ON RESIDUATED APPROXIMATIONS

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1. INTRODUCTION AND BACKGROUND

Let $P \xrightarrow{f} Q \xrightarrow{g} P$ be order-preserving maps on posets. (f, g) is called a **residuated-residual pair for** (P, Q) in case the following holds:

$$\forall\, p \in P,\ \forall\, q \in Q, \quad f(p) \leq q \Leftrightarrow p \leq g(q).$$

If (f, g) is such a pair, then $f(p) = \bigwedge\{\, q \in Q \mid p \leq g(q)\,\}$ and $g(q) = \bigvee\{\, p \in P \mid f(p) \leq q\,\}$. Thus f uniquely determines g and vice versa. f is called **residuated** if there exists such a g and g is called **residual** (and is the **residual of** f) if there exists such an f. (One sometimes denotes g by f^+ and f by g^-.)

Notice that (f, g) is a residuated-residual pair for (P, Q) iff (g, f) is a residuated-residual pair for (Q^*, P^*), where $(\ )^*$ denotes the poset with the inverse order. Because of this duality, we focus on residuated maps (as is traditional) noting that the corresponding dual statements about residual maps can be straightforwardly inferred.

Also, if (f, g) is a residuated-residual pair for (P, Q), then (f, g, P, Q^*) is what historically has been called a **Galois connection**. Thus, for most purposes, the studies of Galois connections and of residuated-residual pairs are equivalent. The "original" Galois connection discovered by Galois is the familiar one between the family of extensions of a subfield K of a field L and the family of all subgroups of the group of automorphisms of L fixing K (where both families are ordered by inclusion).

There has been much work in the area of Galois connections and residuated-residual pairs over the last several decades (cf. e.g., [GH]). More recently, residuated maps are playing an important role in theoretical computer science, where they are used in various areas such as investigations concerning compiler correctness [W], the theory of data type coercions [MS], and in programming semantics and domain theory [S]. They are also used is other settings, such as the emerging theories of cluster analysis [J1], [J4], numerical taxonomy [J3], and automatic classification [J2].

Not all order-preserving maps are residuated. In fact, it is not difficult to show that

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$f : P \to Q$ is residuated if and only if the inverse image of every principal ideal of Q is principal, i.e.,

$$\forall\, q \in Q \ \ \exists\, p \in P \ \text{ such that } f^{-1}(\downarrow q) = \downarrow p.$$

In fact, when f is residuated, the above p is uniquely determined by $p = f^{+}(q)$. Also, it is well-known [P] that a map $f : P \to Q$ between complete lattices is residuated if and only if it preserves arbitrary joins; i.e., $f(\bigvee K) = \bigvee\{\, f(k) \mid k \in K\,\}$.

Motivated by the theory of probabalistic metric spaces (cf. [SS]) Janowitz [J4], in his investigation of cluster analysis, has made the following (unpublished) observation which induced us to consider approximations of order-preserving maps by residuated ones:

1.1 PROPOSITION

If $f : P \to Q$ is an order-preserving map from a complete chain to a complete lattice, then the function $\ell^{-} f$ given by:

$$\ell^{-} f(x) = \begin{cases} \bigvee\{\, f(t) \mid t < x\,\}, & \text{if } x = \bigvee\{\, t \mid t < x\,\} \\ f(x) & \text{if } x \neq \bigvee\{\, t \mid t < x\,\} \end{cases}$$

is residuated.

Clearly, $\ell^{-} f$ is **dominated** by f (i.e., $\ell^{-} f(x) \leq f(x)$ for each $x \in P$). It is not hard to show that it is the largest such residuated map. Below, we show that every map between complete lattices has such a residuated approximation. Also, for arbitrary order-preserving maps f between complete lattices, we construct a map σ_f (called the *shadow* of f), which we believe can be calculated more efficiently than the residuated approximation; and which therefore, may be more appropriate for computer science applications. Our main result is that the shadow and the residuated approximation coincide precisely for those cases in which Q is a completely distributive complete lattice.

2. MAIN RESULTS

The first two results below (2.1 and 2.2) may be considered as folklore. They are included only for completeness.

2.1 LEMMA

Let P and Q be complete lattices and let $\mathcal{F}$ be any family of residuated maps from P to Q. The pointwise join $j_{\mathcal{F}} : P \to Q$ defined by

$$j_{\mathcal{F}}(p) = \bigvee_{f \in \mathcal{F}} f(p),$$

is a residuated map from P to Q.

