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Filters and Supports in Orthoalgebras

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An orthoalgebra, which is a natural generalization of an orthomodular lattice or poset, may be viewed as a "logic" or "proposition system" and, under a well-defined set of circumstances, its elements may be classified according to the Aristotelian modalities: necessary, impossible, possible, and contingent. The necessary propositions band together to form a local filter, that is, a set that intersects every Boolean subalgebra in a filter. In this paper, we give a coherent account of the basic theory of orthoalgebras, define and study filters, local filters, and associated structures, and prove a version of the compactness theorem in classical algebraic logic.

1. INTRODUCTION

In algebraic logic, propositions are represented by elements of an algebraic structure L. For classical logic, L is a Boolean algebra; for intuitionistic logic, L is a Heyting algebra; for quantum logic, L is an orthomodular lattice or a generalization thereof. In the classical and intuitionistic cases, the propositions in L that are syntactically or semantically true, probabilistically certain, or logically provable, form a filter $F \subseteq L$. In quantum logic, suitable versions of filters have a similar role to play.

If the orthomodular lattice L of all projection operators on a separable Hilbert space of dimension three or more is regarded as a proposition system for a quantum mechanical entity (Birkhoff and von Neumann (1936), then each state (σ-additive probability measure) μ on L enjoys the well-known

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Jauch–Piron property (Jauch, 1968; Piron, 1964, 1976)

$$P, Q \in L \quad \text{with} \quad \mu(P) = \mu(Q) = 1 \Rightarrow \mu(P \wedge Q) = 1$$

Thus, in the quantum logic affiliated with the Hilbert space of orthodox quantum mechanics, the set of all propositions that necessarily (i.e., with probability unity) yield the answer "yes" when tested in a given state is an order filter that is closed under the operation $(P, Q) \mapsto P \wedge Q$ of forming greatest lower bounds.

Although the proposition system affiliated with coupled physical entities is often presumed to be represented by a tensor product (Foulis, 1989; Jauch, 1968; Kläy *et al.*, 1987), the tensor product of orthomodular lattices is not necessarily an orthomodular lattice (or even an orthomodular poset) (Kalmbach, 1983, p. 264). The smallest known category of proposition systems containing all unital orthomodular lattices and closed under the formation of tensor products (Foulis and Randall, 1981; Randall and Foulis, 1981*a*), is the category of all unital orthoalgebras. Orthoalgebras are the simplest and most natural structures that can carry orthogonally additive measures, and thus are basic for the rapidly developing field of noncommutative measure theory (Alfsen and Shultz, 1976; Cook, 1985; D'Andrea and De Lucia, 1991; Gudder, 1988; Rüttimann, 1979, 1989; Schindler, 1986).

These considerations suggest the desirability of studying filters in orthoalgebras. In this paper we give a coherent account of the basic theory of orthoalgebras, initiate the study of filters in orthoalgebras, and set the stage for subsequent papers dealing with orthoalgebras and with attributes of physical entities.

2. ORTHOALGEBRAS

In 1666, G. W. Leibniz envisaged a universal scientific language (*Characteristica Universalis*) together with a symbolic calculus (*Calculus Ratiocinator*) for formal logical deduction within this language. In his papers on the logical theory of identity, he introduced the notation $a \oplus b$ for the "logical sum" of the terms a and b. Nearly two centuries later, G. Boole in the *Mathematical Analysis of Logic* (1847) and the *Laws of Thought* (1854) developed a calculus of logic in which the notation $a+b$ was used for what we now call the union of the classes a and b—*but only for the case in which a and b are disjoint classes*. Indeed, Boole was concerned with founding a mathematical theory of probability and, for a probability P, the condition $P(a+b) = P(a) + P(b)$ is required to hold only when a and b are disjoint.

In defining an orthoalgebra, we follow Boole and restrict the domain of definition of sums to certain pairs which eventually are called *orthogonal pairs*. (For a Boolean algebra, these are precisely the disjoint pairs, but this need not be the case for a general orthoalgebra.) However, we use the Leibniz notation $a \oplus b$ to avoid confusing the orthogonal sum with the inclusive sum (or disjunction) $a+b$ later introduced for Boolean algebras by Jevons, Peirce, and Schröder. Orthoalgebras were originally defined in Randall and Foulis (1979, 1981*a*). In Hardegree and Frazer (1981) and Lock and Hardegree (1984*a*,*b*) they were called *associative orthoalgebras*. The simplified definition that follows is due to Golfin (1987).

Definition 2.1. An *orthoalgebra* (OA) is a set L containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ subject to the following conditions for all $p, q, r \in L$:

(i) (Commutativity) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
(ii) (Associativity) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
(iii) (Orthocomplementation) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.
(iv) (Consistency) If $p \oplus p$ is defined, then $p = 0$.

If the hypotheses of (ii) are satisfied, we write $p \oplus q \oplus r$ for the element $(p \oplus q) \oplus r = p \oplus (q \oplus r)$ in L.

We note that a Boolean algebra L forms an OA if we agree that $p \oplus q$ is defined iff $p \wedge q = 0$, in which case $p \oplus q = p \vee q$. More generally, if R is any ring with unity 1 and L is the set of all idempotents in R, then L becomes an orthoalgebra if we define $e \oplus f = e + f$ iff $ef = fe = 0$.

Definition 2.2. Let L be an OA and let $p, q \in L$.

(i) We say that p is *orthogonal* to q and write $p \perp q$ iff $p \oplus q$ is defined in L.
(ii) If there exists an element $r \in L$ such that $p \perp r$ and $q = p \oplus r$, then we write $p \leq q$.
(iii) The unique element q such that $p \perp q$ and $p \oplus q = 1$ is called the *orthocomplement* of p and is written as p'.

