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Completions of Orthomodular Lattices

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Abstract. If $\mathscr{K}$ is a variety of orthomodular lattices generated by a finite orthomodular lattice the MacNeille completion of every algebra in $\mathscr{K}$ again belongs to $\mathscr{K}$.

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The MacNeille completion of an ortholattice (abbreviated: *OL*) *L* carries a natural orthocomplementation, which is unique if one requires that it extends the orthocomplementation of *L*, cf. MacLaren [12]. We always consider the MacNeille completion of an *OL* as an *OL* with this unique orthocomplementation. We are interested here in the MacNeille completion of an orthomodular lattice (abbreviated: *OML*). Very little is known about it. The only positive result we are aware of is due to Janowitz [9]. It states that the MacNeille completion of an indexed *OML* is again an indexed *OML*. The only other result is negative: The MacNeille completion of even a modular *OL* is not necessarily an *OML*.

An example of this (essentially, the only one known) can be constructed using a theorem first stated by Piron [Theorem 22] and proved in its entirety by Amemiya and Araki [2]; this folklore was first noted in print by Adams [1]. If V is a (real or complex) inner product space and $A \subseteq V$, define $A^{\perp}$ to be the set of all elements of

V which are orthogonal to A and define $\mathscr{L}(V, \perp) = \{A \subseteq V \mid A^{\perp\perp} = A\}$. $\mathscr{L}(V, \perp)$ is an *OL*. Amemiya and Araki proved that $\mathscr{L}(V, \perp)$ is an *OML* iff the space V is complete, hence a Hilbert space. Consider now an inner product space V which is not complete and let L be the modular *OL* of all finite-dimensional or cofinite-dimensional subspaces of V. It follows easily from the characterization of the MacNeille completion given at the beginning of the next section, that $\mathscr{L}(V, \perp)$ is the MacNeille completion of L. It is not an *OML* by the theorem of Amemiya–Araki–Piron. As a consequence of this, one obtains that the MacNeille completion of an *OML* L need not again be an *OML*, and *a fortiori* it need not belong to the variety generated by L. This is, of course, not surprising in view of the well-known misbehaviour of the MacNeille completion of general lattices with respect to equations, see, for example, Funayama [6]. In contrast to this general situation is the main result of this paper:

THEOREM. *If $\mathscr{K}$ is a variety of OMLs which is generated by a finite OML then the MacNeille completion of every member S of $\mathscr{K}$ again belongs to $\mathscr{K}$.*

It follows from Jónsson [10] that every subvariety of a variety generated by a finite *OML* is again generated by a finite *OML*. As a consequence of our theorem we thus obtain the following:

COROLLARY. *If an OML L belongs to a variety generated by a finite OML then the MacNeille completion of L belongs to the variety generated by L.*

If $\mathscr{K}$ is generated by the two-element Boolean algebra then our theorem reduces to the well-known fact that the MacNeille completion of a Boolean algebra is again a Boolean algebra. This seems to be the only known special case of our theorem.

After some preliminary observations and remarks in Section 1 we prove our theorem in Section 2 for the special case that S is a Boolean power of a finite *OML*. In this section and in Section 4, general results from the theory of Boolean powers, see [4, 5], could have been brought in more prominently. We convinced ourselves, however, that this would not have shortened our presentation which has the advantage of being comprehensible without a previous knowledge of Boolean powers. In Section 3, which does not require knowledge of Section 2, we construct some polynomials which will be needed later to extend the result of Section 2. In Section 4 this is done for a subdirect power of a finite simple lattice and in Section 5 for the general case. We conclude with a discussion of the more general problem of embeddings of *OML*s into complete lattices. We show that the *OML* given above, the MacNeille completion of which is not an *OML*, can still be embedded in a complete *OML*. As general background information the reader is referred to [11] for *OML*s, [8] for Boolean algebras, [5] for universal algebra and [7] for general lattice theory.

1. Preliminaries

The MacNeille completion of a partially ordered set P was first introduced by MacNeille [13] using a specific construction, the well-known completion by cuts. Later [3, 15] it was noticed that the MacNeille completion of a poset P is determined up to a unique isomorphism over P as a complete poset C containing P as a join-dense and meet-dense sub-poset, meaning that every element of C is the join and meet of elements of P. It is this last characterization that we take as the definition of the MacNeille completion P.

