Algebra Universalis, 27 (1990) 1–9
0002-5240/90/010001-09$01.50 + 0.20/0

Blocks and commutators in orthomodular lattices

GÜNTER BRUNS AND RICHARD GREECHIE

In Memory of Evelyn Nelson

An orthomodular lattice (abbreviated: OML) is block-finite, [2], if the set $\mathfrak{A}_L$ of all its blocks is finite, it is commutator-finite, [5], if the set com L of all its commutators is finite. The class of all commutator-finite OMLs properly contains the class of all block-finite OMLs, see [5]; and, every finitely generated block-finite OML is finite, see [1]. The question thus arises whether every finitely generated commutator-finite OML is also finite. We answer this question affirmatively in this paper. For this purpose we introduce a new concept, the Block Extension Property.

We say that an OML L has the *Block Extension Property* if for every finite set $\mathfrak{B}$ of blocks of L there exists a block-finite full subalgebra S of L containing $\bigcup \mathfrak{B}$. A subalgebra S of L is said to be *full* if every block of S is a block of L. The main result of this paper is:

THEOREM. *Every commutator-finite OML L has the Block Extension Property.*

We, in fact, construct an extension S in such a way that com S = com L. Since every finite subset of a commutator finite OML is trivially contained in the union of finitely many blocks, it follows from our Theorem that it is contained in a block-finite subalgebra. Hence, by [1], the subalgebra generated by it is finite. We thus obtain:

COROLLARY. *Every finitely generated commutator-finite OML is finite.*

In the first section we generalize two results concerning paths in OMLs, which were proved in [2] for block-finite OMLs, to fit our present needs. In the second part we describe how blocks can be constructed from horizontal summands of intervals. We also state a general form of an *Excision Lemma*. A specification of it

Presented by Alan Day. Received January 7, 1988 and in final form July 5, 1988.

is given in the next sections. It is the main tool used to prove the above theorem in section 4. In the concluding section we mention some open problems connected with our investigations.

We are grateful to L. Herman for stimulating conversations concerning the subject matter of this paper. We also gratefully acknowledge direct and indirect support by the Natural Science and Engineering Research Council of Canada.

1. Paths

In this section we show that two theorems concerning paths in OMLs, (4.4) and (4.5) of [2], which were proved in [2] for block-finite OMLs, hold under a weaker assumption, the second in a slightly weakened form. Both generalizations will be needed frequently in this paper. We first recall some relevant definitions and facts, see [2, 3, 4, 6]. The commutator $x*y$ of elements x, y of an OML is defined by

$$x*y=(x\vee y)\wedge(x\vee y')\wedge(x'\vee y)\wedge(x'\vee y').$$

If the union of two distinct blocks (maximal Boolean subalgebras) A, B of L is a subalgebra of L then there exists a unique commutator $e\in A\cap B$ such that $A\cap B=(A\cup B)\cap S_e$, where $S_e=[0,e']\cup[e,1]$. We express this fact briefly by $A\sim_e B$ or, if e is irrelevant, by $A\sim B$. A *path* is a sequence $B_0, B_1, \ldots, B_n$ $(n\geq 0)$ of blocks such that $B_i\sim B_{i+1}$ holds whenever $0\leq i<n$. The path is said to *join* the blocks B_0 and B_n. The path is said to be *strictly proper* if $B_i\cap B_{i+1}\neq C(L)$ holds for all i, it is said to be *proper* if it is strictly proper or if $n=1$. The OML L is defined to be *path-connected* if any two blocks of L can be joined by a proper path.

