ELSEVIER

Topology and its Applications 56 (1994) 165–173

TOPOLOGY
AND ITS
APPLICATIONS

Representations of locally compact orthomodular lattices *

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(Received 11 December 1992)

Abstract

We generalize the results of the compact case to locally compact orthomodular lattices. (= OML). We first show that any locally compact OML is totally disconnected, and prove that any locally compact, locally convex OML has a local basis consisting of small OMLs. Secondly a locally compact, locally convex complete Boolean algebra is completely characterized in terms of power set Booleans and discrete Boolean algebras. Finally, we obtain necessary and sufficient conditions for a locally compact, locally convex OML to have a representation by a product of finitely many compact OMLs and discrete OMLs.

Key words: Topological lattice; Orthomodular lattices

AMS (MOS) Subj. Class.: Primary 06B30

Orthomodular lattices are the most tractible abstract models for the quantum logic approach to the foundations of quantum mechanics. They have been studied principally as algebras, but also from a combinatorial point of view and from a logical perspective [9]. In [5] we studied topological aspects of orthomodular lattices, focusing on compact topological orthomodular lattices. This investigation stimulated the unveiling of new and interesting nontopological characterizations of classes of topological orthomodular lattices [12,13]. In this paper, we generalize some of the results of the compact case in [5] to locally compact orthomodular lattices. We first show that any locally compact orthomodular lattice is totally

* The authors acknowledge the direct and indirect support of an NSERC and a TGRC-KOSEF respectively.

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SSDI 0166-8641(93)E0065-V

disconnected. Further, we prove that any locally compact, locally convex orthomodular lattice has a local basis consisting of small interval subalgebras.

It is well known [3,13] that any compact Boolean algebra is algebraically and topologically isomorphic to a power set lattice $(2^A, \tau_p)$, where τ_p is the product topology, and 2 is the two-element discrete chain. We prove that if B is a locally compact, locally convex complete Boolean algebra, then B is algebraically and topologically isomorphic to $(2^{A_1}, \tau_p) \times (2^{A_2}, \delta) \times D$, where δ is the discrete topology on 2^{A_2}, and D is a discrete atomless Boolean algebra. We then obtain necessary and sufficient conditions for a locally compact, locally convex orthomodular lattice to have a representation by a product of finitely many compact orthomodular lattices and finitely many discrete orthomodular lattices.

By a topological orthomodular lattice (abbreviated TOML), we mean a pair (L, τ), where L is an orthomodular lattice, and τ is a Hausdorff topology on L for which the two lattice operations and the orthocomplementation operation are continuous.

Note that in any TOML, if one of the lattice operations and the orthocomplementation operation are continuous, then the other lattice operation is also continuous. We shall use $A \wedge B$ and $A \vee B$ for subsets A, B of a lattice $(L, \wedge, \vee)$, to denote the set $\{a \wedge b \mid a \in A$ and $b \in B\}$ and $\{a \vee b \mid a \in A$ and $b \in B\}$ respectively. For an element $a \in L$, we write $a \wedge L = \{a\} \wedge L$, which is the closed interval $[0, a]$ in L, where 0 is the zero element, and dually for $a \vee L$. We use the notations $a \wedge L$ and $[0, a]$ interchangeably.

For other terminology, notation and definitions we mainly follow those in [5,9] for OMLs and those in [3,14] for topological lattices.

1. Locally compact TOMLs

Let (L, τ) be a TOML. (L, τ) is said to be *locally convex*, if for any nbd U of any $x \in L$, there exists a convex nbd V of x such that $V \subset U$, where V is convex iff $V = (V \wedge L) \cap (V \vee L)$, equivalently, for $a, b, c \in L$, $a \leqslant c \leqslant b$ and $a, b \in V$ imply $c \in V$.

(L, τ) is said to be *locally compact* if τ is a locally compact topology, i.e., for any nbd U of any $x \in L$, there exists a compact nbd V of x such that $V \subset U$.