It is one of the basic facts of the MacNeille completion of a poset P that it preserves all joins and meets existing in P; in particular, every lattice is a sublattice of its MacNeille completion. Since the natural orthocomplementation of the MacNeille completion of an OL L extends the orthocomplementation of L, the MacNeille completion of an OL L can simply be described as a complete OL C containing L as a join-dense sub-OL. As is easy to see, in the case of OMLs the join dense condition simplifies even more: A subset M of an OML C is join-dense in C iff every non-zero element of C is greater or equal to a non-zero element of M. This remark will be useful later.

The MacNeille completion of the product of two posets is in general not isomorphic to the product of the MacNeille completions of the factors, the standard example being the product of the open unit interval of the reals with itself. OLs (more generally bounded posets) are much better behaved. The reader will easily verify the following:

PROPOSITION 1. *If $(L_x)_{x\in X}$ is a family of OLs, if S is a join-dense sub-OL of $\Pi_{x\in X} L_x$ and if $\bar{L}_x$ is the MacNeille completion of L_x then $\Pi_{x\in X}\bar{L}_x$ is the MacNeille completion of S (as always, up to isomorphism).*

A slightly more elaborate form of this is:

PROPOSITION 2. *Let L be an OML and let M be a set of pairwise orthogonal central elements of L satisfying $\bigvee M = 1$. For every $\alpha \in M$ let L_α be the MacNeille completion of $[0, \alpha]$. Then there exists an OL-embedding of L as a join-dense sublattice into $\Pi_{\alpha\in M} L_\alpha$, thus $\Pi_{\alpha\in M} L_\alpha$ is the MacNeille completion of L.*

Proof. Since the $\alpha \in M$ are central the map $a \to \alpha \wedge a$ is an OL-homomorphism of L onto $[0, \alpha]$. Thus the map $\varphi : L \to \Pi_{\alpha\in M}[0, \alpha]$ defined by $\varphi(a)(\alpha) = \alpha \wedge a$ is an OL-homomorphism. Since the elements $\alpha \in M$ are central, $a = a \wedge 1 = \alpha \wedge \bigvee M = \bigvee_{\alpha\in M}(a \wedge \alpha)$ and φ is an embedding. By Proposition 1 it is enough to show that $\varphi(L)$ is join-dense in $\Pi_{\alpha\in M}[0, \alpha]$. But if $\psi \in \Pi_{\alpha\in M}[0, \alpha]$ then, for every $\alpha \in M$, $\psi(\alpha) \in [0, \alpha] \subseteq L$. It follows easily that $\psi = \bigvee_{\alpha\in M}\varphi(\psi(\alpha))$, proving the claim.

2. OMLs as Lattices of Continuous Functions

Algebras of continuous functions from a topological space X into an algebra A with the discrete topology have been studied by several authors, notably in the case of a Boolean space X, see [4, 5] and the literature quoted there. The lattice of continuous functions from a Boolean space into an algebra A is also known as *Boolean power* of A. Using this terminology the main result of this section can be stated as follows: If S is a Boolean power of an OML L then the MacNeille completion of S is also a Boolean power of L.

If L is a finite OML with the discrete topology, X a topological space and $f: X \to L$ then $\{f^{-1}(a) \mid a \in L\}$ is a finite partition of X. It follows that f is continuous iff $f^{-1}(a)$ belongs to the field (defined below) of open-closed subsets of X. To make our results meld into the sequel we work here with an arbitrary field of sets instead of the field of open-closed sets of a topological space.

By a ring of subsets of a set X we mean a sublattice $\mathscr{F}$ of the power set of X. The ring $\mathscr{F}$ is said to be a field of subsets of X if the complement $A' = X - A$ of every $A \in \mathscr{F}$ also belongs to $\mathscr{F}$ and if $\mathscr{F}$ is not empty. If $\mathscr{F}$ is a field of subsets of X and if L is a finite OML define $C(\mathscr{F}, L)$ as follows:

$$C(\mathscr{F}, L) = \{f: X \to L \mid f^{-1}(a) \in \mathscr{F} \text{ for every } a \in L\}.$$

Since, for any functions $f, g: X \to L$, $(f \wedge g)^{-1}(a) = \bigcup\{f^{-1}(b) \cap g^{-1}(c) \mid b \wedge c = a\}$ and $(f')^{-1}(a) = f^{-1}(a')$, $C(\mathscr{F}, L)$ is a sub-OL of the full power L^X. Our OML $C(\mathscr{F}, L)$ only seems to be more general than the Boolean power construction. It follows from basic results concerning Boolean powers that our OMLs $C(\mathscr{F}, L)$ can indeed be represented as Boolean powers of L. Our considerations, however, do not require this fact.