In a Boolean algebra, regarded as an OA, $p \perp q$ holds iff p and q are disjoint (i.e., the meet exists and is 0), $p \leq q$ holds iff $p = p \wedge q$, and p' is the unique complement of p. In the orthoalgebra L of all idempotents in a ring R with unity, $p \perp q$ holds iff $pq = qp = 0$, $p \leq q$ holds iff $p = pq = qp$, and $p' = 1 - p$.

Henceforth, we assume that L is an orthoalgebra.

Lemma 2.3. Let $p, q \in L$. Then:

(i) $p \perp q \Leftrightarrow q \perp p$.
(ii) $p \perp p \Rightarrow p = 0$.

(iii) $p \perp 1 \Rightarrow p = 0$.
(iv) $p'' = p$.
(v) $1' = 0$ and $0' = 1$.
(vi) $p \perp 0$ and $p \oplus 0 = p$.

Proof. (i) follows from the commutativity condition and (ii) follows from the consistency condition in Definition 2.1. To prove (iii), assume that $p \perp 1$ and let $q = (1 \oplus p)'$. Then $(1 \oplus p) \oplus q = 1$; hence, by the associativity condition in Definition 2.1, $1 \oplus (p \oplus q) = 1$. Therefore, $1 = 1 \oplus (q \oplus p) = (1 \oplus q) \oplus p$; hence, since $1 \perp p$, $((1 \oplus q) \oplus p) \oplus p$ is defined, and it follows from the associativity condition that $p \oplus p$ is defined. Therefore, $p = 0$ by the consistency condition. (iv) is an obvious consequence of the orthocomplementation condition. To prove (v), note that $1' \perp 1$, so $1' = 0$ by part (iii). Therefore, $0' = 1'' = 1$. To prove (vi), note that $1 = 1 \oplus 1' = (p' \oplus p) \oplus 0 = p' + (p \oplus 0)$; hence, $p \oplus 0 = p'' = p$. ■

Lemma 2.4. Let $p, q \in L$ and suppose that $p \perp q$. Then

$$p \perp (p \oplus q)' \quad \text{and} \quad p \oplus (p \oplus q)' = q'$$

Proof. Let $r = (p \oplus q)'$. Then

$$1 = (p \oplus q) \oplus r = p \oplus (q \oplus r) = (q \oplus r) \oplus p = q \oplus (r \oplus p) = q \oplus (p \oplus r)$$

Hence, $p \oplus r = q'$. ■

Corollary 2.5. For $p, q \in L$, $p \perp q \Leftrightarrow p \leq q'$.

Proof. If $p \perp q$, then $p \leq q'$ by Lemma 2.4 and Part (ii) of Definition 2.2. Conversely, suppose that $p \leq q'$. Then there exists $r \in L$ with $p \perp r$ and $p \oplus r = q'$. Thus, $1 = (p \oplus r) \oplus q = (r \oplus p) \oplus q = r \oplus (p \oplus q)$, so $p \oplus q$ is defined, and therefore $p \perp q$. ■

Theorem 2.6 (Orthomodular Identity). For $p, q \in L$ with $p \leq q$,

$$q = p \oplus (p \oplus q')'$$

Proof. Suppose that $p \leq q = q''$. Then $p \perp q'$ by Corollary 2.5; hence, by Lemma 2.4, $q = q'' = p \oplus (p \oplus q')'$. ■

Lemma 2.7 (Cancellation Law). Let $p, q, r \in L$ with $p, q \perp r$. Then:

(i) $p \oplus r = q \oplus r \Rightarrow p = q$.
(ii) $p \oplus r \leq q \oplus r \Rightarrow p \leq q$.

Proof. To prove (i), assume that $p \oplus r = q \oplus r$ and let $s = (p \oplus r)' = (q \oplus r)'$. Then, $(p \oplus r) \oplus s = (q \oplus r) \oplus s = 1$, so $p \oplus (r \oplus s) = q \oplus (r \oplus s) = 1$, from which it follows that $p = (r \oplus s)' = q$. To prove (ii), assume that $p \oplus r \leq q \oplus r$.

Then, there exists $t \in L$ with $(p \oplus r) \perp t$ and $(p \oplus r) \oplus t = q \oplus r$. Consequently, $(p \oplus t) \oplus r = q \oplus r$, so $p \oplus t = q$ by part (i). But $p \oplus t = q$ shows that $p \leq q$. ■

Theorem 2.8. $(L, \leq, 0, 1)$ is a bounded poset.

Proof. Let $p, q, r \in L$. That $p \leq p$ follows from part (vi) of Lemma 2.3. To prove that $\leq$ is antisymmetric, suppose that $p \leq q$ and $q \leq p$. We have to prove that $p = q$. There exist $s, t \in L$ such that $p \perp s$, $q \perp t$, $p \oplus s = q$, and $q \oplus t = p$. Consequently, $p \oplus (s \oplus t) = p = p \oplus 0$, and it follows from Lemma 2.7 that $s \oplus t = 0$. Therefore, $s = s \oplus 0 = s \oplus (s \oplus t) = (s \oplus s) \oplus t$, so $s = 0$ and $q = p \oplus s = p$. The proof that $\leq$ is transitive is straightforward, as is the proof that $0 \leq p \leq 1$ for all $p \in L$. ■

If $p, q, r \in L$, we write $r = p \vee q$ (respectively, $r = p \wedge q$) to indicate that r is the least upper bound (respectively, greatest lower bound) of p and q in the poset $(L, \leq)$. Two elements $p, q \in L$ are said to be *disjoint* if $p \wedge q$ exists and equals 0.

Theorem 2.9. The map $p \mapsto p'$ is an orthocomplementation on the bounded poset L; that is, for $p, q \in L$:

(i) $p = p''$.
(ii) $p \leq q \Rightarrow q' \leq p'$.
(iii) $p \wedge p' = 0$.
(iv) $p \vee p' = 1$.