Note that any compact topological lattice is locally convex, and complete [3]. It is known [4] that any locally compact topological Boolean algebra (abbreviated TBA) is totally disconnected. This is true for any locally compact TOML, as we will see by the following theorem:

Theorem 1.1. *Every locally compact TOML L is totally disconnected.*

Proof. Let E be the connected component of 0 in L. Then by [5, Fact 2], E itself is a locally compact connected sublattice of L with respect to its relative topology, because E is closed in L. Suppose that $E \neq \{0\}$. Then we show that for any $a \in E$, $a \wedge E = a \wedge L$. For, if $b \in a \wedge L$, then $b = b \wedge a \in b \wedge E$. Since $b \wedge E$ is the image of E under the continuous map $f: L \to L$ defined by $f(x) = b \wedge x$ for each $x \in L$, $b \wedge E$ is also connected. Thus $b \wedge E \subset E$, since $0 \in b \wedge E$. Hence $b \in E$. Therefore, $b = a \wedge b \in a \wedge E$. It follows that $a \wedge L \subset a \wedge E$, and hence $a \wedge E = a \wedge L$. Again $a \wedge E$ itself is a locally compact connected TOML with respect to its relative topology. It is known [1] that any locally compact connected lattice is locally convex. Therefore, $a \wedge E$ is locally convex. For a compact nbd U of 0 in $a \wedge E$, take a convex nbd V of 0 in $a \wedge E$ with $V \subset U$. Since $a \wedge E$ is nondegenerate connected, i.e., $a \wedge E \neq \{0\}$, we have $V \neq \{0\}$. Now take $c \in V \setminus \{0\}$. Then $c \wedge L \subset V \subset U$ because $a \wedge E = a \wedge L$. Hence $c \wedge E = c \wedge L$ is a nondegenerate compact connected TOML. This is a contradiction to the fact that every compact TOML is totally disconnected [5]. Hence E must be $\{0\}$. Now we show that the connected component E_x of $x \in L \setminus \{0\}$ is also a singleton $\{x\}$. Suppose that $E_x \neq \{x\}$. Take $y \in E_x \setminus \{x\}$. Since $x' \wedge E_x$ is a connected subset of L containing 0, we have $x' \wedge E_x = \{0\}$ and hence $x' \wedge y = 0$. Thus $x \nleqslant y$. $E'_x = \{z' \mid z \in E_x\}$ is connected, because it is the image of E_x under the homeomorphism of the orthocomplementation map. Therefore, $E'_x \wedge x = \{0\}$. Thus $y' \wedge x = 0$, and hence $y \nleqslant x$ since L is an OML. This means that every element y in $E_x \setminus \{x\}$ is incomparable with x. But this contradicts the fact that E_x is a sublattice of L. Hence we have that $E_x = \{x\}$. □

Let L be a TOML. L is said to have *small OMLs* if, for any nbd U of any $x \in L$, there exists a closed interval $[a, b]$ $(= a \vee (b \wedge L)$ and $a \leqslant b)$ which is a nbd of x and is contained in U.

Lemma 1.2. *If L is a locally compact, locally convex TOML then L has small OMLs, which are compact.*

Proof. Let $x \in L$ and let U_0 be any nbd of x. Choose nbds U_1, U_2 of x such that U_1 is convex, U_2 is compact and $U_2 \subset U_1 \subset U_0$. Since U_2 is compact and totally disconnected, there exists an open and closed (hence compact) nbd V of x such that $V \subset U_2$. Now let $A = \{y \in V \mid x \wedge y \in V \text{ and } x \vee y \in V\}$. Then A is also open and closed and hence compact, since $A = f^{-1}(V) \cap g^{-1}(V) \cap V$, where f, g: $L \to L$ are the continuous maps defined for $y \in L$ by $f(y) = x \wedge y$ and $g(y) = x \vee y$. It is clear that $x \in A$, furthermore $x \wedge A \subset A$ and $x \vee A \subset A$. For, whenever $x \wedge y \in x \wedge A$ with $y \in A$, we have $x \wedge (x \wedge y) = x \wedge y \in V$, and we have $x \vee (x \wedge y) = x \in A$ so that $x \wedge A \subset A$. Dually, $x \vee A \subset A$. Now for each $y \in A$, choose a nbd U_y of y and a nbd V_y of x in A such that $V_y \wedge U_y \subset A$ and $V_y \vee U_y \subset A$. This is possible because $x \wedge y \in A$, $x \vee y \in A$ and A is open. Since A is compact, we have that $A = \bigcup_{i=1}^{n} U_{y_i}$ for finitely many y_i. Set $W = \bigcap_{i=1}^{n} V_{y_i}$ and note that W is a nbd of x in A, and hence in L. Denote $W \vee W \vee \cdots \vee W$ (n times) by W^n and $W \wedge \cdots \wedge W$ (n times) by W_n. Then by induction we have $W^n \subset W^{n-1} \vee A \subset A$