Proof: If $q \in Q$ then

$$j_{\mathcal{F}}^{-1}(\downarrow q) = \{ p \in P \mid j_{\mathcal{F}}(p) \leq q \} = \{ p \in P \mid \bigvee_{f \in \mathcal{F}} f(p) \leq q \}$$

$$= \{ p \in P \mid f(p) \leq q \text{ for all } f \in \mathcal{F} \} = \bigcap_{f \in \mathcal{F}} \{ p \in P \mid f(p) \leq q \}$$

$$= \bigcap_{f \in \mathcal{F}} f^{-1}(\downarrow q) = \bigcap_{f \in \mathcal{F}} (\downarrow f^{+}(q)) = \downarrow (\bigwedge_{f \in \mathcal{F}} f^{+}(q)).$$

Thus $j_{\mathcal{F}}$ is residuated and $j_{\mathcal{F}}^{+}(q) = \bigwedge_{f \in \mathcal{F}} f^{+}(q)$. □

2.2 COROLLARY

If $f : P \to Q$ is a map between complete lattices then, among all of the residuated maps dominated by f, there is a largest: ρ_f.

Proof: Let $\mathcal{R}_f$ be the set of all residuated maps from P to Q that are dominated by f, and let $\rho_f = j_{\mathcal{R}_f}$. By the above lemma ρ_f is residuated and is clearly the largest such map that is dominated by f. □

2.3 DEFINITION

Given such an f, we call ρ_f **the residuated approximation of** f.

2.4 PROPOSITION

If P is a complete chain, Q is a complete lattice, and $f : P \to Q$ preserves order, then the map $\ell^{-} f$ of Janowitz is the residuated approximation of f.

Proof: By Proposition 1.1 $\ell^{-} f$ is residuated and, since f is order-preserving, $\ell^{-} f \leq f$. Let $h : P \to Q$ be any residuated map with $h \leq f$. If $p \neq \bigvee\{ t \in P \mid t < p \}$ then $h(p) \leq f(p) = \ell^{-} f(p)$; and if $p = \bigvee\{ t \in P \mid t < p \}$ then, since residuated maps preserve arbitrary joins,

$$h(p) = h(\bigvee\{ t \in P \mid t < p \}) = \bigvee\{ h(t) \mid t < p \} \leq \bigvee\{ f(t) \mid t < p \} = \ell^{-} f(p).$$

□

In practice, utilizing the proof of Corollary 2.2 to compute the residuated approximation of an order-preserving map $P \xrightarrow{f} Q$ between complete lattices is hampered by the fact that one must first calculate all of the residuated maps that are dominated by f. When P is a complete chain, the above proposition shows that $\ell^{-} f$ (which can be calculated in polynomial time) yields the correct result. When P is not a chain, $\ell^{-} f$ can still be computed, but unfortunately, it need not be residuated. The following construction, yields the residuated approximation in every instance in which Q is completely distributive (Proposition 2.7). Although when Q is not completely distributive it need not yield a residuated map (Proposition 2.9), it shares the virtue of the Janowitz approximation in that it has a polynomial-time algorithm for its computation.

2.5 DEFINITION

Given an order-preserving map $f : P \to Q$ between complete lattices, we define a map $\sigma_f : P \to Q$ (called the **shadow of** f) as follows: for each $x \in P$

$$\sigma_f(x) = \bigwedge\{\, q \in Q \mid x \leq \bigvee f^{-1}(\downarrow q)\,\}.$$

For convenience of notation, let $A^f_x = \{\, q \in Q \mid x \leq \bigvee f^{-1}(\downarrow q)\,\}$.

2.6 PROPOSITION

For each order-preserving map $f : P \to Q$ between complete lattices, σ_f is order-preserving and $f \geq \sigma_f \geq \rho_f$. If σ_f is residuated, then $\sigma_f = \rho_f$. Furthermore, if f is residuated, then $f = \sigma_f = \rho_f$.

Proof: Let $x, y \in P$ with $x \leq y$. If $y \leq \bigvee f^{-1}(\downarrow q)$ then $x \leq \bigvee f^{-1}(\downarrow q)$; thus $A^f_y \subseteq A^f_x$ and $\sigma_f(x) \leq \sigma_f(y)$. Therefore σ_f is order-preserving.

Since $f(x) \in A^f_x$, $f(x) \geq \sigma_f(x)$. To see that $\sigma_f(x) \geq \rho_f(x)$ we need only show that $q \geq \rho_f(x)$ whenever $q \in A^f_x$. To this end let $q \in A^f_x$, so that $x \leq \bigvee f^{-1}(\downarrow q)$. Then, since ρ_f is residuated, $\rho_f(x) \leq \rho_f(\bigvee f^{-1}(\downarrow q)) = \bigvee_{w \in f^{-1}(\downarrow q)} \rho_f(w) = \bigvee_{f(w) \leq q} \rho_f(w) \leq \bigvee_{f(w) \leq q} f(w) \leq q$. Thus $f \geq \sigma_f \geq \rho_f$.

