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Logicoalgebraic Structures II. Supports in Test Spaces

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Test spaces are mathematical structures that underlie quantum logics in much the same way that Hilbert space underlies standard quantum logic. In this paper, we give a coherent account of the basic theory of test spaces and show how they provide an infrastructure for the study of quantum logics. If L is the quantum logic for a physical system $\mathscr{S}$, then a support in L may be interpreted as the set of all propositions that are possible when $\mathscr{S}$ is in a certain state. We present an analog for test spaces of the notion of a quantum-logical support and launch a study of the classification of supports.

1. INTRODUCTION

Early in the development of quantum logic, it was hoped that orthomodular lattices, or perhaps orthomodular posets, would provide a mathematical basis for nonclassical probability theory in much the same way that sigma fields of sets provide a foundation for classical probability and statistics. Indeed, this turned out to be true for the probabilities associated with the measurement of variables that are neither compound nor sequential. In the classical case, the field product of sigma fields affords an appropriate formulation for probabilities associated with compound experiments, and it was anticipated that an analogous construction could be found for the more general quantum logics. However, such a construction presents difficulties—even if orthomodular posets are replaced by more general orthoalgebras.

In retrospect, the difficulty is apparent even for orthodox quantum mechanics based on a Hilbert space $\mathscr{H}$, where the standard quantum logic

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L is understood to be the complete orthomodular lattice of all projection operators on $\mathscr{H}$. In the passage from the wave functions comprising $\mathscr{H}$ to the logic L, one loses the phase information essential for dealing with sequential measurements. For an explicit discussion of this difficulty in connection with sequential Stern–Gerlach experiments, see Wright (1978). Mathematical structures that underlie general quantum logics in the same way that Hilbert space underlies standard quantum logic are required to resolve the difficulty. Among the structures that have been proposed for this purpose are *manuals* (Randall and Foulis, 1973; Foulis and Randall, 1978), *spaces* (Gerelle *et al.*, 1974), *hypergraphs* (Gudder *et al.*, 1986), *cover spaces* (Gudder, 1986), *generalized sample spaces* (Kläy, 1988), and *test spaces* (Foulis, 1989).

This paper may be regarded as a sequel to Foulis *et al.* (1992), henceforth abbreviated FGR-I. In FGR-I, we introduced the idea of a *local filter* F in an orthoalgebra L. If the elements of L are regarded as propositions, then F may be viewed as all propositions in L that are *necessarily true* under a specific set of circumstances, e.g., when the system under investigation is in a certain state. Thus, $S := \{p \in L \mid p' \notin F\}$ may be regarded as all propositions in L that are *possible* under these circumstances. Such a subset S of L is called a *support*, and, since $F = \{p \in L \mid p' \notin S\}$, the support S uniquely determines the local filter F. Hence, from a strictly mathematical point of view, it is a matter of indifference whether we focus attention on local filters or on supports in orthoalgebras. Our purpose in the present paper is to extend these considerations to the test spaces that underlie orthoalgebras, where it turns out to be more convenient to deal with supports than with local filters. We begin by giving a coherent account of the basic theory of test spaces and their relationship to orthoalgebras.

2. DEFINITION OF AND HEURISTICS FOR TEST SPACES

The classical mathematical theory of probability is traditionally based on the notion of a *sample space* X, which, for heuristic purposes, is usually regarded as the set of all mutually exclusive and exhaustive outcomes of a certain experiment, measurement, test, physical operation, or other well-defined, reproducible procedure $\mathscr{P}$. The sample space X is conceived of as a mathematical representation of $\mathscr{P}$. Practical experimental programs, however, often involve more than one such procedure and, as a consequence, the sample X for such a program should be regarded as a *union* of the outcome sets of these procedures. This leads us to the following definition:

Definition 2.1. A *test space* is a pair $(X, \mathscr{T})$ consisting of a nonempty set X and a collection $\mathscr{T}$ of subsets of X satisfying the following conditions.

(i) (*Covering*) Each $x \in X$ belongs to at least one set $E \in \mathscr{T}$.
(ii) (*Irredundancy*) If $E, F \in \mathscr{T}$ and $E \subseteq F$, then $E = F$.

If $(X, \mathscr{T})$ is a test space, elements $x \in X$ are called *outcomes*, and sets $E \in \mathscr{T}$ are called *tests* or *operations*. By abuse of notation, we often refer to "the test space X," when what we really mean is "the test space $(X, \mathscr{T})$." We refer to $\mathscr{T}$ as the *quasimanual* of tests or operations for the test space X.

The heuristics for a test space X are as follows: A test $E \subseteq X$ may be thought of as the set of mutually exclusive and exhaustive outcomes of a well-defined, reproducible operation or procedure—in effect, E is a mathematical representation of the procedure. If E (or, more accurately, the operation or procedure corresponding to E) is executed, then one and only one outcome $x \in E$ will be secured as a consequence.

To help fix ideas, we consider the following examples:

Example 2.2. Let $\mathscr{H}$ be a Hilbert space (which may be thought of as the Hilbert space corresponding to a quantum mechanical system $\mathscr{S}$). Let $X := \{\psi \in \mathscr{H} \mid \|\psi\| = 1\}$ be the unit sphere in $\mathscr{H}$, and let $\mathscr{T}$ be the set of all maximal orthogonal subsets of X. Thus, for the test space X, an outcome is a normalized vector in $\mathscr{H}$ (which may be identified with a vector state for $\mathscr{S}$), and a test is an orthonormal basis for $\mathscr{H}$ (which may be thought of as representing a maximal observation on $\mathscr{S}$).

