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The Center of an Effect Algebra

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Abstract. An effect algebra is a partial algebra modeled on the standard effect algebra of positive self-adjoint operators dominated by the identity on a Hilbert space. Every effect algebra is partially ordered in a natural way, as suggested by the partial order on the standard effect algebra. An effect algebra is said to be distributive if, as a poset, it forms a distributive lattice. We define and study the center of an effect algebra, relate it to cartesian-product factorizations, determine the center of the standard effect algebra, and characterize all finite distributive effect algebras as products of chains and diamonds.

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1. Introduction

Motivated by the use of positive operator valued measures in stochastic (or phase space) quantum mechanics [1, 3, 22, 25, 26] and by the study of fuzzy or unsharp quantum logics [4, 5, 15, 20] there has been a recent surge of interest in a new class of orthostructures called *weak* (or *generalized*) *orthoalgebras* [15, 18], *difference* (or *D*) *posets* [7–10, 17, 19, 21, 24], or *effect algebras* [2, 12, 13]. By and large, these structures are mathematically equivalent, and it is mostly a matter of taste which to select. In what follows, we choose to focus on effect algebras because the additive structure of such algebras articulates well with that of the partially ordered abelian groups which often figure in their representation [2]. Thus, we assume that the reader is somewhat familiar with [2, 12, 13], although, for convenience, we reproduce most of the pertinent basic definitions and results.

In this paper we define and study the notion of the *center* of an effect algebra, relate central elements with factorization into cartesian products, determine the center of a standard effect algebra on a Hilbert space, and use central elements to help characterize all finite distributive effect algebras.

2. Effect Algebras

In [12] an *effect algebra* is defined to be an algebraic system $(E, 0, u, \oplus)$ consisting of a set E, two special elements $0, u \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on E that satisfies the following conditions for all $p, q, r \in E$:

(i) [*Commutative Law*] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
(ii) [*Associative Law*] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
(iii) [*Orthosupplementation Law*] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = u$.
(iv) [*Zero-Unit Law*] If $u \oplus p$ is defined, then $p = 0$.

For simplicity, we often refer to E, rather than to $(E, 0, u, \oplus)$, as being an effect algebra.

For the *remainder of this paper, we assume that E is an effect algebra with unit u.* If $p, q \in E$, we say that p and q are *orthogonal* and write $p \perp q$ iff $p \oplus q$ is defined in E. If we write $p \oplus q = r$, we mean that $p \perp q$ and $p \oplus q = r \in E$. If $p, q \in E$ and $p \oplus q = u$, we call q the *orthosupplement* of p and write $p' := q$. (We use the notation $:=$ to mean "equals by definition.")

If $e_1, e_2, \ldots, e_n$ is a finite sequence of elements of E, then, by recursion, we say that the *orthogonal sum* $e_1 \oplus e_2 \oplus \cdots \oplus e_n$ exists in E iff $s := e_1 \oplus e_2 \oplus \cdots \oplus e_{n-1}$ exists in E and $s \perp e_n$, in which case we define $e_1 \oplus e_2 \oplus \cdots \oplus e_n := s \oplus e_n$. Alternative notation for the orthogonal sum is $\bigoplus_{j=1}^{n} e_j$, or simply $\bigoplus_j e_j$ if n is understood. If $\bigoplus_j e_j$ exists in E, we say that $e_1, e_2, \ldots, e_n$ is an *orthogonal sequence* in E. The basic properties of orthogonal sequences, which can be found in [12, Theorem 4.2], will be used without comment in what follows.

If $p \in E$ and $p_1, p_2, \ldots, p_n$ is an orthogonal sequence, where $p_1 = p_2 = \cdots = p_n = p$, we define $np := \bigoplus_j p_j$. If $p \neq 0$ and $2p$ is defined (i.e., if $p \perp p$), we say that p is an *isotropic* element of E. If np is defined, but $(n+1)p$ is not, we refer to n as the *isotropic index* of p and we call $\mathrm{smt}(p) := np$ the *isotropic summit* of p. If np is defined for all positive integers n, we say that p has *infinite* isotropic index. An *orthoalgebra* [6, 10, 11, 14, 16, 17, 21, 24] may be defined as an effect algebra with no isotropic elements.