Proof. We already have (i). To prove (ii), suppose that $p \leq q$. Then, by corollary 2.5, $p \perp q'$, so $q' \perp p$, from which $q' \leq p'$ follows by Corollary 2.5 again. To prove (iii), suppose that $q \leq p, p'$. We have to prove that $q = 0$. Since $q \leq p'$, we have $p = p'' \leq q'$, and so $q \leq q'$; that is, $q \perp q$ by Corollary 2.5. Therefore, $q = 0$. To prove (iv), suppose that $p, p' \leq q$. We have to prove that $q = 1$. But, $q' \leq p', p$; hence $q' = 0$ by part (iii) above, and it follows that $q = 1$. ■

Theorem 2.10. If $p, q \in L$ with $p \perp q$, then $p \oplus q$ is a minimal upper bound for p and q in the poset L.

Proof. That $p, q \leq p \oplus q$ is clear. Suppose that $p, q \leq r \leq p \oplus q$. We have to prove that $r = p \oplus q$. Since $p, q \leq r$, there exist $s, t \in L$ with $r = p \oplus s = q \oplus t$. Since $r \leq p \oplus q$, there exists $u \in L$ with $r \oplus u = p \oplus q$. Now, $p \oplus s \oplus u = r \oplus u = p \oplus q = u \oplus r = u \oplus t \oplus q$. From $p \oplus s \oplus u = p \oplus q = u \oplus t \oplus q$ and cancellation, we find that $s \oplus u = q$ and $p = u \oplus t$, and therefore that $u \leq p, q$. Since $p \perp q$, it follows from Corollary 2.5 that $q \leq p'$; hence, that $u \leq p, p'$. Consequently, $u = 0$ by part (iii) of Theorem 2.9. ■

Corollary 2.11. If $p, q \in L$ with $p \perp q$, and if $p \vee q$ exists in L, then

$$p \oplus q = p \vee q$$

By definition, an *orthomodular poset* (OMP) is a bounded orthocomplemented poset $(P, \leq, ', 0, 1)$ such that, for $p, q \in P$ with $p \leq q'$, $p \vee q$ exists in P, and P satisfies the *orthomodular identity*:

$$p, q \in P \quad \text{with} \quad p \leq q \Rightarrow q = p \vee (p \vee q')'$$

(see, e.g., Foulis, 1962). Evidently, any OMP may be regarded as an OA by defining $p \oplus q = p \vee q$ precisely in the case $p \leq q'$.

Theorem 2.12. For an OA L, the following conditions are mutually equivalent:

(i) $(L, \leq, ', 0, 1)$ is an OMP.
(ii) For p, q, $r \in L$, the conditions $p \perp q$, $p \perp r$, and $q \perp r$ imply that $(p \oplus q) \perp r$.
(iii) For $p, q \in L$, $p \perp q \Rightarrow p \vee q$ exists.

Proof. We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Suppose that L is an OMP and that the hypotheses of (ii) hold. By Corollary 2.11, $p \oplus q = p \vee q \leq r'$, so $(p \oplus q) \perp r$.

(ii) $\Rightarrow$ (iii): Assume (ii) and suppose that $p \perp q$ and that $r \in L$ with $p, q \leq r$. To show that $p \oplus q$ is effective as $p \vee q$, it suffices to show that $p \oplus q \leq r$. But, $p, q \perp r'$; hence, $(p \oplus q) \perp r'$ by (ii), and therefore $p \oplus q \leq r$.

(iii) $\Rightarrow$ (i): This follows from Theorem 2.6. ■

Example 2.13. The simplest OA that is not an OMP is given by

$$L = \{0, 1, a, b, c, d, e, f, a', b', c', d', e', f'\}$$

where, apart from the obvious relations, we let

$$a \oplus b = d \oplus e = c'$$

$$b \oplus c = e \oplus f = a'$$

$$c \oplus d = f \oplus a = e'$$

$$c \oplus e = d', \qquad a \oplus c = b', \qquad e \oplus a = f'$$

We note that $a \oplus c = b'$ and $a, c \leq e'$, but $b' \leq e'$ fails, so $a \oplus c$ is not the least upper bound of a and c. This example is due to R. Wright.

An *orthomodular lattice* (OML) is defined to be an OMP L in which every pair of elements p, q has a least upper bound $p \vee q$. It then follows that every pair of elements p, q has a greatest lower bound given by $p \wedge q = (p' \vee q')'$, so that L is a lattice. The theory of orthomodular lattices is developed in some detail in Beran (1984) and Kalmbach (1983).

Example 2.14. The simplest OMP that is not an OML is given by

$$L = \{0, 1, a, b, c, d, e, f, g, h, a', b', c', d', e', f', g', h'\}$$

where, apart from the obvious relations, we let

$$a \oplus c = b', \quad c \oplus e = d', \quad e \oplus g = f', \quad a \oplus g = h'$$
$$a \oplus b = d \oplus e = c', \quad b \oplus c = g \oplus h = a'$$
$$c \oplus d = f \oplus g = e', \quad e \oplus f = a \oplus h = g'$$

We note that $a' \wedge e'$ fails to exist in L. This example is due to Janowitz (1963) and is often referred to as J_{18}.

It can be shown that a Boolean algebra is the same thing as an OML in which disjoint pairs of elements are orthogonal.

3. SUBORTHOALGEBRAS AND COMPATIBILITY

Definition 3.1. Let L be an OA. A subset L_1 of L is called a *suborthoalgebra* of L if $0, 1 \in L_1$, L_1 is closed under the orthocomplementation map $p \mapsto p'$, and, whenever $p, q \in L_1$ with $p \perp q$, it follows that $p \oplus q \in L_1$.

Clearly, a suborthoalgebra L_1 of an OA L is an OA in its own right. As such, if L_1 is a Boolean algebra, we call it a *Boolean subalgebra* of L. In Example 2.13, $\{0, 1, a, b, c, a', b', c'\}$, $\{0, 1, c, d, e, c', d', e'\}$, and $\{0, 1, e, f, a, e', f', a'\}$ are Boolean subalgebras of L.