Example 2.3. Let $(\Lambda, \mathscr{F})$ be a Borel (or measurable) space; that is, a pair consisting of a nonempty set Λ and a σ-field $\mathscr{F}$ of subsets of Λ. (Λ may be thought of as a classical sample space and $\mathscr{F}$ may be regarded as the set of all events for Λ in the sense of classical probability theory.) Let X be the set of all nonempty sets in $\mathscr{F}$ and let $\mathscr{T}$ be the collection of all partitions of Λ into countably many sets in X. [A test $E \in \mathscr{T}$ may be regarded as an *experiment* (Kolmogorov, 1956, p. 9) and the outcomes $A \in X$ may be viewed as outcomes of such experiments.]

Example 2.4. Let L be an orthoalgebra (FGR-I). (L may be thought of as a set of propositions regarding a physical system $\mathscr{S}$.) Let X be the set of all nonzero elements in L, and let $\mathscr{T}$ be the collection of all finite orthogonal partitions of 1 in L.

The terminology in the next definition is suggested by the notion of an event in the classical theory of probability.

Definition 2.5. A subset A of a test space X is called an *event* if there exists a test $E \subseteq X$ such that $A \subseteq E$. We refer to such a test E as a test *for* A. The set of all events for X is denoted by $\mathscr{E}$.

Suppose that A is an event for the test space X and let $E \subseteq X$ be a test for A. If E is executed and the outcome $x \in E$ is secured, we say that the event A *occurs* if $x \in A$ and that it *nonoccurs* if $x \notin A$. If E, F are both tests for the same event A, it is a matter of indifference whether E or F is executed in determining the occurrence or nonoccurrence of A. Thus, we may speak of the occurrence or nonoccurrence of A without reference to the specific test involved.

In what follows, we assume that X is a test space, $\mathcal{T}$ is the collection of all tests $E \subseteq X$, and $\mathcal{E}$ is the collection of all events $A \subseteq X$.

3. THE CALCULUS OF EVENTS

The introduction of suitable operations and relationships in $\mathcal{E}$ enables us to regard it as a calculus of events for the test space X.

Definition 3.1. A collection of events $\{A_\alpha \mid \alpha \in I\}$ is said to be *compatible* or *simultaneously testable* if there exists a test $E \subseteq X$ such that $A_\alpha \subseteq E$ for all $\alpha \in I$.

We note that $\{A_\alpha \mid \alpha \in I\}$ is a compatible collection of events iff $\bigcup_{\alpha \in I} A_\alpha$ is an event.

Definition 3.2. Let $A, C \in \mathcal{E}$. If A and C are compatible events and $A \cap C = \emptyset$, we say that A and C are *orthogonal* and write $A \perp C$. If $A \perp C$ and $A \cup C$ is a test, we say that A and C are *local complements* of each other.

Every event A has at least one local complement; indeed, if E is a test for A, then $C := E \backslash A$ is a local complement of A. As a consequence of the irredundancy condition in Definition 2.1, a test E has only one local complement, namely, the empty event $\emptyset$.

Definition 3.3. If A and B are events that share a common local complement C, we say that A and B are *perspective* with axis C and we write $A \sim B$.

We note that, if $A \subseteq X$ is an event and $E \subseteq X$ is a test, then $A \sim E$ iff A is a test. Similarly, $A \sim \emptyset$ iff $A = \emptyset$.

Lemma 3.4 (Cancellation Law). If $A, B, C \in \mathcal{E}$ with $A \perp C$ and $B \perp C$, then $A \cup C \sim B \cup C \Rightarrow A \sim B$.

Proof. If D is an axis for $A \cup C \sim B \cup C$, *then* $D \cup C$ is an axis for $A \sim B$. ∎

Definition 3.5. Let $A, B \in \mathcal{E}$. We say that A *implies* B and write $A \leq B$ if there is a finite sequence $C_0, C_1, C_2, \ldots, C_n \in \mathcal{E}$ such that $A = C_0$,

$B = C_n$, and, for every $i = 1, 2, \ldots, n$, either $C_{i-1} \subseteq C_i$ or $C_{i-1} \sim C_i$. If $A \le B$ and $B \le A$, we say that A and B are *equivalent* events and write $A \equiv B$.

The relation $\le$ is reflexive and transitive on $\mathscr{E}$, that is, it is a preorder on $\mathscr{E}$. Consequently, the relation $\equiv$ is an equivalence relation on $\mathscr{E}$. Also, if $A \subseteq X$ is an event and $E \subseteq X$ is a test, then $\varnothing \le A \le E$. We omit the simple proof of the next lemma.

Lemma 3.6. Let $A \subseteq X$ be an event and let $E \subseteq X$ be a test. Then:
(i) $A \le \varnothing \Rightarrow A = \varnothing$.
(ii) $E \le A \Rightarrow A$ is a test.

Lemma 3.7. Let $A, B, A', B' \in \mathscr{E}$ and suppose that A' and B' are local complements of A and B, respectively. Then

$$A \le B \Rightarrow B' \le A'$$

Proof. It will be enough to prove (i) $A \subseteq B \Rightarrow B' \le A'$ and (ii) $A \sim B \Rightarrow B' \le A'$. To prove (i), suppose that $A \subseteq B$. Then $B' \subseteq B' \cup (B \backslash A) \sim A'$ with axis A; hence, $B' \le A'$. To prove (ii), suppose $A \sim B$ with axis C. Then $B' \sim C$ with axis B and $C \sim A'$ with axis A; hence, $B' \le A'$. ■

Example 3.8. The test space $X := \{a, b, c\}$ for which the quasimanual of tests is $\mathscr{T} := \{\{a, b\}, \{b, c\}, \{c, a\}\}$ is called the *little triangle.*

In the little triangle, we note that $\{a\} \perp \{c\}$ and at the same time $\{a\} \sim \{c\}$ with axis $\{b\}$. Thus, we have both $\{a\} \le \{c\}$ and $\{a\} \perp \{c\}$, a situation which seems counterintuitive. This leads us to the following:

Definition 3.9. The test space X is said to be *consistent* if, for all $A, C \in \mathscr{E}$, $A \perp C$ and $A \le C \Rightarrow A = \varnothing$.