and $W_n \subset W_{n-1} \wedge A \subset A$ for any positive integer $n(\neq 1)$. Hence the sublattice $[W]$ generated by W is contained in A. Clearly, $\mathrm{Cl}([W]) \subset A$, where Cl is the closure operation in L. Since $\mathrm{Cl}([W])$ is a compact sublattice of L, there exist $a = \inf \mathrm{Cl}([W])$ and $b = \sup \mathrm{Cl}([W])$. Since $a, b \in A$, and $A \subset U_1$ which is convex, we have that $W \subset \mathrm{Cl}([W]) \subset [a, b] \subset U_1 \subset U_0$. The proof is complete. □

Corollary 1.3. *Let L be a locally compact, locally convex TOML which is not discrete. If U is a nbd of 0, then there exists a nonzero element b such that $b \wedge L$ is open and compact in L and $b \wedge L \subset U$.*

Proof. For any nbd U of 0, by Lemma 1.2 there exists a nonzero element c such that $c \wedge L$ is a compact nbd of 0 and $c \wedge L \subset U$. Choose an open nbd W of 0 with $W \subset c \wedge L$. Let $L_0 = c \wedge L$. Since L_0 is a compact TOML with respect to its relative topology, it is atomic and its topology is generated by $\{a \vee L_0, (a' \wedge c) \wedge L_0 \mid a \in A_0\}$, where A_0 is the set of atoms of L_0 (see [3, Theorem 1]). Since W is an open nbd of 0 in L_0 as well, there exists a basic open nbd of 0 in L_0 say $[\wedge_{k=1}^{n}(a'_k \wedge c)] \wedge L_0 = b \wedge L \subset W$, where $b = \wedge_{k=1}^{n}(a'_k \wedge c)$. Since $b \wedge L$ is open in $W \subset L_0$ and W itself is open in L, we have $b \wedge L$ is open and compact in L, and $b \wedge L \subset U$. This completes the proof. □

The following lemma is one of the most important facts for the sequel.

Lemma 1.4. *Let L be a TOML. Then for an element $a \in L$, $[0, a]$ is an open set in L iff $[0, a']$ is discrete with respect to its relative topology.*

Proof. Necessity: If $[0, a]$ is an open set in L, then it is a nbd of 0 in L. $[0, a']$ is a TOML with respect to its relative topology. Since $[0, a] \cap [0, a'] = \{0\}$, the element 0 in $[0, a']$ is an isolated point. Hence $[0, a']$ is discrete with respect to its relative topology [5].

Sufficiency: Suppose that $[0, a']$ is discrete in the relative sense. Then there exists a nbd $U_{a'}$ of a' in L such that $U_{a'} \cap [0, a'] = \{a'\}$. To show that $[0, a]$ is an open set, take any $y \in [a', 1]$, then $y \wedge a' = a'$. Therefore, we have a nbd U_y of y in L such that $U_y \wedge a' \subset U_{a'}$. For any $x \in U_y$, we have $x \wedge a' \in U_{a'} \cap [0, a']$. Thus $x \wedge a' = a'$ and hence $a' \leqslant x$. It follows that $U_y \subset [a', 1]$. Hence $[a', 1]$ is open in L. Therefore $[0, a] = \varphi^{-1}([a', 1])$ is open in L, where φ is the orthocomplementation map. □

The following corollary is immediate from Lemma 1.4 and Corollary 1.3.

Corollary 1.5. *If L is a locally compact, locally convex TOML with more than two elements, then there exists a nonzero and a nonunit element $a \in L$ such that $a \wedge L$ is discrete with respect to its relative topology.*

Corollary 1.6. *Under the same hypothesis as Corollary* 1.5, *for any nbd* U *of* 0, *if* $a = \sup U$ *exists, then* $b \vee L$ *is discrete with respect to its relative topology for every* $b \geqslant a$.

Proof. By Corollary 1.3, there exists a compact open nbd $c \wedge L$ of 0 such that $c \wedge L \subset U$. Thus $c \leqslant a$. By Lemma 1.4, $c \vee L = \varphi(c' \wedge L)$ is discrete in the relative sense where $\varphi(x) = x'$ for every $x \in L$. For any $b \geqslant a$, $b \vee L$ is a subspace of $c \vee L$. Hence $b \vee L$ is also discrete for every $b \geqslant a$. □

Now we study locally compact, locally convex complete Boolean algebras. The results which we obtain here are not only interesting in themselves, but will be useful in the sequel.