PROPOSITION 3. *Let $\mathscr{F}$ be a field of subsets of a set X and L a finite OML. Then $\mathscr{F}$ is a complete lattice iff $C(\mathscr{F}, L)$ is a complete lattice.*

Note that in Proposition 3 completeness of $\mathscr{F}$ does not require that infinite joins and meets are unions and intersections and completeness of $C(\mathscr{F}, L)$ does not require that infinite joins and meets in $C(\mathscr{F}, L)$ are the same as those in L^X. If $a \in L$ we write $\uparrow a$ for $\{b \in L \mid a \leqslant b\}$. If $A \subseteq X$ let χ_A be the characteristic function of A, i.e., the function from X into L which takes value 1 if $x \in A$ and value 0 if $x \notin A$.

Proof of Proposition 3. Assume that $\mathscr{F}$ is a complete lattice and $M \subseteq C(\mathscr{F}, L)$. For $a \in L$ define $X_a = \bigvee\{f^{-1}(a) \mid f \in M\}$ ($\bigvee$ in $\mathscr{F}$) and define $g: X \to L$ by $g(x) = \bigvee\{a \in L \mid x \in X_a\}$. It follows from this definition that $g(x) = b$ holds iff there exists a set $S \subseteq L$ such that $\bigvee S = b$ and $x \in (\bigcap_{a \in S} X_a) - \bigcup_{a \notin S} X_a$. Thus $g^{-1}(b) = \cup\{(\cap_{a \in S} X_a) - \cup_{a \notin S} X_a \mid \bigvee S = b\} \in \mathscr{F}$, which implies $g \in C(\mathscr{F}, L)$. If $f \in M$ and $f(x) = b$ then $x \in f^{-1}(b) \subseteq X_b$, thus $g(x) = \bigvee\{a \mid x \in X_a\} \geqslant b$ and g is an upper bound of M in $C(\mathscr{F}, L)$. To show that it is the least upper bound assume that $f \leqslant u \in C(\mathscr{F}, L)$ holds for all $f \in M$. Then $f^{-1}(a) \subseteq u^{-1}(\uparrow a)$ holds for all $f \in M$, $a \in L$ and hence $X_a \subseteq u^{-1}(\uparrow a)$ holds for all $a \in L$. Assume now that $g(x) = b$. Then there exists $S \subseteq L$ such that $\bigvee S = b$ and $x \in X_a$ for all $a \in S$. Thus $x \in u^{-1}(\uparrow a)$ holds for all $a \in S$ and hence $u(x) \geqslant b = g(x)$, proving that g is the least upper bound of M in $C(\mathscr{F}, L)$ and hence that $C(\mathscr{F}, L)$ is complete. To prove the converse assume that $C(\mathscr{F}, L)$ is complete and let $\mathfrak{U}$ be a subset of $\mathscr{F}$. Define $f = \bigvee\{\chi_A \mid A \in \mathfrak{U}\}$ ($\bigvee$ in $C(\mathscr{F}, L)$) and $B = f^{-1}(1)$. Then, for all $A \in \mathfrak{U}$, $A = \chi_A^{-1}(1) \subseteq f^{-1}(1) = B$ and hence B is an upper bound of $\mathfrak{U}$ in $\mathscr{F}$. Let C be any upper bound of $\mathfrak{U}$ in $\mathscr{F}$. Then $\chi_A \leqslant \chi_C$ holds for every $A \in \mathfrak{U}$ and hence $f \leqslant \chi_C$. It follows that $B = f^{-1}(1) \subseteq \chi_C^{-1}(1) = C$. Thus B is the least upper bound of $\mathfrak{U}$, completing the proof.

Again, let $\mathscr{F}$ be a field of subsets of a set X and let M be the MacNeille completion of $\mathscr{F}$. We actually assume that M contains $\mathscr{F}$ (and not just an isomorphic copy of $\mathscr{F}$) as a join-dense, and hence meet-dense, sub-OL. Let Ω be the Stone space of M and let $B(\Omega)$ be the complete Boolean algebra of all open-closed subsets of Ω. Thus Ω is the set of all ultrafilters in M and the map $\kappa : M \to B(\Omega)$ defined by $\kappa(c) = \{P \mid c \in P \in \Omega\}$ is an isomorphism between M and $B(\Omega)$. Then $C(B(\Omega), L)$ is the OML of all continuous functions from Ω into L and $C(B(\Omega), L)$ is complete by Proposition 3. If $f \in C(\mathscr{F}, L)$, $P \in \Omega$ then $\cup_{a \in L} f^{-1}(a) = X \in P$. Since P is prime and the sets $f^{-1}(a)$ are pairwise disjoint there exists exactly one $a \in L$ such that $f^{-1}(a) \in P$. We may thus define a map $\alpha : C(\mathscr{F}, L) \to L^{\Omega}$ by

$$\alpha(f)(P) = a \Leftrightarrow f^{-1}(a) \in P.$$

Under the assumptions thus described we claim:

PROPOSITION 6. *α is a join-dense OL-embedding of $C(\mathscr{F}, L)$ into $C(B(\Omega), L)$.*

Proof. Since $f^{-1}(a) \in P$ is equivalent with $P \in \kappa(f^{-1}(a))$ it follows that for every $f \in C(\mathscr{F}, L)$, $a \in L$, $(\alpha(f))^{-1}(a) = \kappa(f^{-1}(a)) \in B(\Omega)$ and hence that α is a map of $C(\mathscr{F}, L)$ into $C(B(\Omega), L)$. If $f, g \in C(\mathscr{F}, L)$ and $\alpha(f \wedge g)(P) = a$ then $\cup\{f^{-1}(b) \cap g^{-1}(c) \mid b \wedge c = a\} = (f \wedge g)^{-1}(a) \in P$ and hence there exist b, c such that $b \wedge c = a$ and $f^{-1}(b), g^{-1}(c) \in P$. It follows that $\alpha(f)(P) = b$, $\alpha(g)(P) = c$ and $\alpha(f)(P) \wedge \alpha(g)(P) = a$, proving $\alpha(f \wedge g) = \alpha(f) \wedge \alpha(g)$. Furthermore, $\alpha(f')(P) = a$ iff $(f')^{-1}(a) \in P$ iff $f^{-1}(a') \in P$ iff $\alpha(f)(P) = a'$. Thus $\alpha(f') = (\alpha(f))'$ and α is an OL-homomorphism. If $\alpha(f) = 0$ then $\alpha(f)(P) = 0$ and hence $f^{-1}(0) \in P$ for all $P \in \Omega$ which implies $f^{-1}(0) = X$ and hence $f = 0$, proving that α is an embedding. To show that α is join-dense note first that every $g \in C(B(\Omega), L)$ is the join of functions of the form $a \wedge \chi_{\kappa(c)}$. Thus, if $g \neq 0$ there exist $a, c \neq 0$ such that $a \wedge \chi_{\kappa(c)} \leqslant g$ and it is enough to show that there exists a non-zero function $f \in C(\mathscr{F}, L)$ such that $\alpha(f) \leqslant a \wedge \chi_{\kappa(c)}$. Since $c \neq 0$ there exists $F \in \mathscr{F}$ such that $\emptyset \neq F \leqslant c$. Put $f = a \wedge \chi_F$. We have $\alpha(f)(P) = b$ iff $f^{-1}(b) \in P$, which is the case if $F \in P$ and $b = a$, or if $b = 0$ and $F \notin P$. Thus $\alpha(f) = a \wedge \chi_{\kappa(F)} \leqslant g$, completing the proof of Proposition 6.

As an easy consequence of this we get the following:

COROLLARY. *If $\mathscr{F}$ is a field of subsets of a set X and if L is a finite OML then the MacNeille completion of $C(\mathscr{F}, L)$ belongs to the variety generated by L.*

3. Some Polynomials

In this section we construct some polynomials which will be crucial later in the paper.

PROPOSITION 5. *Let L be a finite irreducible OML having exactly n elements. Then there exists an n-ary OL-polynomial $p(x_1, x_2, \ldots, x_n)$ with the properties*:

1. *If $b_1, b_2, \ldots, b_n$ are elements of any OML and $b_i = b_j$ for some distinct indices i, j then $p(b_1, b_2, \ldots, b_n) = 0$.*
2. *If $b_1, b_2, \ldots, b_n$ are pairwise distinct elements of L then $p(b_1, b_2, \ldots, b_n) = 1$.*

Proof. For distinct indices $i, j \in \{1, 2, \ldots, n\}$ define $p_{ij} = (x_i \vee x_j) \wedge (x_i' \vee x_j')$. Note that for elements a, b of any OML, $p_{ij}(a, b) = 0$ iff $a = b$. We now use an iteration procedure which will appear again later. For $1 \leqslant m \leqslant n$ define recursively polynomials $p_{ij}^m(x_1, x_2, \ldots, x_n)$ by $p_{ij}^1 = p_{ij}$ (this polynomial is only binary) and

$$p_{ij}^{m+1} = \bigvee_{k=1}^{n} ((p_{ij}^m \vee x_k) \wedge (p_{ij}^m \vee x_k')).$$