It is easy to see that X is consistent iff no nonempty event in X implies any of its local complements. Also, if X is consistent, then the equivalence relation $\equiv$ on $\mathscr{E}$ is the transitive closure of the perspectivity relation $\sim$ on $\mathscr{E}$. In general, the relation $\sim$ is not transitive on the calculus of events $\mathscr{E}$, even for a consistent test space. When it is, we can also prove that perspectivity is additive in the following sense:

Lemma 3.10 (Additivity Lemma). Suppose that $\sim$ is a transitive relation on $\mathscr{E}$ and let $A, B_1, B_2 \in \mathscr{E}$ with $A \perp B_1$ and $A \perp B_2$. Then $B_1 \sim B_2 \Rightarrow A \cup B_1 \sim A \cup B_2$.

Proof. Let C be an axis for $B_1 \sim B_2$ and let P and Q be local complements of $A \cup B_1$ and $A \cup B_2$, respectively. Then, $A \cup P \sim C$ with axis

B_1 and $C \sim A \cup Q$ with axis B_2; hence, $A \cup P \sim A \cup Q$ by transitivity of $\sim$. By the cancellation law (Lemma 3.4), it follows that $P \sim Q$ with axis, say, D. Thus, $A \cup B_1 \sim D$ with axis P and $D \sim A \cup B_2$ with axis Q; hence, $A \cup B_1 \sim A \cup B_2$ by transitivity of $\sim$. ■

4. ALGEBRAIC TEST SPACES

If perspectivity preserves orthogonality in $\mathscr{E}$, we say that X is an *algebraic* test space. More formally:

Definition 4.1. The test space X is *algebraic* if, for all $A, B, C \in \mathscr{E}$ with $A \sim B$, $A \perp C \Rightarrow B \perp C$.

The test spaces in Examples 2.2–2.4 are algebraic. The simplest nonalgebraic test space is the little triangle (Example 3.8).

Theorem 4.2. The following conditions are mutually equivalent:

(i) X is an algebraic test space.

(ii) If $A, B, C \in \mathscr{E}$, $A \le B$, and $B \perp C$, then $A \perp C$.

(iii) If $A, B, C \in \mathscr{E}$, $A \sim B$, and C is a local complement of A, then C is a local complement of B.

Proof. To prove (i) $\Rightarrow$ (ii), assume (i) and the hypotheses of (ii). Since $A \le B$, there is a sequence of events $A_0, A_1, A_2, \ldots, A_n$ such that $A = A_0$, $B = A_n$, and, for each $k = 1, 2, \ldots, n$ either $A_{k-1} \subseteq A_k$ or $A_{k-1} \sim A_k$. If $A_{k-1} \subseteq A_k$ and $A_k \perp C$, it is clear that $A_{k-1} \perp C$. If $A_{k-1} \sim A_k$ and $A_k \perp C$, then $A_{k-1} \perp C$ by (i). Hence, $A = A_0 \perp C$ by induction. To prove (ii) $\Rightarrow$ (iii), assume (ii) and the hypotheses of (iii). Let D be the axis of the perspectivity $A \sim B$. Since $B \sim A$, we have $B \le A$, so, because $A \perp C$, (ii) implies that $B \perp C$. Let P be a local complement of $B \cup C$, so that $C \cup P \sim D$ with axis B. From this, and the facts that $D \sim C$ with axis A and $P \subseteq C \cup P$, we have $P \le C$. But $C \perp P$, so (ii) implies $P \perp P$; hence, $P = \emptyset$ and therefore C is a local complement of B. That (iii) $\Rightarrow$ (i) follows from the observation that if two events are orthogonal, then either event may be enlarged to a local complement of the other. ■

Corollary 4.3. If X is algebraic, then $\sim$ is transitive on $\mathscr{E}$.

Proof. Suppose $A, B, P \in \mathscr{E}$ with $B \sim A$ and $A \sim P$. Let C be an axis for $A \sim P$. By part (iii) of Theorem 4.2, C is a local complement of B; hence, $B \sim P$ with axis C. ■

Corollary 4.4. If X is algebraic, then X is consistent.

Proof. If $A, C \in \mathscr{E}$ with $A \le C$ and $A \perp C$, then $A \perp A$ by part (ii) of Theorem 4.2, hence, $A = \emptyset$. ■

Corollary 4.5. Let X be algebraic and let $A, B \in \mathscr{E}$. Then $A \leq B$ iff there exists $C \in \mathscr{E}$ with $A \perp C$ and $A \cup C \sim B$.

Proof. If $A \perp C$ and $A \cup C \sim B$, then $A \subseteq A \cup C \sim B$, so $A \leq B$. Conversely, suppose $A \leq B$ and let D be a local complement of B in $\mathscr{E}$. Then $A \perp D$ by part (ii) of Theorem 4.2. Let C be a local complement of $A \cup D$ in $\mathscr{E}$. Then, $A \perp C$ and $A \cup C \sim B$ with axis D. ■

Theorem 4.6. (Additivity Theorem). Let X be algebraic and suppose that $A_1, A_2, B_1, B_2 \in \mathscr{E}$ with $A_1 \sim A_2$, $B_1 \sim B_2$, and $A_1 \perp B_1$. Then, $A_2 \perp B_2$ and $A_1 \cup B_1 \sim A_2 \cup B_2$.

Proof. That $A_2 \perp B_2$ follows from two applications of the condition in Definition 4.1. Using the additivity lemma (Lemma 3.10) twice, we have $A_1 \cup B_1 \sim A_1 \cup B_2 \sim A_2 \cup C_2$. ■

5. THE LOGIC OF A TEST SPACE

The relation $\leq$ is a preorder on $\mathscr{E}$, and therefore it induces a partial order relation on the quotient space $\mathscr{E}/\equiv$ as in the following definition.