Lemma 3.2. Let $p, q, r \in L$ with $p \perp q$ and $(p \oplus q) \perp r$. Then each of the following is a Boolean subalgebra of L:

(i) $\{0, 1, p, p'\}$.
(ii) $\{0, 1, p, q, p \oplus q, p', q', (p \oplus q)'\}$.
(iii) $\{0, 1, p, q, r, p \oplus q, p \oplus r, q \oplus r, p \oplus q \oplus r, p', q', r', (p \oplus q)', (p \oplus r)', (q \oplus r)', (p \oplus q \oplus r)'\}$.

Proof. The proof is a straightforward computation. ■

As a consequence of Lemma 3.2, every orthoalgebra L can be regarded as a union of Boolean algebras, each of which is a subalgebra of L; that is, the unique $\oplus$ in these Boolean algebras matches the $\oplus$ of L. In this sense, an orthoalgebra is "locally Boolean." [Actually, any orthocomplemented poset is a union of four-element Boolean algebras, so it is really Part (ii) of Lemma 3.2 that distinguishes orthoalgebras.]

Definition 3.3. Let $C \subseteq L$.

(i) Elements of C are said to be *jointly compatible* (and C is called a *compatible* subset of L) iff there is a Boolean subalgebra L_1 of L with $C \subseteq L_1$.
(ii) Elements of C are said to be *jointly orthogonal* (and C is called an *orthogonal* subset of L) iff the elements of C are jointly compatible and pairwise orthogonal.

As a consequence of Lemma 3.2, the empty set and any singleton subset of L are compatible (and orthogonal by default). Also, if $p \perp q$, then $\{p, q\}$ is a compatible (hence, orthogonal) set. We note that $\{p, q, r\}$ is an orthogonal subset of L iff $p \perp q$ and $(p \oplus q) \perp r$. In Example 2.13, $\{a, c, e\}$ is a pairwise orthogonal set that is not an orthogonal set. It can be shown that L is an OMP iff every finite pairwise orthogonal subset of L is an orthogonal set. Even if L is an OMP, there may be subsets of L in which the elements are pairwise compatible, but not jointly compatible (Ramsay, 1966). However, pairwise compatible subsets of an OML are compatible subsets.

We say that the elements a and $b \in L$ are *compatible* iff $\{a, b\}$ is a compatible set. The following result is a corollary of Part (iii) of Lemma 3.2.

Lemma 3.4. The elements $a, b \in L$ are compatible iff there exist jointly orthogonal elements $a_1, b_1, d \in L$ such that $a = a_1 \oplus d$ and $b = b_1 \oplus d$.

If L is a Boolean algebra, then any two elements of L are compatible. The following example shows that the converse is false.

Example 3.5. Let

$$L = \{0, 1, a, b, c, d, e, f, g, a', b', c', d', e', f', g'\}$$

Organize L into an OA by imposing the following relations:

$$a \oplus b = f \oplus g = d \oplus e = c', \qquad b \oplus c = d \oplus g = e \oplus f = a'$$

$$a \oplus c = e \oplus g = d \oplus f = b', \qquad c \oplus d = b \oplus g = a \oplus f = e'$$

$$c \oplus e = b \oplus f = a \oplus g = d', \qquad a \oplus e = c \oplus g = b \oplus d = f'$$

$$a \oplus c = c \oplus f = e \oplus b = g'$$

Then L is an OA in which every pair of elements is compatible, but L is not an OMP because, for instance, $\{a, c, e\} \subseteq L$ is pairwise orthogonal, but not orthogonal. We note that the subsets of $\{a, b, c, d, e, f, g\}$ that form jointly orthogonal triples correspond to the lines in the seven-point Fano projective plane.

4. DIFFERENCES AND SUMS IN AN ORTHOALGEBRA

We omit the proofs of the theorems in this section, since they follow in a straightforward way from the results already obtained.

Definition 4.1. If $p, q \in L$ with $p \leq q$, define $q - p = (p \oplus q')'$.

Lemma 4.2. Let $p, q \in L$. Then:

(i) $p \leq q \Rightarrow q = p \oplus (q - p)$ (orthomodular identity).
(ii) $p \perp q \Rightarrow p = (p \oplus q) - q$.
(iii) $p \leq q \Rightarrow p = q - (q - p)$.
(iv) If $p \leq q$, then $p = q \Leftrightarrow q - p = 0$.
(v) $p' = 1 - p$.
(vi) $p = p - 0$.

Lemma 4.3. Let $p, q, r \in L$ with $p \leq q \leq r$. Then:

(i) $(r - q) \oplus (q - p) = r - p$.
(ii) $(r - p) - (q - p) = (r - (q - p)) - p = r - q$.
(iii) $(p \oplus (r - q)) - p = r - q$.
(iv) $r - (p \oplus (r - q)) = q - p$.
(v) $p \leq p \oplus (r - q) \leq r$.

If $p, q \in L$, we write $p < q$ to mean that $p \leq q$ and $p \neq q$.

Definition 4.4. A finite set $D \subseteq L$ is called a *difference set* if either D is empty or there exists a strictly increasing sequence

$$p_0 < p_1 < p_2 < \cdots < p_{n-1} < p_n$$

in L such that

$$D = \{p_k - p_{k-1} \mid k = 1, 2, \ldots, n\}$$

We denote by $\#D$ the number of elements in D.

Lemma 4.5. Let

$$p_0 < p_1 < p_2 < \cdots < p_n \qquad \text{and} \qquad q_0 < q_1 < q_2 < \cdots < q_m$$

be two strictly increasing sequences in L that give rise to the same difference set

$$D = \{p_k - p_{k-1} \mid k = 1, 2, \ldots, n\} = \{q_j - q_{j-1} \mid j = 1, 2, \ldots, m\}$$

Then $n=m=\#D$ and $p_n-p_0=q_m-q_0$.