Lemma 1.7. *Let* (B, τ) *be a locally compact, locally convex complete atomic Boolean algebra. Then* (B, τ) *is algebraically and topologically isomorphic to* $(2^{A_1}, \tau_p) \times (2^{A_2}, \delta)$, where $A = A_1 \cup A_2$ *(disjoint union) is the set of atoms* B, τ_p *is the product topology of discrete two-point chains and* δ *is the discrete topology on* 2^{A_2}.

Proof. Clearly $B \cong 2^A$ algebraically. If B is discrete, then $B \cong (2^{A_2}, \delta)$, $A = A_2$ and $A_1 = \emptyset$. Suppose that B is not discrete. Then by Corollary 1.3 there exists an open compact nbd $b \wedge L$ of 0 in B, where $b \neq 0$. By Lemma 1.6, $b' \wedge L$ is discrete with respect to its relative topology. Clearly $b \wedge L$ is itself a compact Boolean algebra, we have $b \wedge L \cong (2^{A_1}, \tau_p)$ algebraically and topologically, for some subset A_1 of A. Clearly $b' \wedge L \cong 2^{A_2}$ algebraically for the set $A_2 = A \backslash A_1$. Endowing 2^{A_2} with the discrete topology δ, we have that $b' \wedge L \cong (2^{A_2}, \delta)$ topologically as well. In all, the map $f: B \to (b \wedge L) \times (b' \wedge L)$ defined by $f(x) = (x \wedge b, x \wedge b')$ is bicontinuous, namely, f and f^{-1} are both continuous. The proof is complete. □

Theorem 1.8. *Let* (B, τ) *be a locally compact, locally convex complete TBA. Then* (B, τ) *has a representation as follows*: (B, τ) *is isomorphic algebraically and topologically to* $(2^{A_1}, \tau_p) \times (2^{A_2}, \delta) \times D$, *where* $A = A_1 \cup A_2$ *is the set of atoms of* B, τ_p *is the product topology,* δ *the discrete topology and* D *is a discrete atomless Boolean algebra.*

Proof. Let A be the set of atoms of B. If $A = \emptyset$ then the underlying algebra B is an atomless Boolean algebra. We show that in this case (B, τ) is discrete. For, if not then by Corollary 1.3, there exists a compact open nbd $b \wedge B$ of 0, where $b \neq 0$. Then $b \wedge L$ is itself a compact TBA. Thus it has at least one atom which is also an atom of B, which is a contradiction. Hence $(B, \tau) \cong D$. Now suppose that $A \neq \emptyset$, let $a = \sup A$. If $a = 1$, then B is complete and atomic. By Lemma 1.7, we have that $(B, \tau) \cong (2^{A_1}, \tau_p) \times (2^{A_2}, \delta)$. If $a \neq 1$, then the underlying algebra B is isomorphic to $(a \wedge B) \times (a' \wedge B)$, algebraically. But $a \wedge B$ is itself a locally compact, locally convex complete atomic TBA so that, by Lemma 1.7, we have that $a \wedge L \cong (2^{A_1}, \tau_p) \times (2^{A_2}, \delta)$. □

2. Representations of locally compact TOMLs

In this section, we obtain necessary and sufficient conditions for a locally compact, locally convex TOML to have a representation by a product of finitely many compact TOMLs and finitely many discrete TOMLs.

The following lemma is easy to prove by [5, Corollaries 1 and 2].