Definition 5.1. If $A \in \mathscr{E}$, we define $\pi(A) := \{B \in \mathscr{E} \mid A \equiv B\}$ and we refer to $\pi(A)$ as the *proposition* affiliated with A. The set $\Pi(X) := \{\pi(A) \mid A \in \mathscr{E}\}$, partially ordered by $\pi(A) \leq \pi(B)$ iff $A \leq B$, is called the *logic* of the test space X. If $\pi(A) \leq \pi(B)$, we say that $\pi(A)$ *implies* $\pi(B)$. We define $0, 1 \in \Pi(X)$ by $0 = \pi(\emptyset)$ and $1 = \pi(E)$, where $E \subseteq X$ is any test.

If X is understood, we usually write Π rather than $\Pi(X)$. Also, if $x \in X$, we usually write $\pi(x)$ rather than $\pi(\{x\})$.

Suppose that $A, C, D \in \mathscr{E}$ and that both C and D are local complements of A. Then $C \sim D$ with axis A, and it follows that $\pi(C) = \pi(D)$. This observation leads us to our next definition.

Definition 5.2. If $A \in \mathscr{E}$, we define the *negation* $\pi(A)'$ of the proposition $\pi(A) \in \Pi$ by $\pi(A)' := \pi(C)$, where C is any local complement of A in $\mathscr{E}$.

Evidently $(\pi(A)')' = \pi(A)$, $0' = 1$, and $1' = 0$. Also, by Lemma 3.7, if $A, B \in \mathscr{E}$, then $\pi(A) \leq \pi(B) \Rightarrow \pi(B)' \leq (A)'$.

The heuristics for Definition 5.1 and 5.2 are as follows: If $A \in \mathscr{E}$, then a test $E \subseteq X$ is called a *test for the proposition* $\pi(A)$ if there is an event $B \in \pi(A)$ with $B \subseteq E$. If B occurs as a consequence of an execution of E, we say that the proposition $\pi(A)$ is *confirmed*, otherwise we say that $\pi(A)$ is *refuted*. We note that $\pi(A)$ is refuted iff $\pi(A)'$ is confirmed.

Lemma 5.3. The following conditions are mutually equivalent:
(i) X is a consistent test space.
(ii) If $A, D \in \mathscr{E}$, then $\pi(D) \le \pi(A), \pi(A)' \Rightarrow \pi(D) = 0$.
(iii) $(\Pi, \le, ', 0, 1)$ is an orthocomplemented poset.

Proof. Assume (i) and the hypothesis of (ii). Since $\pi(D) \le \pi(A)'$, we have $\pi(A) = (\pi(A)')' \le \pi(D)'$, so $\pi(D) \le \pi(A) \le \pi(D)'$. Let B be a local complement for D. Thus, $\pi(D) \le \pi(D)' = \pi(B)$, so that $D \le B$. But, $D \perp B$; hence, $D = \varnothing$ by (i). To prove that (ii) $\Rightarrow$ (iii), note that, if (ii) holds, then, for every $A \in \mathscr{E}$, the infimum $\pi(A) \wedge \pi(A)'$ exists in Π and is equal to 0; hence, by the de Morgan law, the supremum $\pi(A) \vee \pi(A)'$ exists and equals 1. That (iii) $\Rightarrow$ (i) is clear. ■

If $A, C \in \mathscr{E}$, then, obviously, $A \perp C \Rightarrow \pi(A) \le \pi(C)'$. As an immediate corollary of Theorem 4.2, we have the following result:

Lemma 5.4. X is an algebraic test space iff, for $A, C \in \mathscr{E}$, $A \perp C \leftrightarrow \pi(A) \le \pi(C)'$.

As a consequence of Lemma 5.4 and the additivity theorem (Theorem 4.6), the *orthogonal sum*, introduced in the next definition, is well defined.

Definition 5.5. Let X be an algebraic test space and let $A, C \in \mathscr{E}$. Then, if $\pi(A) \le \pi(C)'$, we define the *orthogonal sum* $\pi(A) \oplus \pi(C)$ in Π by $\pi(A) \oplus \pi(C) = \pi(A \cup C)$.

Lemma 5.6. Let X be an algebraic test space and let $A, B \in \mathscr{E}$. Then $\pi(A) \le \pi(B)$ iff there exists $C \in \mathscr{E}$ with $\pi(A) \le \pi(C)'$ such that $\pi(A) \oplus \pi(C) = \pi(B)$.

Proof. Corollary 4.5 and Definition 5.5. ■

In the following theorem, we establish contact between algebraic test spaces and orthoalgebras. The proof of the theorem is a simple matter of checking the conditions in Definition 2.1 of FGR-I.

Theorem 5.7. If X is an algebraic test space, then the logic Π of X forms an orthoalgebra with respect to the partially defined binary operation $\oplus$.

In Theorem 5.7, it is plain that the orthocomplement of a proposition in Π coincides with its negation as in Definition 5.2. Furthermore, by Lemma 5.6, the partial order on Π induced by the orthogonal sum coincides with the implication relation given in Definition 5.1.

In Example 2.2, the logic Π is isomorphic to the complete orthomodular lattice of all closed linear subspaces of the Hilbert space $\mathscr{H}$; in Example 2.3, it is isomorphic to the Boolean algebra of sets formed by the

σ-field $\mathcal{F}$; and in Example 2.4, it is isomorphic to the orthoalgebra L. Thus, in particular, every orthoalgebra is isomorphic to the logic of an algebraic test space.

6. COHERENT TEST SPACES

Let X be an algebraic test space with logic Π. In this section, we develop conditions guaranteeing that Π is an orthomodular poset (OMP) or an orthomodular lattice (OML).