If σ_f is residuated then, since $\sigma_f \leq f$, we have $\sigma_f \leq \rho_f$, so that $\sigma_f = \rho_f$. Finally, if f is residuated, then clearly $f = \rho_f$, so that $f = \sigma_f = \rho_f$. □

2.7 PROPOSITION

If P and Q are complete lattices with Q completely distributive, and if $f : P \to Q$ is an order-preserving map, then the shadow σ_f of f is residuated and therefore is equal to the residuated approximation ρ_f of f.

Proof: Let $K \subseteq P$. For simplicity, let $A_x = A^f_x$ and $g[K] = \{\, g(k) \mid k \in K\,\}$. Moreover, we use ΠA_k to mean $\Pi_{k \in K} A_k$. Then $\sigma_f(\bigvee K) = \bigwedge A_{\bigvee K} = \bigwedge(\bigcap_{k \in K} A_k)$.

Moreover, by distributivity, we have $\bigvee_{k \in K} \sigma_f(k) = \bigvee_{k \in K} \bigwedge A_k = \bigwedge_{g \in \Pi A_k} \bigvee g[K]$.

Since, for each $g \in \Pi A_k$, $g(k) \in A_k$ for all $k \in K$, and since the fact that f preserves order ensures that A_k is closed under super-elements, $\bigvee g[K] \in A_k$. Hence $\bigvee g[K] \in \bigcap_{k \in K} A_k$. It follows that $\sigma_f(\bigvee K) = \bigwedge(\bigcap_{k \in K} A_k) \leq \bigvee g[K]$ for every $g \in \Pi A_k$ so that $\sigma_f(\bigvee K) \leq \bigwedge_{g \in \Pi A_k} \bigvee g[K]$, i.e., $\sigma_f(\bigvee K) \leq \bigvee_{k \in K} \sigma_f(k)$.

The fact that $\sigma_f(\bigvee K) \geq \bigvee_{k \in K} \sigma_f(k)$ follows immediately from the fact that σ_f preserves order. Thus $\sigma_f(\bigvee K) = \bigvee_{k \in K} \sigma_f(k)$ and, therefore, σ_f is residuated. By Proposition 2.6, $\sigma_f = \rho_f$. □

2.8 COROLLARY

If $f : P \to Q$ is an order-preserving map from a complete chain to a completely distributive complete lattice, then $\ell^- f = \sigma_f = \rho_f$, i.e., the approximation of Janowitz, the shadow, and the residuated approximation are all the same. □

Proof: By Proposition 2.7 $\sigma_f = \rho_f$. By Proposition 2.4 $\ell^- f = \rho_f$, so that all three approximations coincide. □

2.9 PROPOSITION

If Q is a complete lattice that is not completely distributive, then there exists a complete lattice P and an order-preserving map $f : P \to Q$ for which the shadow, σ_f, is not residuated.

Proof: By hypothesis there exists some family $(Q_i)_{i\in I}$ of nonempty subsets of Q such that

$$a = \bigvee_{i\in I}(\bigwedge Q_i) \neq \bigwedge_{g\in\Pi Q_i}(\bigvee g[I]) = b.$$

Construct a complete lattice P as follows: P is the complete Boolean algebra 2^I of all subsets of I with the following extra elements adjoined:
for each $i \in I$ and $q \in Q_i$ let $(1, q, i)$ and $(2, q, i)$ be distinct atoms of P with $(1, q, i) \vee (2, q, i) = \{i\}$ and $(1, q, i) \wedge (2, q, i) = \emptyset$. [Thus if Q_i has n elements, this will generate a set M_i of $2n$ atoms of P with $\bigvee M_i = \{i\}$]. We require that if $i \neq j$ then there is no order relationship between (n, q, i) and $\{j\}$, $n = 1, 2$.
Claim 1. P is a complete lattice.

Since 2^I is complete and we have only introduced new disjoint families M_i of atoms each of which has a join that was an atom of 2^I, an easy case analysis shows that the meet of any subset of P exists.

Let K denote the set of all atoms of P together with $\emptyset$. Define $P \xrightarrow{f} Q$ by:

$$\begin{aligned} f(J) &= 1 \quad \text{if} \quad \emptyset \neq J \in 2^I \\ f(\emptyset) &= 0 \\ f((n, q, i)) &= q \quad \text{for} \quad i \in I,\ q \in Q_i \quad \text{and} \quad n = 1, 2. \end{aligned}$$

Claim 2. f is order-preserving.

This is clear since f takes on the value 1 everywhere except on K and no two non-zero members of K are ordered.
Claim 3. $\sigma_f(\{i\}) = \bigwedge Q_i$.

Let $q \in Q_i$. Then $(n, q, i) \in f^{-1}(q)$, $n = 1, 2$, so $\{i\} = (1, q, i) \vee (2, q, i) \leq \bigvee f^{-1}(\downarrow q)$. Thus $q \in A^f_{\{i\}}$. Hence $Q_i \subseteq A^f_{\{i\}}$, so that $\bigwedge Q_i \geq \bigwedge A^f_{\{i\}} = \sigma_f(\{i\})$.