A subset F of E is called a *sub-effect algebra* of E iff (i) $0, u \in F$, (ii) $p \in F \Rightarrow p' \in F$, and (iii) $p, q \in F$ with $p \perp q \Rightarrow p \oplus q \in F$. Evidently, such an F is an effect algebra in its own right under the restriction of $\oplus$ to F.

If E and F are effect algebras, a mapping ϕ: $E \to F$ is called a *morphism* iff it maps the unit of E to the unit of F and preserves orthogonal sums in the sense that, if $p, q \in E$ with $p \perp q$, then $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$. Evidently, if ϕ: $E \to F$ is a morphism, then $\phi(0) = 0$ and $\phi(p') = \phi(p)'$. An *isomorphism* ϕ: $E \to F$ is defined to be a bijective morphism such that, for $p, q \in E$, $\phi(p) \perp \phi(q) \Rightarrow p \perp q$. Evidently, a bijective morphism ϕ: $E \to F$ is an isomorphism iff ϕ^{-1}: $F \to E$ is also a morphism.

3. The Order Structure of an Effect Algebra

It is shown in [12] that the relation $\leqslant$ defined for $p, q \in E$ by $p \leqslant q$ iff $\exists r \in E$ with $p \oplus r = q$ is a partial order on E and $0 \leqslant p \leqslant u$ holds for all $p \in E$. It is also shown that the orthosupplementation mapping $p \mapsto p'$ is an order-reversing involution and that $q \perp p$ iff $q \leqslant p'$. Furthermore, E satisfies the following *cancellation law*: If $p \oplus q \leqslant r \oplus q$, then $p \leqslant r$. Note that, if F is a sub-effect algebra of E, then the partial order on F is the restriction to F of the partial order on E.

If F is an effect algebra and ϕ: $E \to F$ is a morphism, then, for all $p, q \in E$, $p \leqslant q \Rightarrow \phi(p) \leqslant \phi(q)$. Furthermore, a surjective morphism ϕ: $E \to F$ is an isomorphism iff, for all $p, q \in E$, $\phi(p) \leqslant \phi(q) \Rightarrow p \leqslant q$.

If we say that the effect algebra E has a certain order-theoretic property, we mean that the poset $(E, \leqslant)$ has that property. For instance, if we say that E is lattice ordered, we mean that $(E, \leqslant)$ is a lattice. We use the usual notation $p \wedge q$ and $p \vee q$ for the infimum (greatest lower bound) and supremum (least upper bound) of elements p, q in E *when they exist*. If we write, for instance, $p \wedge q = r$, we mean that the infimum $p \wedge q$ exists and equals r. Two elements $p, q \in E$ are said to be *disjoint* iff $p \wedge q = 0$.

If $e \in E$, we define the *interval* $E[0, e] := \{p \in E \mid p \leqslant e\}$. As usual, a nonzero element $a \in E$ is called an *atom* iff $E[0, a] = \{0, a\}$. We say that E is *atomic* iff, for $0 \neq p \in E$, there is an atom $a \in E$ with $a \leqslant p$. If E is finite, or more generally, if E satisfies chain conditions, then E is atomic and every nonzero element in E is a finite orthogonal sum of atoms [12, Theorem 4.11].

3.1. DEFINITION. A nonempty subset I of E is called an *ideal* in E iff for all $p, q \in E$ with $p \perp q$, $p \oplus q \in I$ iff $p, q \in I$.