Definition 6.1. Let $x, y \in X$ and let $M, N \subseteq X$. If $\{x\} \perp \{y\}$, we say that x is *orthogonal* to y and write $x \perp y$. If $x \neq y$ and x is not orthogonal to y, we write $x \natural y$. We define $M^{\perp} := \{x \in X \mid x \perp y \text{ for all } y \in M\}$ and $M^{\perp\perp} := (M^{\perp})^{\perp}$. If $x \perp y$ holds whenever $x, y \in N$ with $x \neq y$, then N is said to be a *pairwise orthogonal* set.

We note that any event is a pairwise orthogonal set. The following example shows that the converse is false.

Example 6.2. Let $X := \{a, b, c, x, y, z\}$, $\mathcal{T} := \{E, F, G\}$, where $E := \{a, z, b\}$, $F := \{b, x, c\}$, and $G := \{c, y, a\}$. Then $(X, \mathcal{T})$ is an algebraic test space, called the *Wright triangle*, and $N := \{a, b, c\}$ is a pairwise orthogonal subset of X that is not an event.

Definition 6.3. The test space X is *coherent* if every pairwise orthogonal subset of X that is contained in the union of finitely many tests is an event.

Examples 2.2 and 2.3 are coherent test spaces. The test space in Example 2.4 is coherent iff the orthoalgebra L is an OMP. A simple inductive argument yields the following result:

Lemma 6.4. X is coherent iff, for $A, C \in \mathcal{E}$, $A \subseteq C^{\perp} \Rightarrow A \perp C$.

The test space X is both algebraic and coherent iff, for $A, C \in \mathcal{E}$, the three conditions $\pi(A) \leq \pi(B)'$, $A \perp B$, and $A \subseteq B^{\perp}$ are mutually equivalent.

Theorem 6.5. The logic Π of a coherent algebraic test space X is an orthomodular poset. Furthermore, if $A, B \in \mathcal{E}$, the following conditions are mutually equivalent:

(i) $\pi(A) \leq \pi(B)$.
(ii) $B^{\perp} \subseteq A^{\perp}$.
(iii) $A^{\perp\perp} \subseteq B^{\perp\perp}$.

Proof. Let $A, B, C \in \mathcal{E}$ with $A \perp B$, $B \perp C$, and $A \perp C$. Then, $A \cup B \subseteq C^{\perp}$, and it follows from Lemma 6.4 that $A \cup B \cup C \in \mathcal{E}$. Therefore the orthoalgebra Π is an OMP by Theorem 2.12 of FGR-I. To prove (i) $\Rightarrow$ (ii), assume that $\pi(A) \leq \pi(B)$ and that $x \in B^{\perp}$. By coherence, $B \perp \{x\}$ and, hence, $A \perp \{x\}$ by Theorem 4.2, from which it follows that $x \in A^{\perp}$. That (ii) $\leftrightarrow$ (iii) is clear, so it will be enough to prove that (ii) $\Rightarrow$ (i). Thus, assume that $B^{\perp} \subseteq A^{\perp}$. Let B' be a local complement of B and note that $B' \subseteq B^{\perp}$, so $B' \subseteq A^{\perp}$; hence $B' \perp A$ by coherence. Therefore, $\pi(B)' \leq \pi(A)'$, so $\pi(A) \leq \pi(B)$. ■

Lemma 6.6. Let X be algebraic. Then X is coherent iff, for every $A \in \mathscr{E}$, the supremum $\bigvee_{x \in A} \pi(x)$ exists in Π and equals $\pi(A)$.

Proof. Randall and Foulis (1983, Lemma 41). ■

Definition 6.7. A *square* in X is an ordered four-tuple (a, b, c, d) of outcomes in X such that $a \perp b$, $b \perp c$, $c \perp d$, $d \perp a$, $a \natural c$, and $b \natural d$. The test space X is *square-free* if it admits no squares.

Lemma 6.8. Let X be a coherent, algebraic, square-free test space, let $A, B \in \mathscr{E}$, and suppose that the infimum $\pi(A) \wedge \pi(B)$ fails to exist in the OMP Π. Let $D := A^{\perp\perp} \cap B^{\perp\perp}$. Then D is an orthogonal set that is not an event.

Proof. First suppose that D is not an orthogonal set. Then there exist $c, d \in D$ with $c \natural d$. Because $d \in A^{\perp\perp}$, it follows that $A^{\perp} \subseteq \{d\}^{\perp}$; hence, that $\pi(d) \leq \pi(A)$. Therefore, by Corollary 4.5, there exists $N \in \mathscr{E}$ such that $\{d\} \perp N$ and $\{d\} \cup N \sim A$. Thus, by Theorem 6.5, we have $(\{d\} \cup N)^{\perp\perp} = A^{\perp\perp}$; hence, we can replace A by $\{d\} \cup N$ and thus assume, without loss of generality, that $d \in A$. Likewise, replacing B by an equivalent event if necessary, we can assume that $d \in B$. Thus, $d \in A \cap B$. Let $A', B' \in \mathscr{E}$ be local complements of A and B, respectively. If A' were compatible with B', then $\pi(A)'$ would be compatible with $\pi(B)'$ in the OMP Π (FGR-I); hence, $\pi(A)$ would be compatible with $\pi(B)$ in the OMP Π. However, in an OMP, two compatible elements always have an infimum. Since $\pi(A) \wedge \pi(B)$ does not exist, it follows that $A' \cup B'$ cannot be an event. Since X is coherent, it follows that there exist $e \in A' \subseteq A^{\perp}, f \in B' \subseteq B^{\perp}$ such that $e \natural f$. Because $d \in A \cap B$, we have $e, f \perp d$. Since $c \in D = A^{\perp\perp} \cap B^{\perp\perp}$, we also have $e, f \perp c$. Thus, (c, f, d, e) is a square, contradicting the hypothesis and proving that D is an orthogonal set. If D were an event, the fact that $D = A^{\perp\perp} \cap B^{\perp\perp}$ would imply that $\pi(D) = \pi(A) \wedge \pi(B)$, again contradicting the hypothesis. ■

The following theorem generalizes the well-known Loop Lemma for orthomodular lattices (Greechie, 1971; Kalmbach, 1983, p. 43). We use $\# M$ to denote the cardinal number of a set M.