THEOREM 1. *Every commutator-finite OML L is path-connected.*

Proof by induction on $n=|\mathrm{com}\,L|$. If $n=1$, L is Boolean and the claim is trivially true. Assume $n\geq 2$. By Theorem 14 of [5] and the fact that the product of two path-connected OMLs is again path-connected, (4.3) of [2], we may assume that L is irreducible. Let A, B be distinct blocks of L. Assume first that $A\cap B\neq\{0,1\}$. We show that in this case A and B can be joined by a strictly proper path. Choose $a\in A\cap B$, $a\neq 0,1$. If $\mathrm{com}\,C(a)\neq\mathrm{com}\,L$ then, by inductive hypothesis, A and B can be joined by a proper path in $C(a)$, and hence, since $a\notin C(L)$, by a strictly proper path in L. We may thus assume that $\mathrm{com}\,C(a)=\mathrm{com}\,L$, in particular that $\bigvee\mathrm{com}\,C(a)=\bigvee\mathrm{com}\,L=1$. This implies by Theorem 4 of [5] that $C(a)$ has no non-trivial Boolean factor. But $C(a) \simeq [0, a] \times [0, a']$. Since $[0, a]$, $[0, a']$ are not Boolean they have fewer commutators than $C(a)$, hence are path-connected by inductive hypothesis. Thus A, B can be joined by a strictly proper path in L. The case $A \cap B = \{0, 1\}$ reduces to the previous one as in (4.4) of [2].

An OML L is the *horizontal sum* of a family $(L_i)_{i \in I}$ of at least two subalgebras, see [2, 7], if $\bigcup L_i = L$, $L_i \nsubseteq L_j$ and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied. The three conditions relevant for our purposes are:

1. If $x \in L_i - L_j$, $y \in L_j - L_i$ then $x \vee y = 1$;
2. Every block of L belongs to some L_i;
3. If S_i is a subalgebra of L_i then $\bigcup S_i$ is a subalgebra of L.

THEOREM 2. *Let L be a path-connected OML without non-trivial Boolean factor which contains at least two blocks which cannot be joined by a strictly proper path. Define a relation* $\equiv$ *in* $\mathfrak{A}$ *by*

$$A \equiv B \Leftrightarrow A \text{ and } B \text{ can be joined by a strictly proper path.}$$

Then $\equiv$ *is an equivalence relation in* $\mathfrak{A}_L$. *If* $\mathfrak{B}$ *is an equivalence class of* $\mathfrak{A}_L$ *modulo* $\equiv$ *then* $\bigcup \mathfrak{B}$ *is a subalgebra of L with* $\mathfrak{A}_{\bigcup \mathfrak{B}} = \mathfrak{B}$ *and L is the horizontal sum of these subalgebras.*

Proof. If L were the direct product of non-degenerate OMLs L_1 and L_2 then, considering how the paths in L are related to the paths in the factors, see [2], it would follow that any two blocks in L_1 and any two blocks in L_2 could be joined by a (not necessarily proper) path. Thus any two blocks in L could be joined by a strictly proper path, contrary to the assumptions of Theorem 2. Thus L is irreducible.

To complete the proof first note the following:

$$\text{if } A, B \in \mathfrak{A}_L \text{ and } A \cap B \neq \{0, 1\} \text{ then } A \equiv B. \qquad (*)$$

For, A and B may be joined by a proper path. If one of these paths has length $n \geq 2$ it is strictly proper and hence $A \equiv B$. If $n = 1$ then the path $A \sim B$ is strictly proper since $A \cap B \neq \{0, 1\} = C(L)$ and again $A \equiv B$, proving $(*)$. Now let $\mathfrak{B}$ be an equivalence class of $\mathfrak{A}_L$ modulo $\equiv$ and let $a, b \in \bigcup \mathfrak{B}$. We show $a \vee b \in \bigcup \mathfrak{B}$. This is clear if $a \in \{0, 1\}$. We may thus assume that $a \neq 0, 1$. There exist blocks $A \in \mathfrak{B}$ and $B \in \mathfrak{A}_L$ such that $a \in A$ and $a, a \vee b \in B$. Since $a \in A \cap B$ it follows from $(*)$ that $B \in \mathfrak{B}$ and hence $a \vee b \in \bigcup \mathfrak{B}$, making $\bigcup \mathfrak{B}$ a subalgebra. The rest of the claim follows easily from $(*)$.