Suppose I is an ideal in E, $p, r \in E$, $p \leqslant r$ and $r \in I$. Since $p \leqslant r$, $\exists q \in E$ with $p \oplus q = r$, and it follows that $p \in I$. Therefore, I is automatically an order-ideal in the poset $(E, \leqslant)$. In particular, since I is not empty, we have $0 \in I$. The collection $\mathfrak{I}$ of all ideals in E is closed under the formation of arbitrary set-theoretic intersections, so it forms a complete lattice under set-theoretic inclusion. Consequently, if $X \subseteq E$, there will be a smallest ideal $I_X \subseteq E$ with $X \subseteq I_X$. We call I_X the ideal *generated* by X. In particular, the ideal I_e generated by a single element $e \in I$ is called a *principal ideal* in E.

If E is an orthomodular poset and $e \in E$, then the principal ideal I_e is the interval $E[0, e]$. If E is only an orthoalgebra, or more generally, an effect algebra, then the interval $E[0, e]$ may be a proper subset of I_e. Indeed, the necessary and sufficient condition that $I_e = E[0, e]$ is that e is *principal* in the sense of the following definition.

3.2. DEFINITION. If $e \in E$, then e is *principal* iff, for $p, q \in E$, $p \perp q$ and $p, q \leqslant e \Rightarrow p \oplus q \leqslant e$.

By [12, Theorem 5.3], E is an orthomodular poset iff every element in E is principal. If e is a principal element of E, then the corresponding principal ideal $E[0, e]$ is an effect algebra with unit e under the restriction to $E[0, e]$ of $\oplus$ and with $p \mapsto (p \oplus e')'$ as the orthosupplementation mapping. If e is principal and $p, q \in E[0, e]$, then $p \perp q$ in E iff $p \perp q$ in $E[0, e]$, and likewise for the relation $\leqslant$.

3.3. LEMMA. *If $q \in E$ is principal then $q \wedge q' = 0$.*

Proof. If q is principal and $p \leqslant q, q'$, then $p \perp q$ and $p, q \leqslant q$; hence, $p \oplus q \leqslant q = 0 \oplus q$, so $p = 0$ by the cancellation law. □

In the next lemma, we do not necessarily assume that e is principal. Thus the interval $E[0, e] := \{p \in E \mid p \leqslant e\}$ is not necessarily an ideal in E, although it is a poset under the partial order inherited from E.

3.4. LEMMA. *Let $0 \neq e \in E$ and define τ: $E[0, e] \to E[0, e]$ by $\tau(p) := (p \oplus e')'$ for all $p \in E[0, e]$. Then τ is an order-reversing involution on $E[0, e]$, $\tau(0) = e$, $\tau(e) = 0$, and the following de Morgan-type laws hold*:

(i) *If $p, q \in E[0, e]$ and $p \vee q$ exists in E, then $\tau(p) \wedge \tau(q)$ exists in E and $\tau(p \vee q) = \tau(p) \wedge \tau(q)$.*

(ii) *If $p, q \in E[0, e]$ and $\tau(p) \vee \tau(q)$ exists in E, then $p \wedge q$ exists in E and $\tau(p \wedge q) = \tau(p) \vee \tau(q)$.*

Proof. For $p, q \in E[0, e]$, denote by $p \wedge_e q$ and $p \vee_e q$ the infimum and supremum, when they exist, of p and q in $E[0, e]$. If $p \in E[0, e]$, then $\tau(p)$ is the unique element in E for which $p \oplus \tau(p) = e$; hence, $\tau(\tau(p)) = p$. If $p, q \in E[0, e]$ with $p \leqslant q$, then $p \oplus e' \leqslant q \oplus e'$, and it follows that $\tau(q) = (q \oplus e')' \leqslant (p \oplus e')' = \tau(p)$. Thus, τ is an order-reversing involution on the poset $E[0, e]$; hence, the de Morgan laws hold for τ on $E[0, e]$ with respect to $\wedge_e$ and $\vee_e$.