Theorem 6.9. Let X be a coherent, square-free, algebraic test space. Suppose that there exists a nonnegative integer m such that $\#(E \cap F) \leq m$ holds for all tests $E, F \subseteq X$ with $E \neq F$. Then Π is an orthomodular lattice (OML).

Proof. Suppose that Π is not an OML. Then there exist $A, B \in \mathscr{E}$ such that $\pi(A) \wedge \pi(B)$ fails to exist in Π. By Lemma 6.8, $D := A^{\perp\perp} \cap B^{\perp\perp}$ is an orthogonal subset of X that is not an event. Suppose that C is any event contained in D. Arguing as in the proof of Lemma 6.8, we can replace A and B by equivalent events, if necessary, and thus assume without loss of generality that $C \subseteq A \cap B$. Let $E, F \subseteq X$ be tests for A, B, respectively. If $E = F$, then A and B are compatible, contradicting the fact that $\pi(A) \wedge \pi(B)$ fails to exist. Therefore, $E \neq F$, and it follows from the fact that $C \subseteq A \cap B \subseteq E \cap F$ that $\#C \leq m$. By coherence, every finite subset of D is an event. Hence, no finite subset of D contains more than m elements, and it follows that D is finite, contradicting the fact that D is not an event. ■

7. SUPPORTS AND THE CANONICAL MAP

In the context of quasimanuals, supports were originally introduced in Foulis *et al.* (1983). A support $S \subseteq X$ may be regarded as the set of all outcomes in X that are possible under a specific set of circumstances.

Definition 7.1. A *support* for the test space X is a subset S of X that enjoys the following *exchange property*: If $E, F \subseteq X$ are tests, then $S \cap E \subseteq F \Rightarrow S \cap F \subseteq E$.

Note that the empty set and X itself are supports and that a nonempty support must have a nonempty intersection with every test. Also, if $\emptyset \neq S \subseteq X$, then S is a support iff $(S, \mathscr{T}_S)$ is a test space, where $\mathscr{T}_S := \{E \cap S \mid E \in \mathscr{T}\}$. [The exchange property enforces the irredundancy condition for $(S, \mathscr{T}_S)$.] In this way, each nonempty support S can be regarded as a test space in its own right.

Definition 7.2. We denote by $\mathscr{S}$ the set of all supports $S \subseteq X$.

Clearly, the set-theoretic union of supports is again a support; hence, partially ordered by $\subseteq$, $\mathscr{S}$ forms a complete lattice. We refer to $\mathscr{S}$ as the *support lattice* of the test space X.

Lemma 7.3. Let $A, B \in \mathscr{E}$ with $A \sim B$, let $S \in \mathscr{S}$, and suppose that $E, F \subseteq X$ are tests for A, B, respectively. Then, $S \cap E \subseteq A \Leftrightarrow S \cap F \subseteq B$.

Proof. Let C be a common local complement for A and B and assume that $S \cap E \subseteq A$. Since E and $A \cup C$ are tests for A and $S \cap E \subseteq A \subseteq A \cup C$, we have $S \cap (A \cup C) \subseteq E$ by the exchange property. Therefore, $S \cap (A \cup C) \subseteq S \cap E \subseteq A$ and, since $A \cap C = \emptyset$, it follows that $S \cap C = \emptyset$; hence, $S \cap (B \cup C) = S \cap B \subseteq B \subseteq F$. But $(B \cup C)$ and F are tests for B, so $S \cap F \subseteq B \cup C$ follows from another application of the exchange property. Because $S \cap C = \emptyset$, we conclude that $S \cap F \subseteq B$. ■

Theorem 7.4. Let $A, B \in \mathscr{E}$ with $A \leq B$, let $S \in \mathscr{S}$, and suppose that $E, F \leq X$ are tests for A, B, respectively. Then, $S \cap E \subseteq A \Rightarrow S \cap F \subseteq B$.

Proof. In view of Lemma 7.3, it is sufficient to prove the theorem for the special case in which $A \subseteq B$. If $A \subseteq B$, E is a test for A, and F is a test for B, then both E and F are tests for A; hence, by Lemma 7.3, $S \cap E \subseteq A \Rightarrow S \cap F \subseteq A \subseteq B$. ■

Definition 7.5. If $A \in \mathscr{E}$, we define $[A] := \bigcup \{S \in \mathscr{S} \mid S \cap E \subseteq A\}$, where E is any test for A.

As a consequence of Lemma 7.3, $[A]$ is independent of the choice of the test E for A and, because the union of supports is a support, $[A] \in \mathscr{S}$. The map $[\cdot]: \mathscr{E} \to \mathscr{S}$ defined by $A \mapsto [A]$ is called the *canonical map*, and a support having the form $[A]$ for some $A \in \mathscr{E}$ is called a *principal support*. We note that, if E is a test for A, then $[A] \cap E \subseteq A$.

Theorem 7.6. The principal supports are meet-dense in the complete lattice $\mathscr{S}$; that is, every support is the infimum of the principal supports that contain it.

Proof. Foulis *et al.* (1983, Lemma 38). ■

Theorem 7.7. If $A, B \in \mathscr{E}$ with $A \leq B$, then $[A] \subseteq [B]$.

Proof. Theorem 7.4. ■

If $A, B \in \mathscr{E}$ with $A \equiv B$, then $[A] = [B]$ by Theorem 7.7. Therefore we can "lift" the map $[\cdot]: \mathscr{E} \to \mathscr{S}$ to a map $[\cdot]: \Pi \to \mathscr{S}$, also called the *canonical map*, by defining $[\pi(A)] = [A]$ for all $A \in \mathscr{E}$. Thus, $[\cdot]: \Pi \to \mathscr{S}$ is an isotone map of the poset Π onto a meet-dense subset of the complete lattice $\mathscr{S}$.

