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Sites and Tours in Orthoalgebras and Orthomodular Lattices

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Introduction.

Orthoalgebras (defined below) play an important role in the empirical logic approach to the foundations of quantum mechanics initiated by D.J. Foulis and C.H. Randall. Such an algebra appears in that theory as the (empirical) logic of associated with a manual of operations [6]. The logic of a manual is an orthoalgebra and every orthoalgebra is isomorphic to the logic of some manual. The standard quantum logic of all closed subspaces of a Hilbert space and Boolean algebras are orthoalgebras, in fact, they are the prototypical examples. Any orthomodular lattice or orthomodular poset is also an orthoalgebra (in the sense that the axioms may be verified). We give reasons below why we consider orthoalgebras to be a viable generalization of the previously studied models for quantum logic.

We introduce the notions of sites and tours in orthoalgebras. A site is the intersection of a pair of distinct blocks. A tour is a certain type of subset of the set of all blocks of L. The main features of a tour are that the union of a tour is a subalgebra of L, and each minimal tour in a site finite orthoalgebra has the property that its union is block finite. From this it follows that every site finite orthoalgebra L can be covered by block finite subalgebras each of which has the same center as L.

This allows us to lift results from the class of block finite OMLs to the class of site finite OMLs. For example, it follows that site finite orthomodular lattices are path connected, and every finitely generated site finite OML is finite.

1. Sites and Tours in an Orthoalgebra..

An *orthoalgebra* (called an *associative orthoalgebra* in [10], [11, [12]) is a set L containing distinguished elements $0, 1$ on which there is a partially defined binary operation $\oplus$ satisfying, for all elements $p, q, r \in L$, the following four conditions [8]:

(1) (Commutativity) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(2) (Associativity) If $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined, then $q \oplus r$ is defined, $p \oplus (q \oplus r)$ is defined, and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

(3) (Orthocomplementation) For each $p \in L$ there is a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(4) (Consistency) If $p \oplus p$ is defined, then $p = 0$.

Foulis and Randall have argued in [7] that a reasonable notion of tensor product could not be defined in such a way that the class of orthomodular lattices (or orthomodular posets) is closed under the formation of tensor products. In the same paper they prove that orthoalgebras are closed under the tensor product which they introduced. Actually they define tensor products for manuals but this tensor product can be lifted to a tensor product of orthoalgebras as follows: given two orthoalgebras L_1 and L_2, for $i = 1, 2$ let $\mathfrak{M}_i$ be the manual of all finite partitions of unity in L_i and define the tensor product of the orthoalgebras L_i to be the logic of the tensor product of the manuals $\mathfrak{M}_i$. (We gloss over the question of the existence of the tensor product since it exists in the "nice" cases. And we ignore the fact that there are other manuals having the same logics L_i which have a different tensor product so that the above definition may not be the lasting one. The important fact is that there is a meaningful definition. For details see [7],[14] and [15].)

Orthoalgebras provide what appears to be the minimal vehicle on which to efficaciously define and manipulate orthogonally additive measures or states. This has been done, for example, by Rüttimann in [17].

If L is an orthoalgebra and $p \in L$, then the unique element $q \in L$ corresponding to p in condition (3) above is denoted by $q = p'$ and is called the *orthocomplement* of p. If $p, q \in L$, we define $p \leq q$ to mean that there exists $r \in L$ such that $p \oplus r$ is defined and $p \oplus r = q$. It is not difficult to check that L is partially ordered by $\leq$, that $0 \leq p \leq 1$ holds for all $p \in L$, that $p \oplus q$ is defined if and only if $p \leq q'$, and that $(L, \leq, 0, 1, ')$ is an orthoposet whenever L is an orthoalgebra. (See, for example, [8].)

A subset A of an orthoalgebra L is called a *subalgebra* of L if $0, 1 \in A$ and, whenever $p, q \in A$ and $p \oplus q$ is defined in L, it follows that $p' \in A$ and $p \oplus q \in A$. A subalgebra of an orthoalgebra is, of course, an orthoalgebra in its own right. If, as a partially ordered set with 0 and 1, a subalgebra B of L is a Boolean algebra, we call B a *Boolean* subalgebra. A *block* in an orthoalgebra is a maximal Boolean subalgebra. Let $\mathfrak{A}_{\mathcal{L}}$ denote the set of all blocks of L. Since for each $p \in L$ $\{p, p', 0, 1\}$ always forms a Boolean subalgebra of L, $L = \bigcup \mathfrak{A}_L$.