If $p, q \in E[0, e]$ and the supremum $p \vee q$ of p and q exists in E, it is clear that $p \vee_e q$ exists in $E[0, e]$ and $p \vee_e q = p \vee q$ (although $p \vee_e q$ can exist when $p \vee q$ fails to exist). On the other hand, $p \wedge_e q$ exists in $E[0, e]$ iff $p \wedge q$ exists in E and if they exist, they coincide.

Assume the hypotheses of (i). Then $p \vee q = p \vee_e q \in E[0, e]$, so $\tau(p \vee q) = \tau(p) \wedge_e \tau(q) = \tau(p) \wedge \tau(q)$. Replacing p and q by $\tau(p)$ and $\tau(q)$ in (i) and using the fact that τ is an involution, we obtain (ii). □

The following theorem, which was independently obtained by Riečanová [23], is similar to a well-known result in the theory of lattice-ordered abelian groups.

3.5. THEOREM. *If $p, q \in E$, $p \perp q$, and $p \vee q$ exists in E, then $p \wedge q$ exists in E, $p \wedge q \leqslant (p \vee q)' \leqslant (p \wedge q)'$ and $p \oplus q = (p \wedge q) \oplus (p \vee q)$.*

Proof. Let $e := p \oplus q$ and let τ be the mapping in Lemma 3.4. Then $\tau(p) = q$, $\tau(q) = p$, $p \wedge q = q \wedge p = \tau(p) \wedge \tau(q)$ exists in E, and $\tau(p \vee q) = \tau(p) \wedge \tau(q) = p \wedge q$. Also, $p \wedge q \leqslant q \leqslant p'$ and $p \wedge q \leqslant p \leqslant q'$, so $p \wedge q \leqslant p' \wedge q' = (p \vee q)'$. Furthermore, $p, q \leqslant p \oplus q$, so $p \vee q \leqslant p \oplus q = e$, and it follows that

$$p \oplus q = e = \tau(p \vee q) \oplus (p \vee q) = (p \wedge q) \oplus (p \vee q). \qquad \square$$

3.6. COROLLARY. *Let $p, q \in E$ with $p \perp q$ and suppose $p \vee q$ exists in E. Then $p \vee q \leqslant p \oplus q$ with equality iff p and q are disjoint.*

3.7. COROLLARY. *If a is an atom in E, $q \in E$, $a \nleqslant q$, $a \perp q$, and $a \vee q$ exists in E, then $a \oplus q = a \vee q$.*

3.8. COROLLARY. *Let a be an atom in E, let m be a positive integer and suppose that ma is defined. Then, if $0 \neq q \in E$, $q \perp a$, $q \vee a$ exists, and $q \leqslant ma$, it follows that $a \leqslant q$.*

Proof. Assume the hypotheses, but suppose $a \nleqslant q$. By Corollary 3.7, $q \oplus a = q \vee a$. Let n be the smallest positive integer for which na is defined and $q \leqslant na$. If $n = 1$, then $0 \neq q \leqslant a$, so $q = a$, contradicting $a \nleqslant q$. Therefore, $n > 1$. Since $q, a \leqslant na$, we have $q \oplus a = q \vee a \leqslant na$, so, by the cancellation law, $q \leqslant (n-1)a$, contradicting our choice of n. □

3.9. THEOREM. *If E is lattice-ordered, a is an atom, m is a positive integer, ma is defined, $0 \neq q \in E$, $q \perp a$ and $q \leqslant ma$, then $q = na$ for some positive integer $n \leqslant m$.*

Proof. By Corollary 3.8, $a \leqslant q$. Let n be the largest positive integer for which na is defined and $na \leqslant q$. Then $\exists r \in E$ with $na \oplus r = q$. Consequently, $r \perp a$ and $r \leqslant q \leqslant ma$, so either $r = 0$ or $a \leqslant r$ by Corollary 3.8. If $a \leqslant r$, then $\exists s \in E$ with $a \oplus s = r$, so $na \oplus a \oplus s = q$, and $(n+1)a \leqslant q$, contradicting our choice of n. Therefore $r = 0$ and $na = q$. Since $na = q \leqslant ma$, it follows that $n \leqslant m$. □