Motivated by Definition 8.5 in FGR-I, we make the following definition:

Definition 7.8. If $x \in X$, we define $[x] = [\{x\}]$, and we say that x is *modal* if $[x] \neq \emptyset$. If every $x \in X$ is modal, we say that the test space X is *modal.*

The test spaces in Examples 2.2–2.4 are modal. The following example shows that not every test space is modal.

Example 7.9. Let $X := \{a, b, c, x, y, z, v\}$, let $\mathscr{T} := \{E, F, G, H\}$, where $E := \{a, x, v\}$, $F := \{v, y, b\}$, $G := \{v, z, c\}$, and $H := \{a, b, c\}$. Then $(X, \mathscr{T})$ is an algebraic test space, but v is not a modal element of X. This test space is called the *wedge*.

Lemma 7.10. (i) $x \in X$ is a modal iff $x \in [x]$.
(ii) X is modal iff $A \subseteq [A]$ for every $A \in \mathscr{E}$.

Proof. (i) Suppose that $[x] \neq \emptyset$ and let E be a test for $\{x\}$. Then $\emptyset \neq [x] \cap E \subseteq \{x\}$, and so $x \in [x]$. (ii) If $A \subseteq [A]$ for every event A, then $x \in [x]$ for every $x \in X$. Conversely, suppose that $x \in [x]$ holds for all $x \in X$ and let $A \in \mathscr{E}$. Then, for $x \in A$, we have $\{x\} \leq A$, and it follows from Theorem 7.7 that $x \in [x] \subseteq [A]$. ■

8. SUPPORTS IN X AND IN Π

In this section we discuss the connection between supports in an algebraic test space X and supports in the orthoalgebra Π (FGR-I, Definition 7.1). Note that to say that $S_+ \subseteq \Pi$ is a support in Π means that whenever $A, B \in \mathscr{E}$ with $A \perp B$, $\pi(A \cup B) \in S_+$ iff $\pi(A) \in S_+$ or $\pi(B) \in S_+$. In particular, if S_+ is a support in Π, then it is an order filter in Π. We omit the straightforward proof of the following lemma.

Lemma 8.1. Let $A, B \in \mathscr{E}$ with $A \leq B$. Then, for all $S \in \mathscr{S}$:
(i) $S \cap A = \emptyset \Leftrightarrow S \subseteq [\pi(A)']$.
(ii) $S \cap B = \emptyset \Rightarrow S \cap A = \emptyset$.

Definition 8.2. If $S \in \mathscr{S}$, we define $S_\Pi \subseteq \Pi$ by

$$S_\Pi := \{\pi(A) \mid A \in \mathscr{E} \text{ and } S \cap A \neq \emptyset\}$$

Lemma 8.1 guarantees that the condition $S \cap A \neq \emptyset$ in Definition 8.2 is independent of the choice of the representative A of the proposition $\pi(A) \in \Pi$.

Theorem 8.3. If X is algebraic and $S \in \mathscr{S}$, then S_Π is a support in the orthoalgebra Π and $S = \{x \in X \mid \pi(x) \in S_\Pi\}$.
Furthermore, if $S, T \in \mathscr{S}$, then $S \subseteq T \Leftrightarrow S_\Pi \subseteq T_\Pi$.

Proof. Let $A, C \in \mathscr{E}$ with $A \perp C$. We have to prove that $\pi(A \cup C) = \pi(A) \oplus \pi(C) \in S_\Pi$ iff at least one of the propositions $\pi(A)$ or $\pi(C)$ belongs to S_Π. But this translates into the obvious condition that $S \cap (A \cup C) \neq \emptyset$ iff $S \cap A \neq \emptyset$ or $S \cap C \neq \emptyset$. Also,

$$S = \{x \in X \mid S \cap \{x\} \neq \emptyset\} = \{x \in X \mid \pi(x) \in S_\Pi\}.$$

That $S \subseteq T \Leftrightarrow S_\Pi \subseteq T_\Pi$ is obvious. ■

Example 8.4. If X is a nonempty set and $\mathcal{T} := \{X\}$, then $(X, \mathcal{T})$ is an algebraic, coherent, square-free test space, $\mathcal{E} = \mathcal{S}$ is the set of all subsets of X, $[A] = A$ for all $A \in \mathcal{E}$, and the map $\pi: \mathcal{E} \to \Pi$ given by $A \mapsto \pi(A)$ is an isomorphism of the Boolean algebra $\mathcal{E}$ onto Π. We refer to $(X, \{X\})$ as a *classical* test space. Suppose that X is an infinite set and define $S_* := \{\pi(A) \mid A \text{ is an infinite subset of } X\}$, noting that S_* is a support in Π. Clearly, there is no support $S \subseteq X$ for which $S_* = S_\Pi$.

As Example 8.4 shows, the order monomorphism $S \mapsto S_\Pi$ from test-space supports $S \subseteq X$ to orthoalgebra supports $S_\Pi \subseteq \Pi$ is not necessarily surjective. However, we do have the following result:

Theorem 8.5. Let X be an algebraic test space in which every test is a finite set. Then, the map $S \mapsto S_\Pi$ is an isomorphism of the lattice $\mathcal{S}$ onto the lattice $\mathcal{S}_\Pi$ of all supports in the orthoalgebra Π.

