldots \oplus [0, e_n]$ *where* e_0 *is the largest central abelian element of L, each* $e_i \in C(L)$ *and each* $[0, e_k]$, $1 \leq k \leq n$, *is simple and nonBoolean.*

Proof. By Theorem 4 CA(L) has a largest element e_0 and $e_0' = \vee\text{com } L$. Since e_0 is central $L = [0, e_0] \oplus [0, e_0']$. By Remark 3, com L = com $[0, e_0']$, and if $[0, e_0'] = \oplus \Sigma_{i \in I}$ $[0, e_i]$ with each $e_i > 0$, then each summand has at least two commutators so that $|I| <$ $|\text{com } L|$. We may assume that I has maximal cardinality among all such decompositions of $[0, e_0']$. Then each interval $[0, e_i]$ is irreducible and, hence, by Corollary 13 is simple. Moreover each $e_i (i \geq 1)$ is an atom of $C(L)$. Since any such decomposition of $[0, e_0']$ is determined by the atoms of $C(L) \cap [0, e_0']$, the decomposition is unique.

We now turn our attention to irreducible modular commutator-finite OMLs. We show that none of them has height greater than 2. We shall need to cite the following two important results.

LEMMA 15. (1) (*R. Baer*) *There is no finite orthocomplemented projective plane.*

(2) (*G. Bruns*) *Every nonzero abelian element of a subdirectly irreducible modular ortholattice is an atom.*

Proof. For (1) see [1], Theorem 5. For (2) see [3], Theorem 1.

Recall that MO(n) is the horizontal sum of n copies of the four element Boolean algebra.

THEOREM 16. *If L is a commutator-finite modular irreducible OML then* $L \simeq 2^0, 2^1$ *or* MO(n) *for some cardinal number* $n \geq 2$.

Proof. We may assume that $|L| > 2$. By Corollary 13, L is simple and therefore subdirectly irreducible. By Lemma 15, part (2), every nonzero abelian element is an atom and so, by Corollary 8, L is atomic. If $ht(L) \geq 3$, then there exists an x in L with $ht(x) = 3$. The interval $[0, x]$ is an orthocomplemented projective plane and so, by Lemma 15, part (1), it is infinite. Every coatom of $[0, x]$ is a commutator of $[0, x]$ and also of L, contradicting the fact that L is commutator-finite. Hence, $ht(L) \leq 2$.

Recall that an AC-*lattice* is an atomic lattice satisfying the covering property: If a is an atom and $x \wedge a = 0$, then $x \vee a$ covers x.

COROLLARY 17. *Let L be an irreducible commutator-finite OML. These conditions are equivalent:*

(1) *L is M-symmetric with at least one atom.*

(2) *L is an AC-lattice.*

(3) $L \simeq 2^1$ *or* $MO(n)$ *for some cardinal number n.*

Proof. Let J be the set of all joins of finitely many atoms together with 0. Clearly J is closed under finite joins.

Suppose (1) holds. Since L is M-symmetric, it has the covering property [7], Theorem 7.6. We claim that J is a nontrivial p-ideal. To show this it suffices to show that J is closed under Sasaki projections. Let $x \in L$. We shall show that $J\phi_x \subseteq J$; Let a be any atom of L; it suffices to show that $a\phi_x \in J$. Since $x' \, C \, a\phi_x$, we have $[0, a\phi_x] = [x' \wedge a\phi_x, a\phi_x] \simeq [x', x' \vee a\phi_x] = [x', a \vee x']$. However, since $(y, a)M$ for each $y \in L$, we have that $(a, y)M$ for each $y \in L$ by M-symmetry. Thus $(a, y)M^*$ for each $y \in L$ [7], Lemma 1.2. In particular, $(a, x')M^*$. It follows from $(x', a)M$ and $(a, x')M^*$ that $[x', a \vee x'] \simeq [a \wedge x', a]$. Therefore $a\phi_x$ is either an atom or is 0. In either case, $a\phi_x \in J$. Thus J is a nontrivial p-ideal. Then by Corollary 13, $J = L$ so that each nonzero element of L is a join of finitely many atoms. Thus, L is an AC-lattice.

Now assume (2) holds. By [7], Theorem 7.10, L has the atomic exchange property. Thus by [11], Lemma 12, Theorem 13, L contains a nontrivial p-ideal which is a modular sublattice of L. However, by Corollary 13 again, L contains no proper nontrivial p-ideals. Thus, L is modular. We have therefore, from the previous Theorem, that $L \simeq 2^1$ or MO(n) for some cardinal number n. The inference that (3) implies (1) is obvious.

Let $\mathfrak{C}$ denote the class of all irreducible, atomless, M-symmetric and commutator finite OMLs. It is not difficult to see that $\mathfrak{C}$ contains any horizontal sum of a family of atomless Boolean algebras. The horizontal sum of finite products of elements of $\mathfrak{C}$ are again in $\mathfrak{C}$ and the horizontal sum of an element of $\mathfrak{C}$ with an atomless Boolean algebra is again in $\mathfrak{C}$. No other element of $\mathfrak{C}$ are evident.

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