4. Cartesian Products of Effect Algebras

If $(E_\alpha)_{\alpha \in J}$ is a family of effect algebras, the cartesian product $E := \times_{\alpha \in J} E_\alpha$ is organized into an effect algebra in the obvious way using coordinatewise operations and relations [12]. The projection mappings π_α: $E \to E_\alpha$ are surjective effect-algebra morphism for all $\alpha \in J$.

In dealing with infinite cartesian products, one needs an extended notion of orthogonal sum for arbitrary families of elements. Although appropriate definitions can be found in [7, 11, 16], we prefer to avoid this minor complication and focus our attention here on *finite* cartesian products. The proof of the following theorem is a routine verification.

4.1. THEOREM. *For $j = 1, 2, \ldots, n$, let E_j be an effect algebra with unit u_j, let $E := E_1 \times E_2 \times \cdots \times E_n$, and define $e_j := (0, 0, \ldots, 0, u_j, 0, \ldots, 0)$, with u_j in the jth coordinate position. Then:*

(i) *Each e_j is principal in E.*
(ii) *The principal ideal $E[0, e_j]$ in E is isomorphic to E_j under the restriction to $E[0, e_j]$ of the projection homomorphism π_j: $E \to E_j$.*
(iii) *$e_1, e_2, \ldots, e_n$ is an orthogonal sequence in E and $\bigoplus_j e_j$ is the unit element in E.*
(iv) *Every element $p \in E$ can be represented uniquely in the form $p = \bigoplus_j p_j$ with $p_j \in E[0, e_j]$; indeed, $p_j = p \wedge e_j$ for $j = 1, 2, \ldots, n$.*

Theorem 4.1 gives us the clue for factoring an effect algebra E as an "internal direct product" of ideals generated by orthogonal principal elements. Specifically, we have the following theorem.

4.2. THEOREM. *Let $e_1, e_2, \ldots, e_n$ be a finite orthogonal sequence of nonzero, principal elements of E. Define ϕ: $\times_j E[0, e_j] \to E$ by $\phi(p_1, p_2, \ldots, p_n) := \bigoplus_j p_j$ for all $(p_1, p_2, \ldots, p_n) \in \times_j E[0, e_j]$. Then ϕ is an isomorphism iff it is surjective. Furthermore, if ϕ is an isomorphism, then, for $p \in E$, the infima $p \wedge e_1, p \wedge e_2, \ldots, p \wedge e_n$ exist in E and $p = \bigoplus_j (p \wedge e_j)$.*

Proof. Since $e_1, e_2, \ldots, e_n$ is an orthogonal sequence of nonzero principal elements of E, it follows that ϕ: $\times_j E[0, e_j] \to E$ is well-defined and preserves inequalities and orthogonal sums. If ϕ is an isomorphism, it is surjective by definition.

Conversely, suppose ϕ: $\times_j E[0, e_j] \to E$ is surjective and let u be the unit for E. Then $\exists (u_1, u_2, \ldots, u_n) \in \phi^{-1}(u)$, and we have $u = \bigoplus_j u_j$ and $u_j \leqslant e_j$ for all $j = 1, 2, \ldots, n$. Therefore,

$$e_1' \leqslant u_1' = \bigoplus_{j=2}^{n} u_j \leqslant \bigoplus_{j=2}^{n} e_j \leqslant e_1'.$$

(Notice that the last inequality follows from the orthogonality of the sequence $e_1, e_2, \ldots, e_n$, and does not require that e_1' be principal.) Hence, $e_1' = u_1'$, so $u_1 = e_1$, and by symmetry, $u_j = e_j$ for all j. Consequently, $\phi(e_1, e_2, \ldots, e_n) = u$, so ϕ: $\times_j E[0, e_j] \to E$ is a surjective morphism.