Lemma 2.1. *Every locally compact, locally convex atomless TOML is discrete.*

Theorem 2.2. *Let L be a locally compact, locally convex TOML whose center $C(L)$ is nondiscrete and compact. Then L admits the following representation: $L \cong C_0 \times L_1 \times L_2 \times \cdots \times L_n$, algebraically and topologically, where C_0 is a compact TOML, and each L_k is an irreducible nowhere dense principal p-ideal of L, $k = 1,2,\ldots,n$.*

Proof. Since the center $C(L)$ of L is a compact TBA with respect to its relative topology, we have $C(L) \cong (2^A, \tau_p)$, algebraically and topologically, where A is the set of atoms of $C(L)$ (see [3, Corollary 1]). Clearly L is not discrete. Then there exists a compact open nbd $b \wedge L$ of 0 in L where $b \neq 0$. Now consider $W_0 = (b \wedge L) \cap C(L)$. Then W_0 is a compact open nbd of 0 in $C(L)$. Clearly $W_0 \neq \{0\}$ because $C(L)$ is not discrete. Since $(2^A, \tau_p)$ is a compact TBA, the topology τ_p is generated by $\{p \vee L, p' \wedge L \mid p \in A\}$. Moreover, since W_0 is an open set containing 0 in $C(L)$, it contains a basic open nbd of 0 in $C(L)$, namely, there exist finitely many atoms $p_1,p_2,\ldots,p_n$ of $C(L)(\cong 2^A)$ such that $(\bigwedge_{k=1}^n p_k') \wedge C(L) \subset W_0$. Now let $a = \bigwedge_{k=1}^n p_k'$. Then $a \in C(L)$ and $a \leqslant b$. Therefore, $a \wedge L \subset b \wedge L$, and hence $a \wedge L$ is compact. For any $q \in A_0 = A \setminus \{p_1,p_2,\ldots,p_n\}$, we have $q \wedge L \subset a \wedge L$. It follows that $q \wedge L$ is compact for $q \in A_0$. Clearly $a = \sup A_0$ and $a' = \bigvee_{k=1}^n p_k$. We have that $L \cong (a \wedge L) \times \prod_{k=1}^n (p_k \wedge L)$ algebraically and topologically. Setting $C_0 = a \wedge L$ and $L_k = p_k \wedge L$, $1 \leqslant k \leqslant n$, we have the desired representation if we show that each L_k is nowhere dense in L. Suppose that $L_k (= p_k \wedge L)$ is not nowhere dense in L. Then we have a nonvoid open set U in L with $U \subset p_k \wedge L$. By [5, Fact 3], $U \wedge L$ is an open set in L containing 0. Take a compact open nbd $c \wedge L$, which is contained in $U \wedge L$, where $c \neq 0$. This is possible by Corollary 1.3 since L is not discrete. But $c \leqslant p_k$, i.e., $p_k' \leqslant c'$. Thus $p_k' \wedge L \subset c' \wedge L$ which is discrete by Lemma 1.4. This is impossible because the nondiscrete dense compact factor $C_0 = a \wedge L$ is contained in $p_k' \wedge L$. Hence each $p_k \wedge L$ is nowhere dense in L, $0 \leqslant k \leqslant n$. The proof is complete. □

The following lemma provides the necessary and sufficient conditions for a locally compact, locally convex TOML with compact center to admit a representation of the form the product of compact OML and discrete OML.

Lemma 2.3. *Let L be a locally compact, locally convex TOML whose center is compact, so that $C(L) \cong (2^A, \tau_p)$. Then L is isomorphic algebraically and topologically to $C_0 \times L_1 \times \cdots \times L_n$, for some compact TOML C_0 and discrete $L_k = p_k \wedge L$ for $p_k \in A$ $(k = 1,2,\ldots,n)$ iff $(\bigwedge_{k=1}^{n} p_k') \wedge L$ is compact open in L.*

Proof. Suppose that L has such a representation. We have $(\bigvee_{k=1}^{n} p_k) \wedge L$ is discrete with respect to its relative topology, since $L_1 \times \cdots \times L_n$ is discrete. By Lemma 1.4, we get that $(\bigwedge_{k=1}^{n} p_k') \wedge L$ is an open set in L. Since $C_0 \cong (\bigwedge_{k=1}^{n} p_k') \wedge L$, it is compact as well, and hence it is compact open in L. Conversely, if $(\bigwedge_{k=1}^{n} p_k') \wedge L$ is compact open, then $(\bigvee_{k=1}^{n} p_k') \wedge L$ is discrete. Therefore, each $L_k = p_k \wedge L$ is discrete for $k = 1,2,\ldots,n$. Setting $C_0 = (\bigvee_{k=1}^{n} p_k) \wedge L$, we have the desired representation. □

Corollary 2.4. *Under the same hypotheses as in Lemma 2.3, if moreover each L_k is also infinite, then the representation is unique up to isomorphism.*