We do not know whether Theorem 2 remains true if the assumption that L has no Boolean factor is dropped and "horizontal sum" in the conclusion is replaced by "weak horizontal sum," see [2]. This would be true if every path-connected OML had a largest Boolean factor. We do not know whether this is the case.

We will frequently use the following consequence of Theorem 2.

COROLLARY 1. *If L is a non-Boolean path-connected OML then every block A of L which is not a horizontal summand of L contains a commutator $v \neq 0, 1$.*

Proof. Since L is not Boolean and path-connected there exists a proper path $A \sim_v B \sim \cdots$ in L. If $v \neq 1$ the claim is proved. If $v = 1$ then, by Theorem 4 of [5], L has no Boolean factor. If A could not be joined with any other block by a strictly proper path it would be a horizontal summand by Theorem 2. Thus there exists a strictly proper path $A \sim_v B \cdots$ and $v \neq 0, 1$.

2. Blocks and subalgebras

For the rest of the paper we assume that L is an OML every subalgebra of which is path-connected. Only Lemma 2 and the Excision Lemma do not require this assumption.

If $A \subseteq L$, $u \in L$ we define $u \wedge A = \{u \wedge x \mid x \in A\}$. It is well-known that if A is a block of L and $u \in A$ then $u \wedge A = [0, u] \cap A$ is a block of the OML $[0, u]$.

LEMMA 1. *If $u \in A \in \mathfrak{A}_L$ and $A \cap \operatorname{com} L \subseteq \{0, u\}$, then $[0, u]$ is Boolean or $u \wedge A$ is a horizontal summand of $[0, u]$.*

Proof. Since $u \wedge A$ contains no commutators $\neq 0, u$ the claim follows from Corollary 1.

COROLLARY 2. *If u is minimal in $(A \cap \operatorname{com} L) - \{0\}$ then $u \wedge A$ is a horizontal summand of $[0, u]$.*

Proof. Since $u \in \operatorname{com}[0, u]$, $[0, u]$ is not Boolean and the claim follows from Lemma 1.

For $X_1, X_2, \ldots, X_n \subseteq L$ define

$$\bigvee_{i=1}^{n} X_i = \left\{\bigvee_{i=1}^{n} x_i \mid x_i \in X_i\right\}.$$

LEMMA 2. *Let* $\{u_1, u_2, \ldots, u_n\}$ *be an orthogonal subset of* L *with* $\bigvee_{i=1}^n u_i = 1$ *and for every* i *let* A_i *be a block of* $[0, u_i]$. *Then* $\bigvee_{i=1}^n A_i$ *is a block of* L.

We leave the easy proof of this to the reader.

LEMMA 3. *Let* L *be commutator-finite,* $A \in \mathfrak{A}_L$ *and let* $\{u_1, u_2, \ldots, u_n\}$ *be a maximal orthogonal subset of the set of all minimal elements of* $(A \cap \operatorname{com} L) - \{0\}$. *Define* $u_{n+1} = \bigwedge_{i=1}^n u_i'$. *Then, for* $1 \le i \le n$, $u_i \wedge A$ *is a Boolean horizontal summand of* $[0, u_i]$, $[0, u_{n+1}]$ *is Boolean and* $A = \bigvee_{i=1}^{n+1} (u_i \wedge A)$.

Proof. The first claim restates Corollary 2. Since $\{u_1, \ldots, u_n\}$ is maximal, $u_{n+1} \wedge A$ contains no commutator $v \neq 0$. In particular, $u_{n+1} \notin \operatorname{com} L$ and hence $[0, u_{n+1}]$ is not a horizontal sum. Thus $[0, u_{n+1}]$ is Boolean by Lemma 1. By Lemma 2, $\bigvee_{i=1}^{n+1} (u_i \wedge A)$ is a block which trivially contains A, proving the last claim.