Definition 4.6. Let D be the difference set corresponding to the strictly increasing sequence $p_0<p_1<p_2<\cdots<p_n$ in L. We define $\bigoplus D=p_n-p_0$, noting that $\bigoplus D$ is well-defined by Lemma 4.5. If D is the empty difference set, we define $\bigoplus D=0$.

Theorem 4.7. If $D\subseteq L$, then D is a difference set iff D is a finite orthogonal set of nonzero elements. Furthermore, if D is a difference set and B is any Boolean subalgebra of L with $D\subseteq B$, then $\bigoplus D$ is effective as the least upper bound of D as calculated in B.

If $C\subseteq L$ is a finite orthogonal set, then $D=C\backslash\{0\}$ is a difference set, and we define $\bigoplus C=\bigoplus D$. (We use the notation $A\backslash B$ for the set of elements in A that are not in B.) Evidently, if B is a Boolean subalgebra of L and C is a finite orthogonal set with $C\subseteq B$, then $\bigoplus C$ is the least upper bound of C as calculated in B. If $p, q\in L$ with $p\perp q$, then $C=\{p, q\}$ is an orthogonal set and $\bigoplus C=p\oplus q$. Also, if $r\in L$ with $(p\oplus q)\perp r$, then $C=\{p, q, r\}$ is an orthogonal set and $\bigoplus C=p\oplus q\oplus r$. More generally, if $C=\{c_1, c_2, \ldots, c_n\}$ is an orthogonal set, we use the notation $c_1\oplus c_2\oplus\cdots\oplus c_n$ for $\bigoplus C$.

Lemma 4.8. If C is a finite orthogonal set and $C=A\cup B$ with $A\cap B=\varnothing$, then $\bigoplus C=\bigoplus A\oplus\bigoplus B$.

Definition 4.9. By a *finite partition of unity* in L, we mean a difference set $E\subseteq L$ such that $\bigoplus E=1$.

Thus, a finite partition of unity is a finite set $E=\{p, q, \ldots, r\}$ of jointly orthogonal nonzero elements of L such that $p\oplus q\oplus\cdots\oplus r=1$. Note that, if $D=\{p, q, \ldots, r\}$ is a finite orthogonal set of nonzero elements of L and $d=p\oplus q\oplus\cdots\oplus r\neq 1$, then $E=D\cup\{d'\}$ is a finite partition of unity.

5. HEURISTICS FOR ORTHOALGEBRAS

In what follows, we assume that the elements of the orthoalgebra L represent true/false propositions regarding a given physical system, a specified entity, or, indeed, any situation concerning which well-defined, testable, two-valued propositions may be formulated (Birkhoff and von Neumann, 1936; Randall and Foulis, 1981*b*). Propositions belonging to a Boolean subalgebra B of L are supposed to be *simultaneously testable* in the sense that it is possible, at least in principle, to ascertain all of their truth values by conducting a single test, performing a single experiment, or making a single observation. Thus, to say that a collection C of propositions in L forms a compatible set is to say that these propositions admit a simultaneous test. We refer to such a test as a test *for* C.

If D is an orthogonal subset of L, it is understood that a test for D will assign the value "true" to at most one of the propositions in D. If D is finite, then $\oplus D$ is to be regarded as the *logical disjunction* of the propositions in D in the sense that, as a consequence of a test of D, $\oplus D$ is assigned the value "true" iff one of the propositions in D is assigned the value "true," and that, otherwise, $\oplus D$ is assigned the value "false." Of course, the proposition $1 \in L$ is assigned the value "true" by any test; hence, if E is a finite partition of unity, each test of E will result in the assignment of the value "true" to one and only one proposition in E.

Suppose that $p, q \in L$ with $p \leq q$. Then, $\{p, q\}$ is a compatible set and, if B is any Boolean subalgebra of L with $\{p, q\} \subseteq B$, we have $p' \vee q = 1$ in the Boolean algebra B. Therefore, we may interpret $p \leq q$ to mean that, as propositions, p *implies* q in the sense that, whenever p and q are tested simultaneously and p turns out to be "true," then q will also be "true."

Incompatible propositions $p, q \in L$ may or may not have a greatest lower bound $p \wedge q$ in L. If $p \wedge q$ exists, it is simply the "greatest" proposition in L that implies both p and q. To interpret $p \wedge q$ as a logical conjunction of the incompatible propositions p and q is misleading and confusing—if p and q do not admit a simultaneous test, the classical notion of logical conjunction is devoid of meaning. Similar remarks apply to the least upper bound $p \vee q$ of incompatible propositions.

6. FILTERS AND LOCAL FILTERS

Insofar as possible, we use standard order-theoretic terminology in connection with the partially ordered set $(L, \leq)$. For instance, a nonempty subset U of L is called an *order filter* if, for all $p, q \in L$ with $p \leq q$, $p \in U$ implies $q \in U$. If U is an order filter, then, since $U \neq \emptyset$, it follows that $1 \in U$; also, $U = L$ if and only if $0 \in U$. If U is an order filter in L and $0 \notin U$, we say that U is a *proper* order filter. A subset D of L is said to be *filtered* (or *downward directed*) if, for all $p, q \in D$, there exists $r \in D$ with $r \leq p, q$.

The following simple result turns out to be the key to the fact that the Stone space of a Boolean algebra is a Hausdorff topological space.