Proof. Let $S_+ \in \mathcal{S}_\Pi$. Since every test is finite, it follows that every event $A \in \mathcal{E}$ is finite; hence, by induction on the number of elements of A, $\pi(A) \in S_+$ iff there exists $a \in A$ such that $\pi(a) \in S_+$. Let $S := \{x \in X \mid \pi(x) \in S_+\}$. We claim that $S \in \mathcal{S}$. Indeed, let $E, F \subseteq X$ be tests such that $S \cap E \subseteq F$. Let $x \in S \cap F$ and suppose that $x \notin E$. Note that $F \backslash E$ is perspective to $E \backslash F$ with axis $E \cap F$. Therefore, $\pi(x) \leq \pi(F \backslash E) = \pi(E \backslash F)$ and, since $\pi(x) \in S_+$, it follows that $\pi(E \backslash F) \in S_+$; hence, there exists $y \in E \backslash F$ such that $\pi(y) \in S_+$. But then, $y \in S \cap (E \backslash F)$, contradicting $S \cap E \subseteq F$, and proving that $S \in \mathcal{S}$. Evidently, $S_+ = S_\Pi$, showing that $S \mapsto S_\Pi$ maps $\mathcal{S}$ onto $\mathcal{S}_\Pi$. ■

9. LOGICAL CLASSIFICATION OF SUPPORTS

We regard a support $S \in \mathcal{S}$ as the set of all outcomes that are possible when tested under a particular set of circumstances. Specifically, if E is a test, then $S \cap E$ is the set of outcomes in E that can be secured and $E \backslash S$ is the set of outcomes in E that are impossible if the test E is executed under the given circumstances. Thus, we can introduce notions corresponding to the classical Aristotelian modalities.

Definition 9.1. Let $S \in \mathcal{S}$, $A \in \mathcal{E}$, and let E be a test for A.

(i) If $S \cap E \subseteq A$, we say that A is *S-necessary*.

(ii) If $S \cap A \neq \varnothing$, we say that A is *S-possible*.

(iii) If $S \cap A = \varnothing$, we say that A is *S-impossible*.

(iv) If A is S-possible but not S-necessary, we say that A is *S-contingent*.

Notice that parts (ii)–(iv) of Definition 9.1 are unavoidable consequences of part (i) and our intuitive notions of these modalities.

The next result, which is a direct consequence of Definition 7.5, shows that $[A]$ is the largest support for which the event A is necessary.

Lemma 9.2. Let $S \in \mathscr{S}$, $A \in \mathscr{E}$. Then A is S-necessary iff $S \subseteq [A]$.

Definition 9.3. Let $S \in \mathscr{S}$.

(i) S is *prime* iff whenever A and C are S-possible events, there exists and S-possible event D with $D \leq A, C$.

(ii) S is *dispersion-free* iff there are no S-contingent events.

(iii) S is *minimal* iff (i) $S \neq \varnothing$ and (ii) $T \in \mathscr{S}$ with $\varnothing \neq T \subseteq S \Rightarrow T = S$.

Suppose that L is an orthomodular lattice and $X = L \backslash \{0\}$ is organized into a test space as in Example 2.4. If I is a proper ideal in L, then $S := L \backslash I$ is a nonempty support for the test space X. Furthermore, I is a prime ideal in the conventional sense that $\pi(A) \wedge \pi(B) \in I \Rightarrow \pi(A) \in I$ or $\pi(B) \in I$ iff S is a prime support, $I = \phi^{-1}(0)$ for a lattice homomorphism ϕ from L onto the two-element Boolean algebra $\{0, 1\}$ iff S is dispersion-free, and I is a maximal proper ideal iff S is minimal. For a Boolean algebra L, these three conditions turn out to be equivalent. More generally, we have the following result:

Lemma 9.4. Suppose X is consistent, let $\varnothing \neq S \in \mathscr{S}$, and consider the conditions; (i) S is prime, (ii) S is dispersion-free and (iii) S is minimal. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii): Suppose that S is prime and let A be an event that is S-possible. We have to prove that A is S-necessary. Let C be a local complement of A. It is enough to prove that C is S-impossible. However, if C were S-possible, then there would exist an S-possible event D with $D \leq A, C$ and since X is consistent, it would follow that $D = \varnothing$, contradicting $D \cap S \neq \varnothing$.

(ii) $\Rightarrow$ (iii): Suppose that S is dispersion-free and let $T \in \mathscr{S}$ with $\varnothing \neq T \subseteq S$. Let $x \in S$. Then $\{x\}$ is S-possible, so $\{x\}$ is S-necessary, and therefore $T \subseteq S \subseteq [\{x\}]$; hence, $x \in T$. ■

In general neither of the implications in Lemma 9.4 can be reversed. For instance, let X be the unit sphere in a three-dimensional Hilbert space and organize X into a test space as in Example 2.2. Choose $\phi \in X$ and let $S := \{\psi \in X \mid \langle \phi, \psi \rangle \neq 0\}$. By Cohen and Svetlichny (1987), S is a minimal support in X, but X admits no dispersion-free supports. An example of a dispersion-free support that is not prime is as follows: Let $X := \{a, b, c, d\}$ and organize X into a test space by taking $E := \{a, b\}$ and $F := \{c, d\}$ as the only tests. Then $S := \{a, c\}$ is a dispersion-free support that is not prime.

A particularly interesting and important class of supports are those singled out by the following definition.

Definition 9.5. A support $S \in \mathscr{S}$ is said to be *Jauch–Piron* if whenever $A, B \in \mathscr{E}$ are both S-necessary, there is a $C \in \mathscr{E}$ such that C is S-necessary and $C \leq A, B$.

Let W be a von Neumann density operator on the Hilbert space $\mathscr{H}$ of Example 2.2. Then the set $S := \{\psi \in X \mid W\psi \neq 0\}$ is a Jauch–Piron support in X. However, in this example, there are infinitely many non-Jauch–Piron supports. In Example 2.3, every support in X is Jauch–Piron. In Example 2.4, if L is a Boolean algebra, then every support in X is Jauch–Piron; otherwise, in general, this is not so. In a forthcoming paper, we shall give a detailed exposition of the theory of Jauch–Piron supports and its connection with Jauch–Piron probability measures on orthoalgebras.

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