Clearly $P_{ij}^m(x_1, x_2, \ldots, x_n) \leqslant p_{ij}^{m+1}(x_1, x_2, \ldots, x_n)$ and if $p_{ij}(b_i, b_j) = 0$ then $p_{ij}^m(b_1, b_2, \ldots, b_n) = 0$ for every m. Define $p = \bigwedge\{p_{ij}^n \mid i, j \in \{1, 2, \ldots, n\}, i \neq j\}$. It is obvious that p satisfies the first condition. Assume now that $b_1, b_2, \ldots, b_n$ are distinct elements of L. Then $p_{ij}(b_i, b_j) \neq 0$ and hence $p_{ij}^m(b_1, b_2, \ldots, b_n) \neq 0$ for every m. If $p_{ij}^m(b_1, b_2, \ldots, b_n) \neq 1$ then, since L is irreducible, there exists $b \in L$ such that $bCp_{ij}^m(b_1, b_2, \ldots, b_n)$ does not hold. Since the b_i are distinct there exists $b_k = b$. It follows that $p_{ij}^{m+1}(b_1, b_2, \ldots, b_n) \geqslant (p_{ij}^m(b_1, b_2, \ldots, b_n) \vee b_k) \wedge (p_{ij}^m(b_1, b_2, \ldots, b_n) \vee b_k') > p_{ij}^m(b_1, b_2, \ldots, b_n)$. Since L has only n elements it follows that $p_{ij}^n(b_1, b_2, \ldots, b_n) = 1$ and hence $p(b_1, b_2, \ldots, b_n) = 1$, completing the proof.

PROPOSITION 6. *Let L be an irreducible OML having exactly n elements $a_1, a_2, \ldots, a_n$. Then there exists an OL-polynomial $q(x_1, x_2, \ldots, x_n)$ with the properties*:

1. *If $b_1, b_2, \ldots, b_n$ are elements of any OML and if $b_i = b_j$ for some distinct indices i, j then $q(b_1, b_2, \ldots, b_n) = 0$.*
2. *If $b_1, b_2, \ldots, b_n \in L$ then $q(b_1, b_2, \ldots, b_n) = 1$ iff the map $a_i \to b_i$ is an automorphism of L and $q(b_1, b_2, \ldots, b_n) = 0$ otherwise.*

Proof. Define $r(x_1, x_2, \ldots, x_n) = \bigwedge\{(x_i \wedge x_j \wedge x_k) \vee ((x_i \wedge x_j)' \wedge x_k') \mid a_i \wedge a_j = a_k\} \wedge \bigwedge\{(x_i \wedge x_j') \vee (x_i' \wedge x_j) \mid a_i = a_j'\}$.

Clearly $r(b_1, b_2, \ldots, b_n) = 1$ means that $(b_i \wedge b_j \wedge b_k) \vee ((b_i \wedge b_j)' \wedge b_k') = 1$ and hence $b_i \wedge b_j = b_k$ whenever $a_i \wedge a_j = a_k$ and that $(b_i \wedge b_j') \vee (b_i' \wedge b_j) = 1$ and hence $b_i = b_j'$ where $a_i = a_j'$ and hence that $a_i \to b_i$ is a homomorphism. So far the b_i can be elements of an arbitrary OML. Again define recursively the polynomial r^m by $r^1 = r$ and $r^{m+1} = \bigwedge_{k=1}^n ((r^m \wedge x_k) \vee (r^m \wedge x_k'))$. Clearly $r^{m+1}(x_1, x_2, \ldots, x_n) \leqslant r^m(x_1, x_2, \ldots, x_n)$. With the polynomial p as in Proposition

3 define $q = p \wedge r^n$. Clearly q satisfies the first condition. Assume $b_1, b_2, \ldots, b_n \in L$. If the map $a_i \to b_i$ is not one-one then, by Proposition 5, $p(b_1, b_2, \ldots, b_n) = 0$ and hence $q(b_1, b_2, \ldots, b_n) = 0$. We may thus assume that the map $a_i \to b_i$ is one-one and hence that $p(b_1, b_2, \ldots, b_n) = 1$. Therefore the map $a_i \to b_i$ is an automorphism of L iff $r(b_1, b_2, \ldots, b_n) = 1$. If $r(b_1, b_2, \ldots, b_n) = 1$ then clearly $r^n(b_1, b_2, \ldots, b_n) = 1$ and hence $q(b_1, b_2, \ldots, b_n) = 1$. If $r(b_1, b_2, \ldots, b_n) \neq 1$ then clearly $r^m(b_1, b_2, \ldots, b_n) \neq 1$ for all m. If $r^m(b_1, b_2, \ldots, b_n) \neq 0$ then, as before, there exists k such that b_k does not commute with $r^m(b_1, b_2, \ldots, b_n)$. Then $r^{m+1}(b_1, b_2, \ldots, b_n) \leqslant (r^m(b_1, b_2, \ldots, b_n) \wedge b_k) \vee (r^m(b_1, b_2, \ldots, b_n) \wedge b_k') < r^m(b_1, b_2, \ldots, b_n)$. It follows that $r^n(b_1, b_2, \ldots, b_n) = 0$ and hence $q(b_1, b_2, \ldots, b_n) = 0$, completing the proof.