Suppose $(p_1, p_2, \ldots, p_n)$, $(q_1, q_2, \ldots, q_n) \in \times_j E[0, e_j]$ with

$$p := \phi(p_1, p_2, \ldots, p_n) \leqslant q := \phi(q_1, q_2, \ldots, q_n)$$

in E. To prove that ϕ is an isomorphism, we have only to show that

$$(p_1, p_2, \ldots, p_n) \leqslant (q_1, q_2, \ldots, q_n)$$

in $\times_j E[0, e_j]$. Let $(r_1, r_2, \ldots, r_n)$ be the orthosupplement of $(q_1, q_2, \ldots, q_n)$ in $\times_j E[0, e_j]$, so that $r := \phi(r_1, r_2, \ldots, r_n) = q'$. Then

$$r_1 \leqslant \bigoplus_j r_j = r \leqslant q',$$

so $q \leqslant r_1'$ and, therefore, $p_1 \leqslant \bigoplus_j p_j = p \leqslant q \leqslant r_1'$. Since e_1 is principal and $p_1, r_1 \leqslant e_1$, it follows that $p_1 \oplus r_1 \leqslant e_1 = q_1 \oplus r_1$; hence, $p_1 \leqslant q_1$ by the cancellation law. By symmetry, $p_j \leqslant q_j$ for all j, so ϕ is an isomorphism.

Again suppose that $(p_1, p_2, \ldots, p_n) \in \times_j E[0, e_j]$ with $p := \phi(p_1, p_2, \ldots, p_n) = \bigoplus_j p_j$. We note that $p_1 \leqslant p, e_1$ and claim that, in fact, $p_1 = p \wedge e_1$. To prove this, suppose that $x \leqslant p, e_1$. We have to prove that $x \leqslant p_1$. Then $(x, 0, 0, \ldots, 0) \in \times_j E[0, e_j]$ with $\phi(x, 0, 0, \ldots, 0) = x \leqslant p = \phi(p_1, p_2, \ldots, p_n)$. Because ϕ is an isomorphism, we conclude that $x \leqslant p_1$ as desired. □

4.3. COROLLARY. *Suppose that $e_1, e_2, \ldots, e_n$ is an orthogonal sequence of principal elements in E such that, for all $p \in E$, there are elements $p_j \leqslant e_j$ for $j = 1, 2, \ldots, n$ with $p = \bigoplus_j p_j$. Then all orthogonal sums of subsequences of $e_1, e_2, \ldots, e_n$ are principal elements of E.*

5. The Center of an Effect Algebra

With Theorem 4.2 in mind, we make the following definition.

5.1. DEFINITION. An element $a \in E$ is *central* iff

(i) $\forall p \in E, \exists q, r \in E, q \leqslant a, r \leqslant a', p = q \oplus r$.
(ii) a and a' are principal.

The *center* $C(E)$ is the set of all central elements in E.

5.2. LEMMA. *If $a \in C(E)$ and $p \in E$, then:*

(i) *$a \wedge p$ and $a' \wedge p$ exist in E.*
(ii) *$p = a \wedge p \oplus a' \wedge p$.*
(iii) *$a \perp p$ iff $a \wedge p = 0$.*
(iv) *a cannot be isotropic.*

Proof. We can assume $a \neq 0$, u, so that a, a' forms an orthogonal sequence of nonzero principal elements of E. Applying Theorem 4.2 to $e_1 := a$ and $e_2 := a'$, we obtain (i) and (ii). Part (iii) is a consequence of (ii). If $a \perp a$, then $a = a \oplus a$ by (ii), and $a = 0$ by the cancellation law. □