A key step in the proof of our main theorem is an application of a specification of the following general result. We write $M \le L$ to indicate that M is a subalgebra of L.

EXCISION LEMMA. *Let* $M \le L$ *and* $\mathfrak{A} \subseteq \mathfrak{A}_L$. *If* $\bigcup(\mathfrak{A}_L - \mathfrak{A}) \cap \bigcup \mathfrak{A} \subseteq M$ *then* $(L - \bigcup \mathfrak{A}) \cup M \le L$.

Proof. Let $L_1 = (L - \bigcup \mathfrak{A}) \cup M$ and $x, y \in L_1$. Since L_1 is closed under orthocomplementation we need only show that $x \vee y \in L_1$. Suppose not; then $x \vee y \in (\bigcup \mathfrak{A}) - M$, so, by hypothesis, $x \vee y \notin \bigcup(\mathfrak{A}_L - \mathfrak{A})$. Hence every block containing $x \vee y$ belongs to $\mathfrak{A}$. Since x, $x \vee y$ belong to some block, $x \in \bigcup \mathfrak{A}$. By symmetry, $y \in \bigcup \mathfrak{A}$. Thus, by hypothesis, $x, y \in M$ and hence $x \vee y \in M$, completing the proof.

3. Inessential blocks of intervals

For $u \in L$ let $\beta(u)$ be the set of all Boolean horizontal summands of $[0, u]$. Note that $\mathfrak{A}_{[0,u]} - \{X\} \neq \varnothing$ whenever $X \in \beta(u)$. An $X \in \beta(u)$ is said to be *essential* if there exists $D \in \mathfrak{A}_L$ such that $u \notin D$ and $X \cap D \neq \{0\}$, and *inessential* otherwise. If $X \in \mathfrak{A}_{[0,u]}$ define

$$L_X = (L - C(X)) \cup S_u.$$

We shall show that if $u \in \operatorname{com} L - \{0\}$ and $X \in \beta(u)$ then L_X is a subalgebra of L iff X is inessential. The subalgebras L_X play a crucial role in the sequel. We leave the proof of the following simple lemma to the reader.

LEMMA 4. *If* $u \in L$ *and if* X *is a block of* $[0, u]$ *then*

(1) $C(X) = C(u) \cap \{x \in L \mid u \wedge x \in X\}$,
(2) $L_X = (L - C(u)) \cup S_u \cup \{x \in L \mid u \wedge x \notin X\}$,
(3) $X \cap L_X = \{0, u\}$ *and* $L - L_X = C(X) - S_u \subseteq C(u) - S_u$,
(4) $[0, u] - X \subseteq L_X$.

LEMMA 5. *Let* $u \in \operatorname{com} L - \{0\}$ *and* $X \in \beta(u)$. *X is inessential iff* $L_X \leq L$.