Lemma 6.1. Let $D, U, V \subseteq L$, suppose that D is filtered, and let U and V be order filters. Then

$$D \subseteq U \cup V \Leftrightarrow D \subseteq U \quad \text{or} \quad D \subseteq V$$

Proof. Suppose that $D \not\subseteq U$ and $D \not\subseteq V$. Then there exist $d_1, d_2 \in D$ such that $d_1 \notin U$ and $d_2 \notin V$. Because D is filtered, there exists $d \in D$ with $d \leq d_1, d_2$. Since U and V are order filters, we cannot have $d \in U$ (else $d_1 \in U$) and we cannot have $d \in V$ (else $d_2 \in V$). Therefore, $D \not\subseteq U \cup V$. This proves that $D \subseteq U \cup V$ implies $D \subseteq U$ or $D \subseteq V$, and the converse implication is obvious. ■

A nonempty subset F of L is called a *filter* in L if it is a filtered order filter. If $q \in L$, then

$$q\uparrow := \{p \in L \mid q \leq p\}$$

is the smallest filter containing q. A filter of the form $q\uparrow$ is called a *principal filter* and q is called its *generator*. If L is an OML, then F is a filter in L if and only if it is an order filter closed under the operation $(p, q) \mapsto p \wedge q$.

Definition 6.2. $F \subseteq L$ is a *local filter* iff, for every Boolean subalgebra B of L, $F \cap B$ is a filter in B.

If the elements of L are regarded as propositions, we may view the elements of a local filter F as those propositions that are *necessarily true* under a certain set of circumstances. In the next lemma, the proof of which is straightforward, we give an alternative characterization of local filters.

Lemma 6.3. $F \subseteq L$ is a local filter iff it is nonempty and has the following property for every jointly orthogonal triple $p, q, r \in L$:

$$p \oplus r, \quad q \oplus r \in F \Leftrightarrow r \in F$$

In an OML, the map $q \mapsto p \wedge (q \vee p')$ is called the *Sasaki projection* of q onto p (Foulis, 1962; Sasaki, 1954). As a consequence of Lemma 6.3, it is easy to see that, for an OML, the local filters are precisely the order filters that are closed under Sasaki projections in the sense of the following:

Corollary 6.4. If L is an OML and $F \subseteq L$ is an order filter, then F is a local filter iff $p, q \in F \Rightarrow p \wedge (q \vee p') \in F$.

In a Boolean algebra, filters and local filters coincide. In an OMP, every filter is a local filter; in fact, L is an OMP iff every principal filter in L is a local filter. Even in the OML of projection operators on a Hilbert space, there are local filters that are not filters; in fact, an atomic OML is a Boolean algebra if and only if all of its local filters are filters.

In an OA that satisfies the descending chain condition (i.e., every strictly descending chain is finite), an order filter F is determined by the set M of minimal elements in F; indeed, $F = \bigcup \{p\uparrow \mid p \in M\}$. The next lemma shows that a minimal element of a local filter F is disjoint from the orthocomplement of every other element of F.

Lemma 6.5 (M. K. Bennett). Let F be a local filter in L and let q be a minimal element of F. Then, for every $p \in F$, $p' \wedge q$ exists in L and $p' \wedge q = 0$.

Proof. Assume the hypotheses and suppose that $t \in L$ with $t \leq p', q$. We have to prove that $t=0$. Since $t \leq q$, there exists $s \in L$ with $t \perp s$ and $t \oplus s = q$. Since $t \leq p'$, we have $p \leq t'$, from which it follows that $t' \in F$. Now, t, s, q' is a jointly orthogonal triple, $t \oplus s = q \in F$, and $q' \oplus s = t' \in F$; hence, $s \in F$ by Lemma 6.3. But $s \leq q$ and q is a minimal element of F, so $s = q$. Therefore, $t \oplus q = q$, so $t = 0$. ■

Corollary 6.6. Let F be a local filter in L and let p, q be minimal elements of F. Then $p' \wedge q = 0$ and $p' \vee q = 1$.

As a consequence of Corollary 6.6, minimal elements of a local filter are *perspective* (i.e., they share a common complement).

Whereas logicians often prefer to deal with filters (which can be presumed to represent the modality of necessity or truth), algebraists generally prefer to think in terms of ideals (which often figure prominently in representation theory). A subset I of L is called an *ideal* if $\{p' \mid p \in I\}$ is a filter, and it is called a *local ideal* if $\{p' \mid p \in I\}$ is a local filter.

7. SUPPORTS

Definition 7.1. A subset S of L is called a *support* iff $0 \notin S$ and, for every orthogonal pair $p, q \in L$,

$$p \oplus q \in S \Leftrightarrow \{p, q\} \cap S \neq \varnothing$$

We note that the empty set $\varnothing$ is a support. A nonempty support is called a *proper* support. Evidently, every proper support is an order filter; hence, a support S is proper iff $1 \in S$.

Theorem 7.2. Suppose that $0 \notin S \subseteq L$. Then the following conditions are mutually equivalent:

(i) S is a support.
(ii) For every difference set D, $\bigoplus D \in S \Leftrightarrow D \cap S \neq \varnothing$.
(iii) For every pair E, E^* of finite partitions of unity in L,

$$S \cap (E \setminus E^*) \neq \varnothing \Rightarrow S \cap (E^* \setminus E) \neq \varnothing$$

(iv) For every pair E, E^* of finite partitions of unity in L,

$$S \cap E \subseteq E^* \Rightarrow S \cap E^* \subseteq E \qquad \text{(exchange condition)}$$

Proof. (i) ⇒ (ii) by Definition 7.1 and mathematical induction. To prove that (ii) ⇒ (iii), assume (ii) and suppose that E, E^* are finite partitions of unity with $S \cap (E \setminus E^*) \neq \varnothing$. Let $d = \bigoplus(E \cap E^*)$, $e = \bigoplus(E \setminus E^*)$, and $e^* = \bigoplus(E^* \setminus E)$. By (ii) we have $e \in S$. Now $e \oplus d = \bigoplus E = 1 = \bigoplus E^* = e^* \oplus d$, and it follows that $e^* = e \in S$. Therefore, by (ii), $S \cap (E^* \backslash E) \neq \emptyset$. An elementary set-theoretic argument shows that (iii)$\Leftrightarrow$(iv). To finish the proof, we show that (iii)$\Rightarrow$(i). Assume (iii) and suppose that $p, q \in L$ with $p \perp q$. We have to prove that $p \oplus q \in S \Leftrightarrow \{p, q\} \cap S$. Obviously, we can assume that $p, q \neq 0$. If $p \oplus q \neq 1$, let $E = \{p \oplus q, (p \oplus q)'\}$, $E^* = \{p, q, (p \oplus q)'\}$; otherwise, let $E = \{p \oplus q\} = \{1\}$, $E^* = \{p, q\}$ in (iii). ■