Proof. Assume that L has two such representations: $L \cong C_0 \times L_1 \times \cdots \times L_n \cong D_0 \times M_1 \times \cdots \times M_N$, where C_0 and D_0 are compact, each $L_i = p_i \wedge L$ and each $M_j = q_j \wedge L$ are infinite irreducible and discrete for atoms p_i and q_j of the $C(L)$. Since L_i is infinite discrete and $D_0 = [0,(\bigvee q_j)']$ is compact we have $p_i \notin D_0$. Thus each $p_i \in \{q_1,\ldots,q_m\}$. It follows that the representation is unique up to isomorphism. □

Lemma 2.5. *Let L be a locally compact, locally convex TOML whose center is complete atomic and discrete, i.e., $C(L) \cong (2^A, \delta)$ where δ is the discrete topology, and let $p_1,\ldots,p_n \in A$ such that $L_i = p_i \wedge L$ is a compact irreducible infinite TOML and D_0 is a discrete TOML. Then $L \cong L_1 \times L_2 \times \cdots \times L_n \times D_0$ algebraically and topologically iff $(\bigvee_{k=1}^{n} p_k) \wedge L$ is compact open in L. Furthermore, such a representation is unique up to isomorphism.*

Proof. Let $a = \bigvee_{k=1}^{n} p_k$, and suppose that L has such a representation. Then $a \wedge L \cong L_1 \times \cdots \times L_n$ is compact and $D_0 \cong a' \wedge L$ is discrete. By Lemma 1.4, $a \wedge L$ is open in L. Hence $a \wedge L$ is compact open. Conversely, we have that $a \wedge L \cong L_1 \times \cdots \times L_n$ is open in L so that $a' \wedge L$ is discrete. Setting $D_0 = a' \wedge L$, we have the desired decomposition. Since no discrete TOML contains an infinite compact subset, an argument similar to that of Corollary 2.4 shows that such representation is also unique up to isomorphism. □

Remark. Under the same hypotheses for L as in Lemma 2.5, we note that any open compact nbd of 0 cannot contain infinitely many central elements, because the center is discrete. Thus any such representation of L cannot contain infinitely many nondegenerate compact factors.

Lemma 2.6. *If L is a locally compact, locally convex TOML whose center is discrete complete and atomless, then L is discrete.*

Proof. Suppose that L is not discrete. Then there exists $b(\neq 0)$ in L such that $b \wedge L$ is an open and compact nbd of 0. We have that either (i) $(b \wedge L) \wedge (C(L) \setminus \{1\}) = \{0\}$ or (ii) $(b \wedge L) \wedge (C(L) \setminus \{1\}) \neq \{0\}$. If (i) holds, then $b \wedge x = 0$ for every $x \in C(L) \setminus \{1\}$. Consider a maximal chain C in $C(L)$. Since $C(L)$ is atomless, we have $\sup(C \setminus \{1\}) = 1$. It follows that $0 = \sup\{b \wedge x \mid x \in C(L) \setminus \{1\}\} = b \wedge \sup(C(L) \setminus \{1\}) = b$ which is a contradiction. Now if (ii) holds, choose $c \in C(L) \setminus \{1\}$ such that $b \wedge c \neq 0$. Again take a maximal chain C from 0 to c in $C(L)$ and let $L_0 = b \wedge L$. Then $b \wedge C$ is a chain in L_0. Note that L_0 is itself a compact TOML itself. Consider a maximal chain E containing $b \wedge C$ in L_0. Then E is a closed set in L_0 because $E = \bigcap_{x \in E}[(x \wedge L_0) \cup (x \vee L_0)]$ and $x \wedge L_0$, $x \vee L_0$ are both closed. Hence E is a compact chain (hence lattice) in L_0. On the other hand, since $C(L)$ is discrete in its relative topology, there exists a nbd U of 0 in L such that $U \cap C(L) = \{0\}$ and $b \wedge c \notin U$, where $b \wedge c \neq 0$. Since L is not discrete, by Corollary 1.3, there exists d in L such that $d \wedge L$ is compact open compact and $d \wedge L \subset U$. Thus $C \cap (d \wedge L) = \{0\}$. Let $e = b \wedge d$ so that $e \wedge L = (d \wedge L) \cap (b \wedge L)$ is also an open and compact nbd of 0 in L. Thus $e \neq 0$, since L is not discrete, and $E \setminus (e \wedge L)$ is a closed chain in L_0. Hence $E \setminus (e \wedge L)$ is a compact chain in L_0. Now let $C_1 = \{x \in C \mid b \wedge x \in E \setminus (e \wedge L)\}$ and let $C_2 = \{y \in C \mid b \wedge y \in E \cap (e \wedge L)\}$. Then we have $C = C_1 \cup C_2$ (disjoint) and furthermore, $c \in C_1$ and $0 \in C_2$ because $b \wedge c \notin e \wedge L$ since $b \wedge c \notin U$. Now let $\inf C_1 = c_1$ and $\sup C_2 = c_2$ in $C(L)$. Clearly $c_1 \geqslant c_2$. But since $C(L)$ is atomless, we have that $c_1 = c_2$. Since E is a compact lattice, $\inf\{b \wedge x \mid x \in C_1\}$ and $\sup\{b \wedge y \mid y \in C_2\}$ exist in E. But $\inf\{b \wedge x \mid x \in C_1\} = b \wedge \inf C_1 = b \wedge c_2 \in E \setminus (e \wedge L)$, because $E \setminus (e \wedge L)$ is itself a compact chain. And $\sup\{b \wedge y \mid y \in C_2\} = b \wedge c_2 \in E \cap (e \wedge L)$. This is a contradiction. Hence L must be discrete. □