4. Subdirect Powers of a Finite Simple *OML*

If L is an OML and X a set let $S \leqslant L^X$ mean that S is a subalgebra of L^X and let $S \leqslant L^X$ (subdirect) mean that in addition every projection pr_x maps S onto L. Throughout this section we assume that L is a finite simple OML. The reader is reminded that a finite OML is simple iff it is (directly) irreducible iff it is subdirectly irreducible. Constant functions will be distinguished from their singleton image by context only.

LEMMA 1. *If $S \leqslant L^X$ and if S contains a subalgebra L_0 isomorphic with L then there exists an OML S_0 isomorphic with S such that $S_0 \leqslant L_0^X$ and S_0 contains all constant functions.*

Proof. Since L, L_0 are finite and simple, for each $x \in X$ the restriction $\mathrm{pr}_x \mid L_0$ of pr_x to L_0 is an isomorphism between L_0 and L. Let q_x be its inverse. Define $\varphi : L^X \to L_0^X$ by $\varphi(f)(x) = q_x(f(x))$. Then φ is obviously an isomorphism between L^X and L_0^X. Define $S_0 = \varphi(S)$. It is easily checked that S_0 has the desired property, the constant functions being $\{\varphi(f) \mid f \in L_0\}$.

PROPOSITION 7. *Assume $S \leqslant L^X$ and that S contains all constant functions. Define $\mathscr{F} = \{f^{-1}(a) \mid f \in S, a \in L\}$. Then $\mathscr{F}$ is a field of subsets of X and $S = C(\mathscr{F}, L)$.*

Proof. If $f \in L^X$ and $a \in L$ then $f^{-1}(a) = ((f \vee a) \wedge (f' \vee a'))^{-1}(0)$. thus $\mathscr{F} = \{f^{-1}(0) \mid f \in S\}$. Since $f^{-1}(1) = (f')^{-1}(0)$ we also have $\mathscr{F} = \{f^{-1}(1) \mid f \in S\}$. Hence for each $A \in \mathscr{F}$ we may thus assume that $A = f^{-1}(1)$ for some $f \in S$. Define recursively polynomials g_i by $g_1 = f$ and $g_{i+1} = \bigwedge_{a \in L} ((g_i \wedge a) \vee (g_i \wedge a'))$. If $g_i(x) \neq 0, 1$ then there exists $a \in L$ such that $aCg_i(x)$ fails, thus $g_{i+1}(x) \leqslant (g_i(x) \wedge a) \vee (g_i(x) \wedge a') < g_i(x)$. Thus (being conservative) if n is the number of elements of L then $g_n = \chi_A \in S$. Since $A \cup B = (\chi_A \vee \chi_B)^{-1}(1)$, $\mathscr{F}$ is closed under finite unions and since $(f^{-1}(1))' = \cup_{a \neq 1} f^{-1}(a)$, $\mathscr{F}$ is closed under complements and hence is a field of sets. Since $f = \bigvee_{a \in L} (a \wedge \chi_{f^{-1}(a)})$ it follows that $S = C(\mathscr{F}, L)$.

PROPOSITION 8. *If $S \leqslant L^X$ (subdirect) then the MacNeille completion of S belongs to the variety generated by L.*