To show the reverse inclusion, let $\hat{q} \in A^f_{\{i\}}$. If there exists some $j \in I$ with $\{j\} \in f^{-1}(\downarrow \hat{q})$, then $f(\{j\}) = 1 \leq \hat{q}$, so $\hat{q} = 1$. If no such j exists, then $f^{-1}(\downarrow \hat{q}) \subseteq K$. But $\hat{q} \in A^f_{\{i\}}$ implies that $\{i\} \leq \bigvee f^{-1}(\downarrow \hat{q})$. Thus $f^{-1}(\downarrow \hat{q})$ contains some atom below $\{i\}$, say (n, q, i). Therefore $f(n, q, i) = q \leq \hat{q}$, where $q \in Q_i$. So in either case there is some $q \in Q_i$ with $q \leq \hat{q}$. Thus $\bigwedge Q_i \leq \bigwedge A^f_{\{i\}} = \sigma_f(\{i\})$, and Claim 3 is established.

Claim 4. $\bigvee_{i\in I}\sigma_f(\{i\}) = a$.

This is immediate from Claim 3 and the definition of a as $\bigvee_{i\in I}(\bigwedge Q_i)$.

Claim 5. For each $g \in \Pi Q_i$, $\bigvee g[I] \in A_1^f$.

For each $i \in I$, $g(i) \in Q_i$, Thus $g(i) = f((n, g(i), i) \leq \bigvee g[I]$ so that $(n, g(i), i) \in f^{-1}(\bigvee g[I])$. Since for each $i \in I$, $(n, g(i), i) \leq \{i\}$, and $(n, g(i), i)$ is an atom, $1 = \bigvee_{i\in I}\{i\} \leq \bigvee f^{-1}(\bigvee g[I])$. Thus $\bigvee g[I] \in A_1^f$.

Claim 6. $\sigma_f(1) \leq b$.

This is immediate from Claim 5, the definition of σ_f, and the definition of b.

Claim 7. $b \leq \sigma_f(1)$.

Let $q \in A_1^f$ and let $C = f^{-1}(\downarrow q)$. Then $1 \leq \bigvee C$.

Case I. $C \not\subseteq K$:

Then there is some $J \subseteq I$, $J \neq \emptyset$ such that $J \in C$. But $f(J) = 1$ implies that $1 \leq q$. Thus $q = 1$.

Case II. $C \subseteq K$:

Since $\bigvee C = 1$, it must be the case that for each $i \in I$, there is some $c_i \in C$ with $c_i < \{i\}$. Thus $c\colon I \to P$. Let $g = f \circ c$; i.e., $g(i) = f(c_i)$. Now $c_i < \{i\} \Longrightarrow f(c_i) \in Q_i$. Hence $g \in \Pi Q_i$, and $\bigvee g[I] = \bigvee\{f(c_i)\} \leq q$ (since $C = f^{-1}(\downarrow q)$). Therefore in either Case I or Case II, for each $q \in A_1^f$ there is some $g \in \Pi Q_i$ with $\bigvee g[I] \leq q$. Thus $b = \bigwedge_{g\in\Pi Q_i}(\bigvee g[I]) \leq \bigwedge A_1^f = \sigma_f(1)$.

Claim 8. σ_f is not residuated.

By Claims 6 and 7, $\sigma_f(1) = b$. Thus $b = \sigma_f(\bigvee_{i\in I}(\{i\})$. But by Claim 4, $\bigvee_{i\in I}\sigma_f(\{i\}) = a$ and by hypothesis $a \neq b$; thus σ_f does not preserve joins. $\square$

Propositions 2.7 and 2.9 combine to characterize the completely distributive lattices among the complete ones. It is also interesting to note that the completely distributive complete lattices are also characterized among the complete ones as being precisely the injective objects in the category with objects all complete lattices and morphisms all residuated (resp. all residual) maps. (See [C].) By duality, they are also precisely all the projective objects in the same categories.

2.10 THEOREM

Let Q be a complete lattice. These are equivalent:

(1) *Q is completely distributive .*

(2) *for every complete lattice P and for every order-preserving map $f : P \to Q$, the shadow of f is residuated.*

(3) *for every complete lattice P and for every order-preserving map $f : P \to Q$, the shadow of f is the residuated approximation of f.*

Proof: (1) implies (2) and (3) by Proposition 2.7. (2) implies (1) by Proposition 2.9. (3) implies (2) since the residuated approximation of f is always residuated. $\square$

The above theorem shows that a complete lattice Q is completely distributive if and only if the shadow of each order-preserving function from each complete lattice P to Q is residuated. Who knows which complete lattices are completely distributive? *The Shadow Knows!*

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