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Quasi-atoms in symmetric orthomodular lattices

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In this paper, we initiate the study of atom-like elements in symmetric orthomodular lattices. A *quasi-atom* is a nonzero element a of L such that $[0, a]$ is a Boolean lattice and a forms a modular pair with every element of L. In a symmetric orthomodular lattice, the set of quasi-atoms, together with 0, is an order ideal closed under perspectivity. We show that an orthomodular lattice with height greater than 2 having a quasi-atom is a projective geometry if and only if it is simple and symmetric. We give a reasonable condition involving modularity which forces a symmetric orthomodular lattice with only finitely many commutators to have height no greater than 2. We investigate the connections between subdirect irreducibility and hyperirreducibility and prove that, for a symmetric orthomodular lattice having a quasi-atom, the two concepts coincide. Finally, we characterize the atoms of symmetric orthomodular lattices using quasi-atoms. A preliminary version of this paper was presented to the NIH Conference on Universal Algebra and Lattice Theory in Washington, D.C., during the summer of 1986.

1. Quasi-atoms

Throughout this paper, (L) will denote an orthomodular lattice, abbreviated OML. Undefined terms may be found in the recent reference book by G. Kalmbach [6].

We write $x \oplus y$ for $x \vee y$ if $x \perp y$ and $y - x$ for $x' \wedge y$ if $x \leq y$. For s and x in an OML, $s\phi_x$ is an abbreviation for the lattice polynomial $(s \vee x') \wedge x$; ϕ_x is called the *Sasaki projection* onto x; it is basic that Sasaki projections preserve joins (cf. [6], p. 143). As usual if elements x and y are perspective, we write $x \sim y$.

Many of the results of this paper arise from a generalization of the concept of an atom. If a is an atom of a lattice, then the interval $[0, a]$ is Boolean. As in the theory of von Neumann algebras, we shall call such elements *abelian*; the set of

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abelian elements of an OML L will be written as $A^{(0)}(L)$. Obviously $A^{(0)}(L)$ is an order ideal of L.

An atom a of L has the additional property that xMa for each $x \in L$; in words, a forms a modular pair with each $x \in L$. Here we are writing xMa to mean $(x \wedge a) \vee a_0 = (x \vee a_0) \wedge a$ for each $a_0 \leq a$. (We also write xM^*y to mean that xMy in the lattice dual to L.)

With these two attributes of atoms in mind, we call a nonzero element a of an OML L a *quasi-atom* if a is abelian and xMa for each $x \in L$. We denote the set of quasi-atoms of L, together with 0, by $A^{(1)}(L)$. It is easy to verify that $A^{(1)}(L)$ is an order ideal of L.

Following M. F. Janowitz [5], for elements x and y in L, we define $xS^{(0)}y$ to mean x and y have no nonzero strongly perspective subelements, $xS^{(1)}y$ to mean x and y have no nonzero perspective subelements, and $xS^{(\infty)}y$ to mean x and y have no nonzero projective subelements. Clearly $L \times \{0\} \cup \{0\} \times L \subseteq S^{(\infty)} \subseteq S^{(1)} \subseteq S^{(0)} \subseteq \perp$. Janowitz proved that $S^{(0)} = S^{(1)}$ if and only if $S^{(0)} = S^{(\infty)}$ [5], Theorem 3.15. In what follows these separation relations play a crucial role. It is important that the reader know the following facts: $xS^{(0)}y$ is equivalent to $x \wedge y = 0$ and $x \in \mathrm{Cen}[0, x \vee y]$, cf. [5], Theorem 3.10; and $S^{(1)}$ coincides with F. Maeda's ∇ relation (recall that $x \nabla y$ means $(x \vee z) \wedge y = z \wedge y$ for each $z \in L$, or equivalently $x \vee t = 1$ implies $y \leq t$, cf. [6], Lemma 2.8.2).

Abelian elements can be characterized using the $S^{(0)}$ relation and quasi-atoms can be characterized using the $S^{(1)}$ relation. This is the content of our first two propositions.

PROPOSITION 1.1. *Let a be an element of an* OML *L. Then these conditions are equivalent*:

(1) $a \in A^{(0)}(L)$.

(2) *Orthogonal subelements of a are in the relation* $S^{(0)}$.

(3) *a contains no nonzero orthogonal strongly perspective subelements.*

(4) *Each subelement of a is in the relation* $S^{(0)}$ *to its relative orthocomplement in* $[0, a]$.