Proof. Let M be a maximal set of pairwise orthogonal central elements α of S with the property: the interval $[0, \alpha]$ in S is isomorphic with a subalgebra of some power L^{X_α} and contains an isomorphic copy of L as a subalgebra. Using Lemma 1, Proposition 7 and the corollary to Proposition 4, the MacNeille completion of each interval $[0, \alpha]$ with $\alpha \in M$ belongs to the variety generated by L. The claim thus follows from Proposition 2 if we can show that $\bigvee M = 1(\bigvee$ in $S)$. If this was not the case there would exist an upper bound $u < 1$ of M in S and there would exist $x \in X$ such that $u(x) < 1$. Let $a_1, a_2, \ldots, a_n$ be the elements of L and let q be the polynomial with the properties stated in Proposition 6. Since $S \leqslant L^X$ (subdirect) we may choose for every $i = 1, 2, \ldots, n$ an element $f_i \in S$ such that $f_i(x) = a_i$ and we may assume that u is one of the f_i. Define $\beta = q(f_1, f_2, \ldots, f_n)$. Since q takes in L the values 0, 1 only, β is central in S. Define $Y = \beta^{-1}(1)$ and $S_0 = \{f|Y| f \in S, f \leqslant \beta\}$. Clearly S_0 is isomorphic with the interval $[0, \beta]$ in S and $S_0 \leqslant L^Y$. By Proposition 6, for every $y \in Y$, the map $a_i \to f_i(y)$ is an automorphism of L. It follows that the map $a_i \to f_i \mid Y$ is an OL-embedding of L into S_0. We show that β is orthogonal with every $\alpha \in M$. If $\alpha(y) = 0$ clearly $\alpha(y) \leqslant \beta'(y)$. If $\alpha(y) \neq 0$ then, since α is central, $\alpha(y) = 1$ and hence $u(y) = 1$; since $u(x) \neq 1$ the map $a_i \to f_i(y)$ is not an automorphism and, again by Proposition 6, $\beta(y) = q(f_1, f_2, \ldots, f_n)(y) = 0$ and $\alpha(y) \leqslant \beta'(y)$. Thus β is orthogonal with all $\alpha \in M$. But $\beta(x) = 1 \nleqslant u(x)$ implies $\beta \notin M$, contradicting the maximality of M. Thus $\bigvee M = 1$, completing the proof.

5. Proof of the Theorem

Extending an earlier notation we define $S \leqslant \Pi_{x \in X} L_x$ (subdirect) to mean that S is a subalgebra of $\Pi_{x \in X} L_x$ and that the projections pr_x map S onto L_x. We first prove:

LEMMA 2. *Assume $S \leqslant \Pi_{x \in X} L_x$ (subdirect), where the L_x come from a finite set $\mathscr{K} = \{k_1, k_2, \ldots, k_m\}$ of finite irreducible OMLs. For $k = 1, 2, \ldots, m$ define $Y_k = \{x \in X \mid L_x = K_k\}$. Assume furthermore that for every k and every $y \in Y_k$ there exists $f \in S$ such that $f \mid \cup_{j \neq k} Y_j = 0$ and $f(y) \neq 0$. Then the MacNeille completion of S belongs to the variety generated by $\mathscr{K}$.*

Proof. We show first:

(*) For every $k \in \{1, 2, \ldots, m\}$ and every $y \in Y_k$ there exists $g \in S$ such that $g \mid \cup_{j \neq k} Y_j = 0$ and $g(y) = 1$.

By assumption there exists $f \in S$ such that $f \mid \cup_{j \neq k} Y_j = 0$ and $f(y) \neq 0$. Let $a_1 = 0, a_2 = f(y), a_3, \ldots, a_n$ be the elements of K_k and let p be a polynomial as in Proposition 5 for the OML K_k. For $i = 1, 2, \ldots, n$ choose $f_i \in S$ such that $f_i(y) = a_i$. We do this in such a way that f_1 is the zero-function and $f_2 = f$. Define $g = p(f_1, f_2, \ldots, f_n)$. If $x \in \cup_{j \neq k} Y_j$ then $f_1(x) = f_2(x) = 0$. It follows from the first condition of Proposition 5 that $g(x) = 0$ and hence that $g \mid \cup_{j \neq k} Y_j = 0$. It follows from the second condition of Proposition 1 that $g(y) = 1$, proving (*). Define now

$$S_0 = \left\{ f \in \prod_{x \in X} L_x \mid \text{for every } k \text{ there exists } g_k \in S \text{ such that } g_k \mid Y_k = f \mid Y_k \right\}.$$

Clearly S_0 is a subalgebra of $\Pi_{x\in X} L_x$ and S is a subalgebra of S_0. We show that S is join-dense in S_0. For this it is obviously enough to show that for every $f \in S_0, y \in X$ there exists $\alpha \in S$ such that $\alpha \leqslant f$ and $\alpha(y) = f(y)$. Assume $y \in Y_k$. By (*) there exists $g \in S$ such that $g \mid \cup_{j \neq k} Y_j = 0$ and $g(y) = 1$. By definition of S_0 there exists $g_k \in S$ such that $g_k | Y_k = f | Y_k$. The function $\alpha = g_k \wedge g$ obviously has the desired properties proving that S is join-dense in S_0. For $k = 1, 2, \ldots, m$ define $S_k = \{f | Y_k | f \in S\}$. Clearly S_0 is isomorphic with $\Pi_{k=1}^{m} S_k$. but $S_k \leqslant K_k^{Y_k}$ (subdirect). The claim thus follows from Proposition 1 and Proposition 8.