A *site* is a subset S of L of the form $S = A \cap B$ where A and B are distinct blocks of L; A and B are said to *determine* the site. Let $\mathfrak{A}_L$ denote the set of all blocks of L, and let S_L denote the set of all sites of L. L is *block finite* (respectively, *site finite*) in case L has only finitely many blocks (respectively, sites). A subalgebra of L is *full* if the blocks of the subalgebra are blocks of L. L has the *block extension property* in case every finite set of blocks can be extended to a full block finite subalgebra of L, i.e. if $A_i \in \mathfrak{A}_L$ for $i = 1, \ldots, n$, then there exists a subalgebra L_1 of L such that L_1 is block finite, each $A_i \in \mathfrak{A}_{L_1}$, and $\mathfrak{A}_{L_1} \subseteq \mathfrak{A}_L$. (This property was introduced for orthomodular lattices in [5].)

For $A \in \mathfrak{A}_L$, let S_A denote the set of all sites determined by A, i.e.

$$S_A = \{A \cap B \mid B \in \mathfrak{A}_L \setminus \{A\}\}.$$

For $A, B \in \mathfrak{A}_L$, define $A \equiv B$ in case $S_A = S_B$. Clearly $\equiv$ is an equivalence relation on $\mathfrak{A}_{\mathcal{L}}$. We say that A is *site equivalent to* B in case $A \equiv B$.

LEMMA 1. Let A and B be blocks of the orthoalgebra L. Then $A \equiv B$ if and only if $A \cap C = B \cap C$ for all $C \in \mathfrak{A}_L \setminus \{A, B\}$.

PROOF. Assume $A \equiv B$ and let $D \in \mathfrak{A}_L \setminus \{A, B\}$. Since $A \cap D \in S_A = S_B$, there exists $D_1 \in A_L \setminus \{B\}$ with $A \cap D = B \cap D_1$ and hence $A \cap D = A \cap B \cap D$. Also $B \cap D \in S_B = S_A$ so there exists $D_2 \in \mathfrak{A}_L \setminus \{A\}$ with $B \cap D = A \cap D_2$ and hence $B \cap D = A \cap B \cap D$. It follows that $A \cap D = B \cap D$, so that the condition holds.

Now assume that the condition holds. Let $M \in S_A$ so that $M = A \cap C$ for some $C \in \mathfrak{A}_L \setminus \{A\}$. If $C = B$ then $M = A \cap B \in S_B$. Therefore, we may assume that $C \neq B$ so that, by assumption, $M = A \cap C = B \cap C \in S_B$. Hence $S_A \subseteq S_B$. By symmetry $S_A = S_B$ so that $A \equiv B$.

A *basic tour of* L is a subset $\mathcal{T} \subseteq \mathfrak{A}_L$ such that for all $A \in \mathfrak{A}_L$ there exists a unique $B \in \mathcal{T}$ with $A \equiv B$. A *tour* of L is a subset $\mathcal{T} \subseteq \mathfrak{A}_L$ which contains a basic tour of L. It is easy to see that a tour $\mathcal{T}$ satisfies the *block exchange property*: if $A \in \mathcal{T}$ and $B \in \mathfrak{A}_L$ with $A \equiv B$ then $(\mathcal{T} \setminus \{A\}) \cup \{B\}$ is also a tour of L. Later we will discuss a similar (and less immediate) block exchange property for paths of blocks.

A tour $\mathcal{T}$ is said to have multiplicity n in case $|(A/\equiv) \cap \mathcal{T}| \leq n$ for all $A \in \mathcal{T}$ and $|(B/\equiv) \cap \mathcal{T}| = n$ for some $B \in \mathcal{T}$. A tour is of *finite multiplicity* in case it has multiplicity n for some finite n. Let $\mu(\mathcal{T})$ denote the multiplicity of a tour $\mathcal{T}$. Note that the basic tours are precisely the tours of multiplicity 1.