Proof. Suppose that (1) holds and that a_1 and a_2 are orthogonal subelements of a. Then since $A^{(0)}(L)$ is an order ideal, we have $a_1 \oplus a_2 \in A^{(0)}(L)$. It follows that $a_1 \in \mathrm{Cen}[0, a_1 \oplus a_2]$ and, therefore, $a_1 S^{(0)} a_2$ by [5], Theorem 3.10

Now assume that (2) holds. If a_1 and a_2 were nonzero orthogonal strongly perspective subelements of a, then $a_1 \not S^{(0)} a_2$, by the definition of $S^{(0)}$. Thus (3) holds.

Now assume that (3) holds and let a_0 be a subelement of a. If $a_0 \not S^{(0)} a - a_0$, then a_0 and $a - a_0$ have nonzero strongly perspective subelements, which contradicts (3). Thus $a_0 S^{(0)} a - a_0$.

Finally, assume that (4) holds and let a_0 be a subelement of a. Then $a_0 S^{(0)} a - a_0$ implies $a_0 \in \text{Cen}[0, a_0 \oplus (a - a_0)] = \text{Cen}[0, a]$. Thus $a \in A^{(0)}(L)$.

PROPOSITION 1.2. *Let a be an element of an OML L. Then these conditions are equivalent:*

(1) $a \in A^{(1)}(L)$.

(2) *Orthogonal subelements of a are in the relation $S^{(1)}$.*

(3) *a contains no nonzero orthogonal perspective subelements.*

(4) *Each subelement of a is in the relation $S^{(1)}$ to its relative orthocomplement in $[0, a]$.*

Proof. Suppose that (1) holds and that a_1 and a_2 are orthogonal subelements of a. Then since $A^{(1)}(L)$ is an order ideal, we have $a_1 \oplus a_2 \in A^{(1)}(L)$. Let $x \in L$. Then $xMa_1 \oplus a_2$ and $[0, a_1 \oplus a_2]$ Boolean imply

$$(x \vee a_1) \wedge a_2 = (x \vee a_1) \wedge (a_1 \oplus a_2) \wedge a_2 = \{[x \wedge (a_1 \oplus a_2)] \vee a_1\} \wedge a_2$$
$$= [x \wedge (a_1 \oplus a_2) \wedge a_2] \vee (a_1 \wedge a_2) = x \wedge a_2.$$

Thus $a_1 S^{(1)} a_2$.

The proofs of (2) implies (3) and (3) implies (4) are similar to those of Proposition 1.1.

Finally, suppose that (4) holds. Then, of course, (4) of Proposition 1.1 holds also and so $a \in A^{(0)}(L)$. It remains to show that xMa for each $x \in L$. To that end, let $x \in L$. Consider any element a_0 with $x \wedge a \le a_0 \le a$. Then, by (4), we have $a_0 S^{(1)} a - a_0$. Hence, by the Foulis–Holland theorem (see [6], Theorem 1.3.5), we may compute

$$(x \vee a_0) \wedge a = (x \vee a_0) \wedge [(a - a_0) \oplus a_0] = [(x \vee a_0) \wedge (a - a_0)] \oplus a_0$$
$$= [x \wedge (a - a_0)] \oplus a_0 = (x \wedge a \wedge a_0') \oplus a_0 = a_0.$$

Thus xMa. We have proved $a \in A^{(1)}(L)$.

It is not difficult to verify that the word "orthogonal" can be replaced by the word "disjoint" in the statements of (2) and (3) in Propositions 1.1 and 1.2. By disjoint elements we mean, of course, elements whose meet is 0.

We write $u\mathring{M}v$ to mean $u \wedge v = 0$ and uMv. It is well known that $\perp \subseteq \mathring{M}$ (cf. [8], Theorem 29.13); in words, orthogonal pairs are modular pairs. By [8], Lemma 1.5, if $u\mathring{M}v$ then $u_0\mathring{M}v_0$ for each $u_0 \le u$ and each $v_0 \le v$.

The usefulness of quasi-atoms is made possible by the following fundamental lemma.

LEMMA 1.3. *Let x and y be elements of an* OML. *If* $y\mathring{M}x'$, *then* $S^{(1)}x \subseteq S^{(1)}y$.