As a consequence of the exchange condition in Part (iv) of Theorem 7.2, a proper support S must have a nonempty intersection with every finite partition of unity in L. In particular, if $p \in L$ and S is a proper support, then at least one of the two elements p, p' must belong to S.

Definition 7.3. If S is a support in L, we define

$$F_S = \{p \in L \mid p' \notin S\}$$

Theorem 7.4. If S is a support in L, then F_S is a local filter in L and the map $S \mapsto F_S$ provides a one-to-one correspondence between supports in L and local filters in L.

Proof. We begin by noting that, if $S = \emptyset$, then $F_S = L$ is a local filter in L. Suppose, then, that S is a proper support and let p, q, r be jointly orthogonal elements in L. We claim that F_S is an order filter. Indeed, suppose $a, b \in L$ with $a \in F_S$ and $a \leq b$. Then $b' \leq a'$ and $a' \notin S$; hence, since S is an order filter, $b' \notin S$, and it follows that $b \in F_S$. This shows that $r \in F_S \Rightarrow p \oplus r, q \oplus r \in F_S$. Conversely, suppose that $p \oplus r, q \oplus r \in F_S$. Let $d = (p \oplus q \oplus r)'$, so that p, q, r, d are jointly orthogonal and $p \oplus q \oplus r \oplus d = 1$. Now, $q \oplus d = (p \oplus r)' \notin S$ and $p \oplus d = (q \oplus r)' \notin S$, from which it follows that $p, q, d \notin S$; hence, that $r' = p \oplus q \oplus d \notin S$. Therefore, $r \in F_S$.

To show that $S \mapsto F_S$ is a one-to-one correspondence between supports and local filters, it suffices to show that every local filter F can be written as F_S for some support S. By an argument similar to that given above, if F is a local filter, then $S := \{p \in L \mid p' \notin F\}$ is shown to be a support, and it is clear that $F = F_S$. ■

If the support S corresponds to the local filter $F = F_S$, and if F is viewed as representing the modality of necessity, then (owing to the fact that a proposition $p \in L$ belongs to S iff its negation p' does not belong to F) we may view S as representing the modality of *possibility*. Thus, if p, q is an orthogonal pair of propositions in L, the fact that $p \oplus q \in S$ iff $\{p, q\} \cap S \neq \emptyset$ may be interpreted to mean that the disjunction $p \oplus q$ is possible if and only if at least one of the propositions p, q is possible. Also, the fact that S is an order filter may be viewed as the condition that if p is possible and $p \leq q$, then q is possible. Part (iii) of the following lemma, the simple proof of which is omitted, may be interpreted as the obvious requirement that a necessary proposition is possible.

Lemma 7.5. Let $S, T \subseteq L$ be supports. Then:

(i) $S \subseteq T \Leftrightarrow F_T \subseteq F_S$.
(ii) S is proper $\Leftrightarrow F_S$ is proper.
(iii) S is proper $\Leftrightarrow F_S \subseteq S$.

If S is a support in L, then $I_S := L \backslash S = \{p \in L \mid p' \in F_S\}$ is a local ideal in L, and $S \leftrightarrow I_S$ provides a one-to-one correspondence between all supports in L and all local ideals in L. The quadruple

$$S, F_S, I_S, S \backslash F_S$$

may be viewed as a classification of the propositions in L according to the classical modalities *possible*, *necessary*, *impossible*, and *contingent*, respectively. This interpretation and its connection with Kripke models of sets of formulas in modal logics is explored in Svetlichny (1986, 1990).

In what follows, we concentrate our attention on supports. This confers a certain mathematical simplicity on our deliberations, and the pertinent facts concerning local filters and local ideals are easily derivable from the corresponding facts about supports.

8. THE SUPPORT LATTICE AND THE CANONICAL MAP

Definition 8.1. We denote by $\mathscr{S} = \mathscr{S}(L)$ the set of all supports in L, partially ordered by set-theoretic inclusion.

Note that $\varnothing$ and $L \backslash \{0\}$ are elements of $\mathscr{S}$ and that $\varnothing \subseteq S \subseteq L \backslash \{0\}$ holds for all $S \in \mathscr{S}$, so that $\mathscr{S}$ is a bounded poset. Evidently, the set-theoretic union of supports is again a support, so $\mathscr{S}$ actually forms a complete lattice under $\subseteq$. We refer to $\mathscr{S}$ as the *support lattice* of the orthoalgebra L. Although, in general, the set-theoretic intersection of supports need not be a support, it is clear from Definition 7.1 that the intersection of an inclusion chain of supports is again a support. Furthermore, since a support S is proper iff $1 \in S$, it is evident that the intersection of an inclusion chain of proper supports is again a proper support. Therefore, by Zorn's lemma, every proper support contains a minimal proper support. This proves the following:

Theorem 8.2. $\mathscr{S}$ is a complete atomic lattice.

Even if L is finite, $\mathscr{S}$ need not be an *atomistic* lattice; that is, there may be proper supports that cannot be written as a union of minimal proper supports. For instance, the support $S = L \backslash \{0, d, f, h\}$ in the OMP of example 2.14 cannot be written as a union of minimal supports. If L is a Boolean algebra, then the support lattice $\mathscr{S}$ of L is an atomistic dual Brouwerian lattice, i.e., an atomistic dual Heyting algebra that is complete as a lattice.