Proof. Assume that X is inessential. We apply the Excision Lemma with $\mathfrak{A} = \mathfrak{A}_{C(X)}$ and $M = S_u$. Since $S_u \leq L$ we need only verify that $(\bigcup (\mathfrak{A}_L - \mathfrak{A}_{C(X)})) \cap C(X) \subseteq S_u$. To this end, let $x \in A \cap B$ where $A \in \mathfrak{A}_{C(X)}$ and $B \in \mathfrak{A}_L - \mathfrak{A}_{C(X)}$. Assume $u \notin B$. Then there exist $y \in B$ such that $y \not\mathrel{C} u$. Since $x \vee y$, $x' \vee y \mathrel{C} u$ would imply $y \mathrel{C} u$ we may assume that $x \vee y \not\mathrel{C} u$. Then there exists a block D containing $u \wedge x$ and $x \vee y$ and hence not containing u. Since X is inessential and $u \wedge x \in X$, it follows that $u \wedge x = 0$; moreover, since $x \mathrel{C} u$, we obtain $x \in S_u$. We may thus assume that $u \in B$. There exists a path $A = A_0 \sim A_1 \sim \cdots \sim A_n = B$ in $C(\{x, u\})$. Let j be the smallest index such that $u \wedge A_{j+1} \neq X$. Then $u \wedge A_j = X$ and $(u \wedge A_{j+1}) \cap X = \{0, u\}$. Let $A_j \sim_v A_{j+1}$. Then $v \leq u$ or $u \leq v'$. But $u \leq v'$ would imply $X \subseteq A_{j+1}$, contrary to our assumption. Thus $v \leq u$, hence $u = v$. It follows that $x \in A_j \cap A_{j+1} \subseteq S_u$. To prove the converse assume $D \in \mathfrak{A}_L$ and $0 \neq x \in D \cap X$. We have to show that $u \in D$. Suppose that $u \notin D$ so that there exists $y \in D$, $y \not\mathrel{C} u$. Since $y = (y \vee x) \wedge (y \vee x')$ and $u' \leq y \vee x' \mathrel{C} u$ it follows that $y \vee x \not\mathrel{C} u$. Since $x \leq u \wedge (x \vee y) < u$ and $X \in \beta(u)$ we obtain $u \wedge (x \vee y) \in X$. But $u \in L_X$ and, since $x \vee y \not\mathrel{C} u$, $x \vee y \in L_X$ by Lemma 4. Thus, by hypothesis, $u \wedge (x \vee y) \in L_X$, contradicting $X \cap L_X = \{0, u\}$. It follows that $u \in D$, completing the proof.

COROLLARY 3. *If* $X \in \beta(u)$ *is inessential then* L_X *is the largest subalgebra* N *of* L *such that* $X \cap N = \{0, u\}$.

Proof. By Lemmas 4 and 5, L_X has the property. Let $N \leq L$ with $X \cap N = \{0, u\}$ and suppose $n \in N - L_X$. By Lemma 4, $n \in C(X) - S_u$, so that $0 < u \wedge n < u$. But $u \wedge n \in N \cap C(X) \cap [0, u] = N \cap X = \{0, u\}$, a contradiction. Hence $N \subseteq L_X$ and L_X is the largest such subalgebra.

LEMMA 6. *If* $0 \neq u \in \operatorname{com} L$ *and if* $X \in \beta(u)$ *is inessential then*

(1) *if* $y \not\mathrel{C} u$ *then* $S_y \subseteq L_X$,
(2) $\operatorname{com} L \subseteq L_X$,
(3) *if* $u \neq v \in \operatorname{com} L - \{0\}$ *and* $Y \in \beta(v)$ *then* $Y \subseteq L_X$,
(4) *if* $v \in \operatorname{com} L$ *and if* $[0, v']$ *is Boolean then* $[0, v'] \subseteq L_X$.

Proof. Assume that there exists $z \in L$ such that $u \nleq y \leq z \notin L_X$. Then $0 < u \wedge z' \leq u \wedge y' < u$. Since $u \wedge z' \in X$ it would follow that $u \wedge y' \in X \cap L_X$, contradicting (3) of Lemma 4. Thus $[y, 1] \subseteq L_X$ and hence $S_y \subseteq L_X$, proving (1). To prove (2) assume $v \in \operatorname{com} L$ and $v \notin L_X$. Then $v \in C(u) - S_u$, $0 < u \wedge v$, $u \wedge v' < u$ and $u \wedge v$, $u \wedge v' \in X$. Since $[0, u \wedge v]$ is Boolean, $u \wedge v$ is not a commutator and it follows from Corollary 6 of [5] that $C(v) \not\subseteq C(u)$. Thus there exists $B \in \mathfrak{A}_L$ such that $u \notin B$ and $v \in B$. Let $u, v \in A \in \mathfrak{A}_L$ and let $A = A_0 \sim A_1 \sim \cdots \sim A_n = B$ be a path in $C(v)$. Let p be an index such that $u \in A_p - A_{p+1}$ and $A_p \sim_w A_{p+1}$. Since $v \in S_w$, $u \wedge v$ or $u \wedge v'$ is in $X \cap A_{p+1}$, contradicting the fact that X is inessential and proving (2). To prove (3) assume that under the assumptions of (3) there exists $y \in Y - L_X$. Then, by (2) of Lemma 4, $0 < u \wedge y < u$ and $u \wedge y \in X$. Since $u \wedge y \leq u \wedge v \leq u$ this implies $u \wedge v \in X$ which, since $v \in L_X$, gives $u < v$. But $0 < u \wedge y \leq y < v$ and $y \in Y$ implies $u \wedge y \in Y$. This with $u \wedge y \leq u < v$ implies $u \in Y$ and the Boolean algebra $[0, u]$ contains a commutator, a contradiction proving (3). Assume finally that v satisfies the assumptions of (4) and there exists $y \leq v'$, $y \notin L_X$. Then $0 < u \wedge y \leq u \wedge v' \leq u$ and $u \wedge y \in X$. Thus also $u \wedge v' \in X$ and, since $v \in L_X$ by (2), $u \leq v'$, contradicting the fact that $[0, v']$ is Boolean.