Proof. Let $w \in S^{(1)}x$. Suppose w_0 and y_0 are subelements of w and y, respectively, and that $t : w_0 \sim y_0$. Then $w_0 S^{(1)}x$ and $w_0 \vee t = 1$ imply $x \leq t$, by [6], Lemma 2.8.2. Moreover, the orthomodular identity implies that $x \vee (t - x) \vee y_0 = t \vee y_0 = 1$. Thus $w \leq (t - x) \vee y_0$. Since yMx', we have also $y_0 Mx'$ by [8], Lemma 1.5. We may thus compute $w \leq [(t - x) \vee y_0] \wedge x' = (t - x) \vee (y_0 \wedge x') = t - x \leq t$. Then $t = w \vee t = 1$ and so we have $w_0 = w_0 \wedge t = 0 = y_0 \wedge t = y_0$. This shows that $wS^{(1)}y$.

2. Irreducibility conditions

A non-empty subset of an OML is called a *subperspectivity set* if it contains every element of the lattice which is perspective to a subelement of one of its elements.

We leave the following result as an exercise for the reader.

LEMMA 2.1. *In any* OML, *these conditions on a non-empty subset S are equivalent*:

(1) *S is a subperspectivity set.*

(2) $s\phi_x \in S$ *for each* $s \in S$ *and each* $x \in L$.

(3) *If* $s \in S$ *and* $s' \wedge y = 0$, *then* $y \in S$.

(4) *S is an order ideal and is closed under perspectivity.*

The next result justifies our interest in subperspectivity sets.

PROPOSITION 2.2. *The ideal generated by a subperspectivity set is a p-ideal. It consists of the joins of finite orthogonal subsets of the subperspectivity set. Moreover, the center of a subperspectivity set lies in the center of the lattice.*

Proof. Let S be a subperspectivity set. The lattice ideal $I(S)$ generated by S consists of all subelements of joins of finite subsets of S, but if $x \leq \bigvee s_i$ with $s_i \in S$, then we have $x = (\bigvee s_i)\phi_x = \bigvee s_i \phi_x$ with $s_i \phi_x \in S$. Thus $I(S)$ consists of all joins of finite subsets of S. Using the fact, once again, that Sasaki projections preserve joins, we see that, by Lemma 2.1, $I(S)$ is also a subperspectivity set. That the elements of $I(S)$ can be expressed as joins of finite orthogonal subsets of S is a consequence of the identity $s \vee t = s\phi_{t'} \vee t$ and a simple induction argument.

By our observations above, we have $C(S) = C(I(S))$. Then by [4], Lemma 10, we see that $\text{Cen}\, S := S \cap C(S) \subseteq I(S) \cap C(I(S)) \subseteq C(L)$.

In the sequel we call a relation R *degenerate* in case $R \subseteq L \times \{0\} \cup \{0\} \times L$. Note that, for $R \in \{S^{(0)}, S^{(1)}, S^{(\infty)}\}$, R is degenerate if and only if equality holds above.

LEMMA 2.3. *Assume that* $S^{(\infty)}$ *is degenerate on an* OML L. *Then*
(1) *every nonzero subperspectivity set is join dense,*
(2) *each atom of* L *is in each nonzero subperspectivity set, and*
(3) *all atoms are mutually projective.*

Proof. Let S be a nonzero subperspectivity set. To prove (1), consider any nonzero x in L. Choose any nonzero element s of S. By hypothesis, $x \not S^{(\infty)} s$. Thus x and s have nonzero subelements x_0 and s_0 which are projective. Thus $x_0 \in S$. It follows that S is join dense. To prove (2), consider an atom a of L. By (1), there exists a nonzero member s of S with $s \leq a$. Thus $a = s \in S$.

Finally, if a and b are atoms then $a \not S^{(\infty)} b$. Thus a and b dominate nonzero projective subelements, which of necessity must be a and b themselves.

We proceed to investigate subdirectly irreducible orthomodular lattices.

PROPOSITION 2.4. *Let* L *be an* OML. *If* L *is subdirectly irreducible, then* $S^{(\infty)}$ *is degenerate. If* L *has an atom, the converse also holds.*

Proof. A careful reading of G. Bruns's proof of (1.1) in [1] shows that $S^{(\infty)}$ is degenerate whenever L is subdirectly irreducible. (His paper is about modular ortholattices but his proof of (1.1) never invokes modularity.)

To prove the converse, let a be any atom of L. By Lemma 2.3, a lies in every nonzero p-ideal; thus the p-ideal generated by a is the smallest nonzero p-ideal of L.