We are now in a position to give a proof of our theorem. If the variety $\mathscr{K}$ is generated by a finite OML then it follows from Jónsson [10] that $\mathscr{K}$ contains (up to isomorphism) only finitely many subdirectly irreducibles $K_1, K_2, \ldots, K_m$ and that these are all finite. Thus, if $S \in \mathscr{K}$ we may assume by Birkhoff's subdirect representation theorem that $S \leqslant \Pi_{x\in X} L_x$ (subdirect) and that the L_x come from $\{K_1, K_2, \ldots, K_m\}$. For $k = 1, 2, \ldots, m$ define $X_k = \{x \in X \mid L_x = K_k\}$. We define recursively subsets Y_k of X_k with the properties.

1. If $f, g \in S, f \neq g$ there exists $x \in Y_1 \cup \cdots \cup Y_k \cup X_{k+1} \cup \cdots \cup X_m$ such that $f(x) \neq g(x)$.
2. If $y \in Y_k$ there exists $f \in S$ such that $f \mid Y_1 \cup \cdots \cup Y_{k-1} \cup X_{k+1} \cup \cdots \cup X_m = 0$ and $f(y) \neq 0$.

Assume that $1 \leqslant k \leqslant n$ and that the Y_j with the described properties have been constructed if $1 \leqslant j < k$. Define an element $y \in X_k$ to be redundant if whenever the restriction of an $f \in S$ to $Y_1 \cup \cdots \cup Y_{k-1} \cup X_{k+1} \cup \cdots \cup X_n$ is 0 then $f(y) = 0$. Let Y_k consist of all $y \in X_k$ which are not redundant. We show that the two conditions are satisfied for this k. This is obvious for the second condition. To show it for the first condition assume f, $g \in S$ and $f \neq g$. By inductive hypothesis there exists $x \in Y_1 \cup \cdots \cup Y_{k-1} \cup X_k \cup \cdots \cup X_m$ such that $f(x) \neq g(x)$. If this is true for some $x \notin X_k$ then Condition 1 is trivially satisfied. If not, the restriction of $(f \vee g) \wedge (f' \vee g')$ to $Y_1 \cup \cdots \cup Y_{k-1} \cup X_{k+1} \cup \cdots \cup X_m$ is 0 and there exists $y \in X_k$ such that $f(y) \neq g(y)$ and hence $((f \vee g) \wedge (f' \vee g'))(y) \neq 0$. it follows that $y \in Y_k$ and hence Condition 1 is satisfied. Define $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_m$ and let S_0 be the projection of S into $\Pi_{y\in Y} L_y$. Then $S \simeq S_0 \leqslant \Pi_{y\in Y} L_y$ (subdirect) and the last subdirect representation has the property assumed in Lemma 2. It follows that the MacNeille completion of S belongs to $\mathscr{K}$, completing the proof.

6. Concluding Remarks

Our results are obviously only a first step in developing a theory of completions of OMLs. The most pressing problem to us seems to be: Can every OML L be embedded as a sub-OL into a complete OML C? Because of the frequency with which completeness is assumed in axiomatic treatments of the foundations of quantum mechanics, e.g. [14], a positive resolution to the above question could widen such developments. There are, of course, several versions of this embedding problem. We can require that L is just a sub-OL of C or that all or some specified sets of joins and meets in L are preserved, or that L admits a full or strong set of states as in a quantum logic. The answer to none of these questions seems to be known.

The example of a modular OL L the MacNeille completion of which is not an OML which we gave in the introduction does not provide a counter-example to this more general embedding problem. The incomplete inner product space V we used to construct L can first be completed. Then, as is not difficult to prove, L can be embedded into the lattice L_1 of finite-dimensional or cofinite-dimensional subspaces of the completion $\bar{V}$ of V. By the theorem of Amemiya–Araki–Piron the MacNeille completion of L_1 is an OML and L is isomorphic with a sub-OL of it.

The simplest unsolved case dealing with the MacNeille completion is the case of the variety $\mathscr{K}$ generated by $MO\omega$, the modular OL consisting of countably many pairwise incomparable elements and the bounds. $\mathscr{K}$ can also be described as the variety generated by all finite modular OLs. It seems that our methods require considerable refinement to deal with this case.

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