A *selection function* for $\equiv$ is a map $\sigma : \mathfrak{A}_L/\equiv \to \mathfrak{A}_L$ with $\sigma(A/\equiv) \in A/\equiv$. Let $(\sigma_\alpha : \alpha \in I)$ be a family of selection functions for $\equiv$. Then $\bigcup \text{image}(\sigma_\alpha)$ is a basic tour for each $\alpha \in I$. Moreover, $X_I := \bigcup_{\alpha \in I} \text{image}(\sigma_\alpha)$ is a tour. X_I is a tour of finite multiplicity if I is a finite set. We shall prove that the union of every tour $\mathcal{T}$ is a subalgebra of L, and that this subalgebra is block finite whenever L is site finite and $\mathcal{T}$ has finite multiplicity. By the *center*, $Cen(L)$, of L we mean the intersection of all the blocks of L:

$$Cen(L) = \bigcap \mathfrak{A}_L.$$

The Cartesian product of orthoalgebras is again an orthoalgebra in the usual way, by defining operations componentwise. An orthoalgebra L is *irreducible* in case it cannot be written as a Cartesian product in a non-trivial way. If $x \in Cen(L)$ then $L[0,x] := \{y \in L | y \leq x\}$ is again an orthoalgebra. Moreover, $x \in Cen(L)$ if and only if $L[0,x]$ and $L[0,x']$ are (induced) orthoalgebras and L is isomorphic to $L[0,x] \times L[0,x']$. (See [13] for the details.) Thus L is irreducible if and only if $Cen(L) = \{0,1\}$.

PROPOSITION 2. Let $\mathcal{T}$ be a tour of L. Then
(1) $\bigcup \mathcal{S}_L \subseteq \bigcup \mathcal{T}$ and $\bigcap \mathcal{S}_L = \bigcap \mathcal{T}$,
(2) $\bigcup \mathcal{T}$ is a subalgebra of L,
(3) $\mathfrak{A}_{\bigcup \mathcal{T}} = \mathcal{T}$, and
(4) $Cen(L) = \bigcap \mathcal{T}$.

PROOF. (1) If $x \in \bigcup \mathcal{S}_L$, then there exist distinct $A, B \in \mathfrak{A}_L$ with $x \in A \cap B$ and there exists $A_1 \in \mathcal{T}$ with $A \equiv A_1$. Now $A \cap B \in S_A = S_{A_1}$ so that $A \cap B \subseteq A_1 \subseteq \bigcup \mathcal{T}$ and $x \in \bigcup \mathcal{T}$ which proves the first containment. To prove the second part, first note that if L is a Boolean algebra then $\mathcal{T} = \mathfrak{A}_L$ the equation follows from the fact that $\bigcap\{L\} = L = \bigcap \phi$. Thus we may assume that L is not Boolean and hence $|\mathcal{T}| \geq 2$ and $\mathcal{S}_L \neq \phi$. Hence $\bigcap \mathcal{T} = \bigcap\{A \cap B | A, B \in \mathcal{T}\} = \bigcap\{A \cap B | A, B \in \mathcal{T}$ and $A \neq B\} = \bigcap\{A \cap B | A, B \in \mathfrak{A}_L$ and $A \neq B\} = \bigcap \mathcal{S}_L$.

(2) Let $a, b \in \bigcup \mathcal{T}$ with $a \perp b$. We shall prove that $a \oplus b \in \bigcup \mathcal{T}$. There exist $A, B \in \mathcal{T}$ with $a \in A$ and $b \in B$. We may assume that $b \notin A$ and $a \notin B$. There exist $C \in \mathfrak{A}_L$ with $\{a, a \oplus b\} \subseteq C$ and there exists $C_1 \in \mathcal{T}$ with $C \equiv C_1$. We may assume that $C \notin \mathcal{T}$ so that $A, B \neq C$. Since $A \cap C, B \cap C \in S_C = S_{C_1}$, $a \in A \cap C \subseteq C_1$ and $b \in B \cap C \subseteq C_1$. Hence $a \oplus b = a \vee_{C_1} b \in C_1 \subseteq \bigcup \mathcal{T}$. Thus $\bigcup \mathcal{T}$ is closed under existing $\oplus$. Since it is also closed under $'$, it is a subalgebra.