Proposition 2.4 has the following immediate consequence. If L is a subdirectly irreducible OML and $S^{(1)} = S^{(\infty)}$, then $S^{(1)}$ is degenerate. If L has a quasi-atom, the converse also holds; this is because, by (5) of Proposition 1.2, $S^{(1)}$ degenerate implies that quasi-atoms are in fact atoms; and, of course, $S^{(1)}$ degenerate implies that $S^{(1)} = S^{(\infty)}$.

An orthomodular lattice is *hyperirreducible* if every interval is irreducible. Since in an OML, whenever $x_0 \leq x$, the mapping $x_1 \mapsto x_1 - x_0$ is an orthoisomorphism of $[x_0, x]$ onto $[0, x - x_0]$, one need only consider principal ideals in order to verify hyperirreducibility. Obviously hyperirreducible and subdirectly irreducible OMLs are irreducible. The horizontal sum of 2^2 and 2^3 is a simple, and so subdirectly irreducible, OML which is not hyperirreducible. The lattice of closed subspaces of an infinite dimensional Hilbert space is hyperirreducible but not simple; it is, however, subdirectly irreducible. At present, we know of no hyperirreducible OML which is not subdirectly irreducible. The following result is one of the many characterizations of hyperirreducibility which appear in E. L. Marsden Jr. [9]. We provide a proof because the journal in which [9] appears is not easily obtained. Other results related to hyperirreducibility appear in [2], [3] and [5].

LEMMA 2.5. *L is hyperirreducible if and only if $S^{(0)}$ is degenerate.*

Proof. Assume that L is hyperirreducible and let $xS^{(0)}y$. Thus $x \wedge y = 0$ and $x \in \mathrm{Cen}[0, x \vee y]$. So $x = 0$ or $x = x \vee y$. If $y \leq x$, then $yS^{(0)}y$ and so $y = 0$. Thus $S^{(0)}$ is degenerate. Conversely, assume that $S^{(0)}$ is degenerate. Suppose L is not hyperirreducible so that there exists x_0 and x in L with $x_0 \in \mathrm{Cen}[0, x] = \mathrm{Cen}[0, x_0 \oplus (x - x_0)]$ and $0 < x_0 < x$. Then $x_0 S^{(0)} x - x_0$. So $x_0 = 0$ or $x_0 = x$, which is a contradiction.

An immediate consequence of Proposition 1.1 and Lemma 2.5 is that the nonzero abelian elements of a hyperirreducible OML are atoms.

PROPOSITION 2.6. *Let L be an* OML. *If L is subdirectly irreducible and $S^{(0)} = S^{(1)}$, then L is hyperirreducible. If L has a nonzero abelian element, the converse also holds.*

Proof. Assume L is subdirectly irreducible and $S^{(0)} = S^{(1)}$. By Proposition 2.4, $S^{(\infty)}$ is degenerate. By the result of M. F. Janowitz [5], Theorem 3.15, we have $S^{(0)} = S^{(\infty)}$ so that $S^{(0)}$ is degenerate. It follows from Lemma 2.5 that L is hyperirreducible.

Conversely, assume that L is hyperirreducible. As mentioned above, the nonzero abelian elements are atoms. By Lemma 2.5, $S^{(0)}$ is degenerate, which in turn implies that $L \times \{0\} \cup \{0\} \times L = S^{(\infty)} = S^{(1)} = S^{(0)}$. By Proposition 2.4 the argument is complete.

Since perspectivity and strong perspectivity agree in modular OMLs, cf. [6], Theorem 2.6.3, $S^{(0)} = S^{(1)}$ in modular OMLs. Thus, by Proposition 2.6, subdirectly irreducible modular OMLs are hyperirreducible. However, the horizontal sum of two atomless Boolean lattices is a simple symmetric OML which is not hyperirreducible. In the next section, we shall show that the hypothesis $S^{(0)} = S^{(1)}$ can be omitted from the statement of Proposition 2.6 for symmetric OMLs with quasi-atoms.

3. Symmetric OMLs

A principal result of this paper is that $A^{(1)}(L)$ is a subperspectivity set in any symmetric OML L. By a *symmetric* OML we mean, of course, an OML in which the relation of being a modular pair is symmetric. It is easy to see that an OML is M-symmetric if and only if it is M^*-symmetric. F. Maeda and S. Maeda [8] have studied extensively elements m of lattices with xMm for each x. We shall call such elements *Maeda-modular* and write $M(L)$ for the set of such elements in an OML L. It is easy to verify that if $m, n \in M(L)$, then $m \wedge n \in M(L)$. Note that quasi-atoms are Maeda-modular.