Definition 8.3. If $p \in L$, we define $[p] \in \mathcal{S}$ by

$$[p] = \bigcup \{S \in \mathcal{S} \mid p \in F_S\}$$

and we refer to the map $[\cdot]: L \to \mathcal{S}$ as the *canonical map.* A support of the form $[p]$ is called a *principal support.*

We may interpret $[p]$ as the largest support that confers the modality of necessity on the proposition p. We omit the straightforward proof of the following lemma.

Lemma 8.4. Let $p, q \in L$ and $S \in \mathcal{S}$. Then:

(i) $[0] = \varnothing$.
(ii) $[1] = L \backslash \{0\}$.
(iii) $p \leq q \Rightarrow [p] \subseteq [q]$.
(iv) $p \in F_S \Leftrightarrow S \subseteq [p]$.
(v) $p' \notin [p]$.
(vi) $[p] \wedge [p'] = \varnothing$.
(vii) $S = \bigwedge \{[p] \mid p \in L, S \subseteq [p]\}$.

By Lemma 8.4, $[\cdot]: L \to \mathcal{S}$ provides an order-preserving map from the poset L onto the meet-dense subset of $\mathcal{S}$ consisting of the principal supports. In example 3.5, $[\cdot]: L \to \mathcal{S}$ maps every atom in the poset L onto the element $\varnothing \in \mathcal{S}$; hence, no element of $\mathcal{S}$ confers the modality of necessity on any atom in L. The following definition is intended to rule out this somewhat undesirable situation.

Definition 8.5. L is *modal* if $[p] \neq \varnothing$ holds for every $p \in L$ with $p \neq 0$.

If L is a modal orthoalgebra and $p \in L$ with $p \neq 0$, 1, there are sufficiently many supports in $\mathcal{S}$ to confer upon p each of the four classical modalities—necessity, possibility, impossibility, and contingency. It is easy to see that L is modal iff $p \in [p]$ holds for all $0 \neq p \in L$. We define $p^\perp = \{q \in L \mid q \perp p\}$ for each $p \in L$. The proof of the next lemma is straightforward.

Lemma 8.6. (i) L is an OMP iff $L \backslash p^\perp \in \mathcal{S}$ holds for every $p \in L$.

(ii) If L is an OMP, then L is modal and $[p] = L \backslash p^\perp$ holds for every $p \in L$.

Every OML is modal, as is every OMP. Also, every OA that admits "sufficiently many" probability measures is modal.

9. A COMPACTNESS THEOREM

The compactness theorem for classical algebraic logic takes the following form for an orthoalgebra L:

Theorem 9.1. Let $X \subseteq L$ have the property that $X \cap S \neq \varnothing$ holds for every proper support $S \in \mathscr{S}$. Then there exists a finite subset X_0 of X with the same property.

Proof. Let Σ denote the set of all minimal proper supports in L. By theorem 8.2, it will suffice to prove that there exists a finite subset X_0 of X such that $X_0 \cap S \neq \varnothing$ holds for all $S \in \Sigma$. Thus, let

$$T = \{e \in L \mid \forall J^{\text{finite}} \subseteq X, \exists S \in \Sigma \text{ with } e \in S \text{ and } S \cap J = \varnothing\}$$

To begin with, we are going to prove that $T \in \mathscr{S}$. Suppose $e, f \in L$ with $e \in T$ and $e \leq f$, and let J be a finite subset of X. Then there exists $S \in \Sigma$ with $e \in S$ and $S \cap J = \varnothing$. Hence, since S is an order filter, $f \in S$, and it follows that $f \in T$. This proves that, if $p, q \in L$ with $p \perp q$, then $\{p, q\} \cap T \neq \varnothing \Rightarrow p \oplus q \in T$.

Conversely, let $p, q \in L$ with $p \perp q$ and suppose that $p \oplus q \in T$, but that $\{p, q\} \cap T = \varnothing$. Then there exist finite sets $J_p, J_q \subseteq X$ such that, for all $S \in \Sigma$,

$$p \in S \Rightarrow S \cap J_p \neq \varnothing \quad \text{and} \quad q \in S \Rightarrow S \cap J_q \neq \varnothing$$

Let $J = J_p \cup J_q$. Since $p \oplus q \in T$, there exists $S \in \Sigma$ such that $p \oplus q \in S$ and $S \cap J = \varnothing$. But, then, one of the conditions $p \in S$ or $q \in S$ must hold, so either $S \cap J_p \neq \varnothing$ or $S \cap J_q \neq \varnothing$, contradicting $S \cap J = \varnothing$. This proves that T is a support.

Suppose that T is a proper support. Then, by Theorem 8.2, there exists $S_0 \in \Sigma$ such that $S_0 \subseteq T$. By hypothesis, there exists $e \in X \cap S_0 \subseteq T$. Let $J_0 = \{e\}$. Since J_0 is a finite subset of X, there exists $S \in \Sigma$ with $e \in S$ and $S \cap J_0 = \varnothing$. Thus, we arrive at the contradiction $e \in S$ and $e \notin S$, from which we may conclude that $T = \varnothing$. Therefore, $1 \notin T$, and it follows that there exists a finite set $X_0 \subseteq X$ such that, for all $S \in \Sigma$, $1 \in S \Rightarrow X_0 \cap S \neq \varnothing$. However, since each $S \in \Sigma$ is an order filter, the condition $1 \in S$ is automatic, and our proof is complete. ■

In terms of the idea that each support $S \in \mathscr{S}$ determines corresponding modalities for all of the propositions $p \in L$, Theorem 9.1 may be paraphrased as follows: If $X \subseteq L$ has the property that for each proper support $S \in \mathscr{S}$, at least one proposition $p \in X$ is possible, then there is a finite subset of X with the same property.

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