(3) To prove that $\mathfrak{A}_{\bigcup \mathcal{T}} = \mathcal{T}$ we need only prove that $\mathfrak{A}_{\bigcup \mathcal{T}} \subseteq \mathcal{T}$. Let $A \in \mathfrak{A}_{\bigcup \mathcal{T}}$. Suppose $A \notin \mathcal{T}$. Now $A \subseteq A_1$ for some $A_1 \in \mathfrak{A}_L$ and $A_1 \notin \mathcal{T}$ (else $A, A_1 \in \mathfrak{A}_{\cup \mathcal{T}}$ would imply $A = A_1 \in \mathcal{T}$). There exists $B \in \mathcal{T}$ with $A_1 \equiv B$. Now $A \not\subseteq B$ (else $A, B \in \mathfrak{A}_{\bigcup \mathcal{T}}$ would imply $A = B \in \mathcal{T}$) so that there exist $x \in A \setminus B$ and $D \in \mathcal{T}$ with $x \in D$. Moreover

$D \neq A_1$ since $D \in \mathcal{T}$ and $A_1 \notin \mathcal{T}$, and $D \neq B$ since $x \in D \setminus B$. But, by Lemma 1, $x \in A_1 \cap D = B \cap D \subseteq B$, which is a contradiction. Hence $A \in \mathcal{T}$. The result follows.

(4) By part (1) we need only prove that $Cen(L) = \bigcap S_L$. The proof of this is quite similar to that of the second part of (1).

COROLLARY 3. An orthoalgebra L is irreducible if and only if the union of every (or, of any) tour of L is an irreducible subalgebra of L.

PROOF: This follows easily from parts (2), (3) and (4) of the Proposition.

THEOREM 4. If L is a site finite orthoalgebra, then L satisfies the block extension property.

PROOF. Let $\mathfrak{A} \subseteq \mathfrak{A}_L$ be a finite set of blocks of L. Then there exists a tour $\mathcal{T}$ with $\mu(\mathcal{T}) \leq |\mathfrak{A}|$ such that each $A_i \in \mathcal{T}$. Now $\mathcal{T}$ is the union of at most $|\mathfrak{A}|$ basic tours $\mathcal{T}_i$, $i = 1, \ldots, k$ with $k \leq |\mathfrak{A}|$. By Proposition 2 $\bigcup \mathcal{T}$ is a subalgebra of L. Moreover $\bigcup \mathcal{T}$ is block finite since $|\mathfrak{A}_{\bigcup \mathcal{T}}| \leq \sum_{i=1}^{k} |\mathfrak{A}_{\bigcup \mathcal{T}_i}| = \sum_{i=1}^{k} |\mathcal{T}_i| = \sum_{i=1}^{k} |\mathfrak{A}_L/\equiv| = k|\mathfrak{A}_L/\equiv| = k|\{S_A \mid A \in \mathfrak{A}_L\}| \leq k(2^{|S_L|}) < \infty$.

If A and B are distinct blocks of L with $A \cup B$ a subalgebra of L then we write $A \sim B$. Given blocks A and B in $\mathfrak{A}_L$, a *path* from A *to* B is a finite sequence $A_0, A_1, \ldots, A_n$ of blocks such that $A_0 = A$, $A_n = B$ and $A_{i-1} \sim A_i$ for each $i = 1, \ldots, n-1$. A path $A_0, A_1, \ldots, A_n$ is *proper* if $n = 1$ or $A_i \cap A_{i+1} \neq Cen(L)$ for each $i = 0, \ldots, n-1$. L is *path connected* in case there exists a proper path from A to B for all pairs of distinct blocks $A, B \in \mathfrak{A}_L$.

LEMMA 5. Let L be a path connected orthoalgebra and let $x \in L \setminus Cen(L)$. Then there exist blocks A, B of L with $A \sim B$ and $x \in A \setminus B$.

PROOF. Since $\bigcup \mathfrak{A}_L = L$ there exists a block D with $x \in D$ and, since $x \notin Cen(L)$ there exists a block E with $x \notin E$. Let $A_0, A_1, \ldots, A_n$ be a path from D to E. Let k be the least integer such that $x \notin A_k$. Then $A := A_{k-1}$ and $B := A_k$ satisfy the stated conditions.

PROPOSITION 6. If A, B and C are blocks of the orthoalgebra L with $A \equiv B$, $A \sim C$ and $C \neq B$ then $B \sim C$.