4. Proof of the main theorem

Assume $0 \neq u \in \operatorname{com} L$ and $X \in \beta(u)$. We say that a commutator v is *associated with* X iff

There exist $A, E \in \mathfrak{A}_L$ such that $u \in A - E$, $u \nleq v$, $A \sim_v E$ and $X \subseteq A$. (*)

LEMMA 7. *If* $0 \neq u \in \operatorname{com} L$ *and if* $X \in \beta(u)$ *is essential then there exists a commutator* v *associated with* X. *If* $X \neq Y \in \beta(u)$ *and* Y *is also essential then the sets of commutators associated with* X *and* Y *are disjoint.*

Proof. Since X is essential there exists $D \in \mathfrak{A}_L$ such that $u \notin D$ and $X \cap D \neq \{0\}$. Let $A_0 \in \mathfrak{A}_L$ contain X and let $A_0 \sim A_1 \sim \cdots \sim A_n = D$ be a path in $C(X \cap D)$. There exists p such that $u \in A_p - A_{p+1}$, $A_p \sim_v A_{p+1}$. Put $A = A_p$, $E = A_{p+1}$. There exists $x > 0$, $x \in X \cap D \subseteq A \cap E$. But $v \leq x$ would imply $v \leq u$, hence $u \in A_{p+1}$, a contradiction. Thus $x \leq u \wedge v'$ and $u \nleq v$. Since $x \in X \cap A$, $x < u$ (since $u \notin D$) and $u \wedge A$ is a block in $[0, u]$ it follows that $u \wedge A = X \subseteq A$. Thus v is associated with X. To prove the second part let v be associated with X and w with Y. Then there exist blocks A, B as above such that $u, v \in A$, $u, w \in B$, $X \subseteq A$ and $Y \subseteq B$. It follows that $u \wedge v' \in X$ and $u \wedge w' \in Y$. It follows that $u \wedge v' \neq u \wedge w'$ and hence $v \neq w$, proving the second part.

As an immediate consequence of this we obtain the following.

COROLLARY 4. *Commutator-finite OMLs contain only finitely many essential Boolean horizontal summands of intervals* $[0, u]$ *with* $u \in \operatorname{com} L$.

COROLLARY 5. *A commutator-finite OML is not block-finite if and only if it contains an interval having infinitely many inessential Boolean horizontal summands.*

The last corollary, together with the applicaion of the Excision Lemma which follows, exhibits a structural feature of commutator-finite OMLs which shows how closely related they are with block-finite OMLs and, by the corollary to the main theorem, with finite OMLs.