LEMMA 3.1. *$M(L)$ is a subalgebra of L whenever L is a symmetric* OML.

Proof. Let $m \in M(L)$. Then xMm for each $x \in L$ implies mMx for each $x \in L$; by [8], Lemma 1.2, it follows that mM^*x for each $x \in L$, that is, $m'Mx'$ for each $x \in L$ and thus $x'Mm'$ for each $x \in L$. We see, therefore, that $m' \in M(L)$ and so $M(L)$ is closed under orthocomplementation. This fact, together with the fact that $M(L)$ is closed under meets of finite subsets, shows that $M(L)$ is a subalgebra of L.

In any lattice with 0, $u\mathring{M}u$ and $w\mathring{M}u \vee v$ imply $u \vee w\mathring{M}v$ by [8], Lemma 1.6. We use this result in the following lemma. If L is an M-symmetric lattice with 0, then $\mathring{M}$ is obviously a symmetric relation.

LEMMA 3.2. *Let L be a symmetric* OML. *If $y_0 \leq y$ with $y\mathring{M}x'$, then $y_0' \wedge y\mathring{M}y_0 \vee x'$ and $y_0\mathring{M}(y_0' \wedge x) \oplus x'$.*

Proof. By symmetry we have $x'\mathring{M}y$. The orthomodular identity yields $y = y_0 \oplus (y_0' \wedge y)$. Then $x'\mathring{M}y_0 \oplus (y_0' \wedge y)$ and $y_0\mathring{M}y_0' \wedge y$ imply $y_0 \vee x'\mathring{M}y_0' \wedge y$. Then, by symmetry once again, we have $y_0' \wedge y\mathring{M}y_0 \vee x'$.

The second conclusion is proved similarly. In fact, $y_0\mathring{M}x'$ and $y_0' \wedge x\mathring{M}y_0 \vee x'$ imply $y_0 \oplus (y_0' \wedge x)\mathring{M}x'$. So, by symmetry, we have $x'\mathring{M}y_0 \oplus (y_0' \wedge x)$; this together with $y_0' \wedge x\mathring{M}y_0$ implies $(y_0' \wedge x) \oplus x'\mathring{M}y_0$. Then, by symmetry once again, we have $y_0\mathring{M}(y_0' \wedge x) \oplus x'$.

THEOREM 3.3. *$A^{(1)}(L)$ is a subperspectivity set whenever L is a symmetric* OML.

Proof. Let $a \in A^{(1)}(L)$ with $y \wedge a' = 0$. By Lemma 3.1, $a' \in M(L)$ and so $y\mathring{M}a'$. Let $y_0 \leq y$. By the previous lemma and Lemma 1.3, it follows that $S^{(1)}y_0\phi_a \subseteq S^{(1)}y_0$ and $S^{(1)}y_0' \wedge a \subseteq S^{(1)}y_0' \wedge y$. By Proposition 1.2, $a \wedge y_0' \in S^{(1)}y_0\phi_a$. Hence $a \wedge y_0' \in S^{(1)}y_0$ and, therefore, $y_0 \in S^{(1)}a \wedge y_0'$ so that $y_0 \in S^{(1)}y \wedge y_0'$. Hence, by

Proposition 1.2, $y \in A^{(1)}(L)$. Appealing to Lemma 2.1, we conclude that $A^{(1)}(L)$ is a subperspectivity set.

Easy examples show that Lemma 3.2 and Theorem 3.3 do not hold for arbitrary OMLs.

The following result shows that an OML with height greater than 2 having a quasi-atom is a projective geometry if and only if it is simple and symmetric. The proof utilizes the result of G. Bruns [1], Theorem 1, that every nonzero abelian element of a subdirectly irreducible *modular* OML is an atom.