PROOF. Suppose not. Then $A \neq B$ and there exist $b \in B \setminus C$ and $c \in C \setminus B$ such that $b \oplus c \notin B \cup C$. There exist blocks D and E distinct from A and B with $\{b, b \oplus c\} \subseteq D$ and $\{c, b \oplus c\} \subseteq E$. Now $A \neq E$; for, if $A = E$ then, by Lemma 1, $c \in E \cap C = A \cap C = B \cap C \subseteq B$ which contradicts the fact that $c \notin B$. Also $A \neq D$; for, if $A = D$ then, by Lemma 1 again since $E \neq A, B$, we have $b \oplus c \in D \cap E = A \cap E = B \cap E \subseteq B$ contradicting the choice of b and c. Two more applications of Lemma 1 yield $b \in B \cap D = A \cap D \subsetneq A$ so that $b \oplus c \in (A \cup C) \setminus C \subseteq A$ (since $A \sim C$) and hence $b \oplus c \in A \cap E = B \cap E \subseteq B$ which is a contradiction, proving the claim.

Proposition 6 yields two interesting corollaries. The first is a *block exchange property for paths.* The second, which follows by repeated applications of the first, states that if there is a path from A to B in L then there is such a path with all intermediary blocks in any predetermined tour $\mathcal{T}$ of L.

COROLLARY 7. If $A_0 \sim \ldots \sim A_{i-1} \sim A_i \sim A_{i+1} \sim \ldots \sim A_n$ is a path in L and $B_i \equiv A_i$ with $B_i \neq A_{i-1}, A_{i+1}$, then $A_0 \sim \ldots \sim A_{i-1} \sim B_i \sim A_{i+1} \sim \ldots \sim A_n$ is also a path in L. If B_i is not distinct from A_{i-1} or A_{i+1}, then exchanging B_i for A_i and deleting repetitions yields a shorter path.

COROLLARY 8. If $\mathcal{T}$ is a tour of L and $A_0, A_1, \ldots, A_n$ is a path in L of length n, then there exists a path (B_i) in L of length m with $m \leq n, B_0 = A_0$, $B_m = A_n$ and $B_i \in \mathcal{T}$ for $i = 1, 2, \ldots, m-1$.

2. Orthomodular Lattices.

In this section we apply the above development to the sub-class of orthoalgebras which are lattices, i.e. to the orthomodular lattices. A commutator in an orthomodular lattice is an element of the form $a * b := (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. L is *commutator finite* if it has only finitely many commutators. Commutator finite orthomodular lattices were investigated in [5] and [9]. We show that the class of all site finite orthomodular lattices fall strictly in between the class of all block finite orthomodular lattices and the class of all commutator finite orthomodular lattices.

In Proposition 2 we proved that an orthoalgebra is irreducible if and only if the union of any tour of L is irreducible. If L is an orthomodular lattice then it is not difficult to see that a similar result holds for the notions of modularity and completeness.

For any $x \in L$ the subalgebra $S_x := [x, 1] \cup [0, x']$ is called the *section* generated by x. Note that x is an atom of S_x. If A and B are distinct blocks in L with $A \sim B$, then $A \cap B = S_\alpha \cap (A \cup B)$ for some commutator α of L with $\alpha \in A \cap B$; write $A \sim_\alpha B$ in this case. Commutators which arise in this way are called *vertices* (as in [16]), and L is *vertex finite* in case L has only finitely many vertices. We write com$L$ for the set of all commutators of L and V_L for the set of all vertices of L. It follows from Lemma 5 and the above comment on the structure of $A \cup B$ when $A \sim B$ that if L is path connected then $\{x \in L | x \geq \alpha \text{ for all vertices } \alpha \text{ of } L\} \subseteq Cen(L)$, and if, in addition, $1 \in comL$ then $\{x \in L | x \geq \alpha \text{ for all vertices } \alpha \text{ of } L\} = \{1\}$. This was first proved in [16].

$V_L = \{\alpha \mid \exists\ A \sim_\alpha B\}$

LEMMA 9. Every site finite othomodular lattice is vertex finite.

PROOF. We need only show that if $A_i, B_i \in \mathfrak{A}_L$ with $A_i \neq B_i$ $(i = 1,2)$ and