We are now in a position to prove the main Theorem. Let L be commutator-finite and $\mathfrak{B}$ a finite set of blocks of L. For every $u \in \operatorname{com} L$ choose a_u, b_u such that $u = a_u * b_u$ and add a block to $\mathcal{B}$ containing a_u and a block containing b_u. Then $\mathcal{B}$ remains finite and $\operatorname{com} S = \operatorname{com} L$ holds for every subalgebra S of L containing $\bigcup \mathfrak{B}$. For every $u \in \operatorname{com} L$, $u \neq 0$, let Ω_u be the set of all those Boolean horizontal summands of $[0, u]$ which are either essential or of the form $u \wedge A$ for some $A \in \mathfrak{B}$ with $u \in A$. By Corollary 4 and the finiteness of $\mathcal{B}$ each Ω_u is finite. Define $\Omega'_u = \beta(u) - \Omega_u$. Define

$$S = \bigcap \{L_X \mid 0 \neq u \in \operatorname{com} L, X \in \Omega'_u\}$$

Clearly S is a subalgebra of L. By (4) of Lemma 4 and (3) of Lemma 6 we have $\bigcup \Omega_u \subseteq S$ whenever $0 \neq u \in \operatorname{com} L$. By Lemma 3 and (4) of Lemma 6 it follows that $\bigcup \mathfrak{B} \subseteq S$. We prove next that every block A of S is a block of L, i.e., that the subalgebra S of L is full. Extend A to a block $\bar{A}$ of L. Since $\operatorname{com} S = \operatorname{com} L$ the commutators in A and $\bar{A}$ are the same. Let u be a minimal element of $(A \cap \operatorname{com} L) - \{0\} = (\bar{A} \cap \operatorname{com} L) - \{0\}$. By Corollary 2, $u \wedge A$ is a Boolean horizontal summand of $[0, u] \cap S$ and $u \wedge \bar{A}$ is a Boolean horizontal summand of $[0, u] \cap S$ and $u \wedge \bar{A}$ is a Boolean horizontal summand of $[0, u]$. In particular there exists $x \in u \wedge A$, $0 < x < u$. It follows from (3) of Lemma 4 that $u \wedge \bar{A} \in \Omega_u$ and hence $u \wedge \bar{A} \subseteq S$. Since $u \wedge A$ is a block in $[0, u] \cap S$ it follows that $u \wedge \bar{A} = u \wedge A$ and therefore $u \wedge A \in \Omega_u$. Now let $\{u_1, u_2, \ldots, u_n\}$ be a maximal orthogonal subset of the set of all minimal elements of $(\bar{A} \cap \operatorname{com} L) - \{0\}$ and let $v = \bigvee_{i=1}^{n} u_i$. By Lemma 3 $[0, v']$ is Boolean; by Corollary 6 of [5] and (4) of Lemma 6 it is contained in S so that $[0, v'] \subseteq \bar{A} \cap S = A$. It now follows from Lemma 2 and 3 that $A = \bar{A}$ and hence that S is a full subalgebra. Since all Ω_u are finite it also follows that S is block-finite, completing the proof.

5. Concluding remarks

It is easy to construct examples of OMLs which have the Block Extension Property (abbreviated: BEP) but are not commutator-finite. It would be of interest to find classes of OMLS having the BEP which properly contain the class of commutator-finite OMLs or, ideally, to find alternative descriptions of OMLs having the BEP. The BEP is trivally preserved under finite products. Which other basic algebraic constructions preserve the BEP? In particular, is the BEP preserved by subalgebras and homomorphic images? The BEP has as an obvious consequence that every subalgebra generated by the union of finitely many blocks is block-finite. We call this last statement the Weak Block Extension Property, abbreviated: WBEP. The WBEP is obviously enough to ensure that every finitely generated subalgebra is finite. How is the WBEP related to the BEP? Is it strictly weaker or equivalent? It is easy to see that the WBEP is preserved under finite products and subalgebras. Is it preserved under homomorphic images? We believe that answers to such questions would prove significant in the developing theory of orthomodular lattices.

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Department of Mathematics
McMaster University
Hamilton, Ontario
L8S 4K1

Kansas State University
Manhattan, Kansas
U.S.A., 66506