The following lemma is immediate by Theorem 1.8:

Lemma 2.7. *If L is a locally compact, locally convex TOML whose center $C(L)$ is complete, then $C(L) \cong (2^{A_1}, \tau_p) \times (2^{A_2}, \delta) \times B_0$ algebraically and topologically, where $A = A_1 \cup A_2$ (disjoint union) is the set of atoms of $C(L)$, τ_p the product topology on 2^{A_1}, δ the discrete topology on 2^{A_2} and B_0 is a discrete atomless Boolean algebra.*

Combining all the results in Lemmas 2.3, 2.5, 2.6 and 2.7, we have the following conclusion: Let L be any locally compact, locally convex TOML whose center $C(L)$ is complete so that $C(L) \cong (2^{A_1}, \tau_p) \times (2^{A_2}, \delta) \times B_0$ as in Lemma 2.6. Calculating in $C(L)$ let $a = \sup A_1$, $b = \sup A_2$ and $c = \sup B_0$. Then we have $L \cong (a \wedge L) \times (b \wedge L) \times (c \wedge L)$ algebraically and topologically. Furthermore, $a \wedge L$, $b \wedge L$ and $c \wedge L$ are all locally compact, locally convex TOMLs with respect to their relative topologies. And their centers are $C(a \wedge L) \cong (2^{A_1}, \tau_p)$, $C(b \wedge L) \cong$

$(2^{A_2}, \delta)$ and $C(c \wedge L) \cong B_0$ which are compact, complete atomic and discrete, and discrete, respectively.

Let $p_1, p_2, \ldots, p_n \in A_1$ and $q_1, \ldots, q_m \in A_2$ such that $L_i = p_i \wedge L$ $(i = 1,2,\ldots,n)$ is irreducible infinite and discrete, and $K_j = q_j \wedge L$ $(j = 1,2,\ldots,m)$ is irreducible infinite and compact, and let C_0 be a compact OML, D_0 a discrete atomic OML and D a discrete atomless OML. By Corollary 2.3, $a \wedge L \cong C_0 \times L_1 \times \cdots \times L_n$ *iff* $(\bigwedge_{i=1}^{n} p_i') \wedge L$ is compact open. And by Lemma 2.5, $b \wedge L \cong K_1 \times K_2 \times \cdots \times K_m \times D_0$ iff $(\bigvee_{j=1}^{m} q_j) \vee L$ is compact open. By Lemma 2.6, $c \wedge L (= D)$ must be a discrete atomless OML.

Hence we have the following main theorem:

Theorem 2.8. *Let L be a locally compact, locally convex TOML whose center C(L) is complete. Then* $L \cong (C_0 \times L_1 \times L_2 \times \cdots \times L_n) \times (K_1 \times \cdots \times K_m \times D_0) \times D_1$ *algebraically and topologically iff* $(\bigwedge_{i=1}^{n} p_i') \wedge L$ *and* $(\bigvee_{j=1}^{m} q_j) \vee L$ *are both compact open. Furthermore, such representation is unique up to isomorphism, provided it exists.*

3. References

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