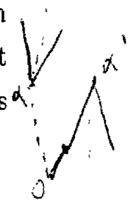
$$A_1 \cap S_{\alpha_1} = A_1 \cap B_1 = A_2 \cap B_2 = A_2 \cap S_{\alpha_2}$$

for $\alpha_1, \alpha_2 \in A_1 \cap B_1$, then $\alpha_1 = \alpha_2$. By symmetry it suffices to prove that $\alpha_2' \leq \alpha_1'$. Suppose not, then $\alpha_2' \geq \alpha_1$ since $\alpha_2 \in S_{\alpha_1}$. Moreover, $\alpha_1 \in A_1 \cap B_1 = A_2 \cap S_{\alpha_2} \subseteq A_2$. Now α_1 is not an atom of A_2. (If it were, then it would be an atom of L and so of A_1 and it would follow that $A_1 = A_1 \cap S_{\alpha_1} = A_1 \cap B_1$ which would imply that $A_1 \subseteq B_1$ so that $A_1 = B_1$ contradicting the assumption that A_1 and B_1 are distinct blocks of L.) Hence there exists $x \in A_2$ with $0 < x < \alpha_1$. But α_1 is an atom of S_{α_1} and $\alpha_1 \in A_1 \cap S_{\alpha_1}$ so that α_1 is an atom of $A_1 \cap S_{\alpha_1} = A_1 \cap B_1$. Hence $x \in A_2 \cap S_{\alpha_2} = A_1 \cap B_1$ and $x < \alpha_1$ imply $x = 0$, which is a contradiction. Hence $\alpha_2' \leq \alpha_1'$ and the result follows.

$S_{\alpha_1} = \alpha_1 \uparrow \cup \alpha_1' \downarrow$

α commutator is not an atom.

Call distinct blocks A and B *contiguous* in case there exists an $x \in A \cap B$ with $A \cap B = S_x \cap (A \cup B)$. The last proof shows that if a site is determined by two contiguous blocks then the section is uniquely determined. A site may be determined by different pairs of blocks; but if the pairs of blocks are contiguous pairs, then the corresponding sections are the same. This fact is independent of whether or not the union of the pair of blocks is a subalgebra.



If α is a commutator of the orthomodular lattice L then we define

$$V_L^\alpha := \{\beta \in V_L | \beta \leq \alpha\}.$$

LEMMA 10. If L is a site finite orthomodular lattice and $\alpha \in \text{com}L$, then $\alpha = \bigvee V_L^\alpha$.

PROOF. Let $\alpha = x * y$ and, for each $\beta \in V_L^\alpha$, let $A_\beta, B_\beta \in \mathfrak{A}_L$ with $A_\beta \sim_\beta B_\beta$. By Lemma 9 $|V_L^\alpha| \leq |V_L| < \infty$ so that there exists a tour $\mathcal{T}$ of $\mathfrak{A}_L$ such that $x, y \in \bigcup \mathcal{T} := L_1$ and $A_\beta, B_\beta \in \mathcal{T}$ for all $\beta \in V_L^\alpha$ and $\mathcal{T}$ has finite multiplicity. Thus L_1 is block finite, $\alpha \in \text{com}L_1$ and $V_{L_1}^\alpha = V_L^\alpha$. Since $L_1[0,\alpha]$ is block finite, it is path connected so, by Lemma 5 and the comment preceding Lemma 9, $\alpha = \bigvee_{L_1[0,\alpha]} V_L^\alpha = \bigvee_L V_L^\alpha$ since L_1 is a subalgebra of L.

THEOREM 11. Let L be an orthomodular lattice. Consider the following properties:

(1) L is block finite,

(2) L is site finite.

(3) L is commutator finite, (4) ⇏ (3)

(4) L is vertex finite.

Then (i) implies (i+1) for i = 1,2,3 and none of the reverse implications hold.

PROOF. (1) implies (2) and (3) implies (4) are trivial, (2) implies (3) follows from Lemma 9 and Lemma 10. The countably infinite ortholattice of height 3, $MO(\omega)$, is site finite but not

(3) ⇏ (2)

MO2 × MOω

S.F is not productive.

block finite; $MO(2) \times MO(\omega)$ ($MO(2)$ is the 6-element ortholattice of height 3) is commutator finite but not site finite; and the ortholattice of all projections in a von Neumann algebra of type II_1 is vertex finite (since it is modular and totally non-atomic so that there are no vertices therein) but not commutator finite.

COROLLARY 12. Let L be a site finite orthomodular lattice. Then (1) L is path connected, and (2) every finitely generated sub-ortholattice of L is finite.

PROOF. By (2) implies (3) of the Theorem, L is commutator finite. Thus (1) follows from Theorem 1 of [5], and (2) follows from the Corollary to the main Theorem of [5].

While the notions of block and of site are readily available in orthoalgebras, the notions of commutator and of vertex have not yet been generalized to this (at present) more recondite setting.

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