COROLLARY 3.4. *Let L be a simple symmetric* OML *with* $A^{(1)}(L) \neq \{0\}$. *Then* $L = 2^1$, *a non-trivial horizontal sum of four-element Boolean lattices or a projective geometry.*

Proof. Because L is simple, $L = I(A^{(1)}(L))$, the lattice ideal generated by $A^{(1)}(L)$, by Theorem 3.3 and Proposition 2.2. Then L is modular by Lemma 3.1. Thus, by Bruns's result, $A^{(1)}(L)$ consists of 0 and atoms. By Proposition 2.2, each nonzero element of $L = I(A^{(1)}(L))$ is a join of a finite orthogonal set of atoms. It follows that L has finite height. If the height of $L \leq 2$, then $L = 2^1$ or is a non-trivial horizontal sum of four-element Boolean lattices; L cannot be 2^2 since L is irreducible. If the height of $L \geq 3$, then L is a projective geometry.

This work evolved from an entirely different direction. In [4], we investigated OMLs having only finitely many commutators; the *commutator* of elements x and y of an OML is the lattice polynomial $(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$. We showed that such OMLs are the direct product of a Boolean lattice and finitely many simple non-Boolean OMLs. Naturally, our interest focused on simple commutator-finite OMLs. We are now in a position to strengthen one of the principal results of that paper.

We write Mac(L) for the set consisting of those $m \in M(L)$ with $[0, m]$ a modular sublattice of L. Such elements were studied extensively in *complete* symmetric OMLs by M. D. MacLaren [7]. It is easy to verify that Mac(L) is an order ideal of L.

COROLLARY 3.5. *Let L be an irreducible symmetric commutator-finite* OML. *If* $\mathrm{Mac}(L) \neq \{0\}$, *then L is* 2^1 *or a non-trivial horizontal sum of four-element Boolean lattices.*

Proof. Since L is irreducible, it is simple by [4], Corollary 13. Let $0 \neq m \in \mathrm{Mac}(L)$. Since $A^{(0)}(L)$ is join dense by [4], Corollary 8, there is a nonzero element $a \in A^{(0)}(L)$ with $a \leq m$. Since, however, Mac(L) is an order ideal, we see that $a \in M(L)$. Thus $a \in A^{(1)}(L)$. Then, by Corollary 3.4, L is modular. The conclusion of the theorem is provided by [4], Theorem 16.

We return to the study of hyperirreducible and subdirectly irreducible OMLs. Following Maeda [8], we say that L has the *covering property* if $a \nleq x$ with a an atom implies that $a \vee x$ covers x. It follows from the aforementioned observation about isomorphic intervals in OMLs (which followed Proposition 2.4) that the set of atoms, together with 0, of an OML is a subperspectivity set if and only if the lattice has the covering property. This is because $[x, a \vee x]$ is orthoisomorphic to $[0, a\phi_{x'}]$ for each atom a and element x in L. Incidentally, this isomorphism result also demonstrates that $A^{(0)}(L)$ is a subperspectivity set if and only if $[x, a \vee x]$ is Boolean for each abelian element a and each element x in L with $a \nleq x$. Maeda [8] defines an *AC-lattice* to be an atomistic lattice with the covering property. Our next result characterizes hyperirreducible AC-OMLs.

THEOREM 3.6. *Let L be an* OML. *Then these conditions are equivalent*:
(1) *L is hyperirreducible, has the covering property and is atomic*;
(2) *L is subdirectly irreducible, symmetric and $A^{(1)}(L) \neq \{0\}$.*

Proof. Assume that (1) holds. Since L has atoms, $A^{(1)}(L) \neq \{0\}$. Moreover, by Proposition 2.6, L is subdirectly irreducible. Finally, since L has the covering property and is atomic, it is symmetric by [8], Theorem 30.2.

Now assume that (2) holds. By Proposition 2.6, to show that L is hyperirreducible, we need only show that $S^{(0)} = S^{(1)}$. By Theorem 3.3, $A^{(1)}(L)$ is a nonzero subperspectivity set. So, by Lemma 2.3, $A^{(1)}(L)$ is join dense in L. Now consider x and y in L with $x\mathcal{S}^{(1)}y$. Since $A^{(1)}(L)$ is join dense, there exist quasi-atoms a and b under x and y, respectively, with $a\mathcal{S}^{(1)}b$, cf. [6], Corollary 2.8.3. Then there exist quasi-atoms a_0 and b_0 under a and b, respectively, with $a_0 \sim b_0$. If t is an axis of perspectivity for a_0 and b_0, it is easy to verify, using the modularity properties of a_0 and b_0, that $t \wedge (a_0 \vee b_0)$ is an axis of strong perspectivity for a_0 and b_0. Thus $x\mathcal{S}^{(0)}y$ since x and y have nonzero strongly perspective subelements. We have shown that $S^{(0)} = S^{(1)}$. Thus L is hyperirreducible by Proposition 2.6; and, therefore, the nonzero abelian elements of L are atoms. In particular, $A^{(1)}(L)$ consists of atoms, together with the 0 element of L. Therefore L has the covering property. Finally, by Lemma 2.3 and Proposition 2.4, L is atomic since $A^{(1)}(L)$ which consists of atoms, together with 0, is a join dense subperspectivity set.

We remark that the covering property hypothesis in the statement of Theorem 3.6 can be replaced by the hypothesis that L is *semimodular* in the sense that x covers $x \wedge y$ if and only if $x \vee y$ covers x, cf. [8], Theorem 7.10. Moreover, an argument similar to (2) implies (1) of Theorem 3.6 shows that if Mac(L) is a join dense subset of an OML L, then $S^{(0)} = S^{(1)}$ and hence all the separation conditions coincide by [5], Theorem 3.15.

Our Theorem 3.6 may be regarded as a generalization of Bruns's result: The nonzero abelian elements of a subdirectly irreducible symmetric OML with at least one quasi-atom are atoms.

In order to motivate our final result, consider an atom a of an OML and a p-ideal J not containing it. For each $x \in J, x\phi_a \in J$. Hence $x\phi_a < a$ and so $x\phi_a = 0$. Therefore $a \perp x$. We have argued that, given any atom a and any p-ideal J, a is in J or is orthogonal to it.

THEOREM 3.7. *Let L be a symmetric* OML. *Let $a \in L$. Then a is an atom of L if and only if the following conditions hold*:

(1) $a \in A^{(0)}(L)$,

(2) *a dominates a quasi-atom, and*

(3) *for each p-ideal, a is either in the p-ideal or is orthogonal to it.*

Proof. By our observation above, an atom clearly has these properties. To prove the converse, assume that the three conditions hold. By Theorem 3.6, we may assume that L is subdirectly reducible. Thus we have a family L_i of subdirectly irreducible OMLs and a family of orthoepimorphisms $f_i : L \to L_i$ each with $\ker f_i \neq \{0\}$ and with $\bigcap \ker f_i = \{0\}$. If a were not an atom, there would exist an element a_0 with $0 < a_0 < a$ and an orthoepimorphism $f_j : L \to L_j$ with $a_0 f_j < af_j$. Since a is an abelian element of L, it is easy to see that af_j is a nonzero abelian element of L_j.

We concentrate on L_j. Our main task is to prove that L_j is symmetric. By Theorem 3.3, Proposition 2.2 and Lemma 3.1, $I(A^{(1)}(L))$, the ideal generated by $A^{(1)}(L)$, is a p-ideal of L which is included in $M(L)$. From the identity $xf_j\phi_{(yf_j)} = (x\phi_y)f_j$ and the fact that the elements of $I(A^{(1)}(L))$ are joins of finite subsets of $A^{(1)}(L)$, it follows that $I(A^{(1)}(L))f_j$ is a p-ideal of L_j which is included in $M(L_j)$. If $I(A^{(1)}(L))f_j = \{0\}$, then by (3), $a \perp I(A^{(1)}(L))$; this, however, contradicts (2). Thus $I(A^{(1)}(L))f_j \neq \{0\}$ and so, by Proposition 2.4 and Lemma 2.3, $I(A^{(1)}(L))f_j$ is a join dense p-ideal of L_j which is included in $M(L_j)$. Hence, by [8], Corollary 35.5, L_j is symmetric. Moreover, $I(A^{(1)}(L))f_j \neq \{0\}$ implies $A^{(1)}(L)f_j \neq \{0\}$, since f_j preserves joins; also it is easy to verify that $A^{(1)}(L)f_j \subseteq A^{(1)}(L_j)$. Thus $A^{(1)}(L_j) \neq \{0\}$.

From Theorem 3.6, then, we know that af_j is an atom of L_j. Thus $a_0 f_j = 0$ and so $a_0 \in \ker f_j$. However, $a \notin \ker f_j$; thus by (3) $a \perp a_0$, which contradicts $a_0 \neq 0$. We conclude that a must be an atom of L.

We close with a direct consequence of Theorem 3.7. Since AC-OMLs are M-symmetric, cf. [8], Theorem 30.2, an atom is the only nonzero abelian element of an AC-OML with the property that, for each p-ideal it is either in the p-ideal or is orthogonal to it.

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