5.3. LEMMA. *If $a \in E$, then $a \in C(E)$ iff there is an orthogonal sequence $e_1, e_2, \ldots, e_n$ of principal elements in E such that, for all $p \in E$, there are elements $p_j \leqslant e_j$ for $j = 1, 2, \ldots, n$, with $p = \bigoplus_j p_j$, and a is an orthogonal sum of a subsequence of $e_1, e_2, \ldots, e_n$.*

Proof. If $a \in C(L)$, let $e_1 := a$ and $e_2 := a'$. Conversely, suppose $e_1, e_2, \ldots, e_n$ is an orthogonal sequence of principal elements in E such that, for all $p \in E$, there are elements $p_j \leqslant e_j$ for $j = 1, 2, \ldots, n$ with $p = \bigoplus_j p_j$, and suppose that a is a sum of a subsequence of $e_1, e_2, \ldots, e_n$. Without loss of generality, we can assume that $a = e_1 \oplus e_2 \oplus \cdots \oplus e_k$. By Theorem 4.2, $a' = e_{k+1} \oplus e_{k+2} \oplus \cdots \oplus e_n$ and, by Corollary 4.3, both a and a' are principal. If $p \in E$, there exist $p_j \leqslant e_j$ for $j = 1, 2, \ldots, n$ with $p = \bigoplus_j p_j$. Then, with $q := p_1 \oplus p_2 \oplus \cdots \oplus p_k \leqslant a$ and $r := p_{k+1} \oplus p_{k+2} \oplus \cdots \oplus p_n \leqslant a'$, we have $p = q \oplus r$. □

5.4. THEOREM. *The center $C(E)$ is a sub-effect algebra of E and, as an effect algebra in its own right, $C(E)$ forms a Boolean algebra. Furthermore, if $a, b \in C(E)$, then $a \wedge b$ and $a \vee b$ as calculated in $C(E)$ are also the infimum and supremum of a and b as calculated in E.*

Proof. Clearly, $0, u \in C(E)$ and $a \in C(E) \Rightarrow a' \in C(E)$. Suppose $a, b \in C(E)$. By Lemma 5.2, applied first to $p = b$ and then to $p = b'$,

$$e_1 := a \wedge b, \quad e_2 := a \wedge b', \quad e_3 := a' \wedge b, \quad \text{and} \quad e_4 := a' \wedge b'$$

all exist in E; $b = e_1 \oplus e_3$, and $b' = e_2 \oplus e_4$, so

$$u = b \oplus b' = e_1 \oplus e_3 \oplus e_2 \oplus e_4,$$

and e_1, e_2, e_3, e_4 forms an orthogonal sequence in E. Since $b \in C(L)$, Lemma 5.2 also implies that $a = e_1 \oplus e_2$. Evidently, if two principal elements have an infimum in E, then that infimum is again principal, so e_1, e_2, e_3, and e_4 are principal elements of E.

If $p \in E$, we have $p = q \oplus r$ with $q \leqslant a$ and $r \leqslant a'$. Also, since $b \in C(E)$, we have $q = p_1 \oplus p_2$ and $r = p_3 \oplus p_4$ with $p_1, p_3 \leqslant b$ and $p_2, p_4 \leqslant b'$. Therefore, $p = q \oplus r = p_1 \oplus p_2 \oplus p_3 \oplus p_4$ with

$$p_1 \leqslant q, b; \quad p_2 \leqslant q, b'; \quad p_3 \leqslant r, b; \quad \text{and} \quad p_4 \leqslant r, b'.$$

Since $q \leqslant a$ and $r \leqslant a'$, it follows that

$$p_1 \leqslant a \wedge b = e_1, \quad p_2 \leqslant a \wedge b' = e_2,$$

$$p_3 \leqslant a' \wedge b = e_3, \quad \text{and} \quad p_4 \leqslant a' \wedge b' = e_4,$$

so e_1, e_2, e_3, e_4 satisfy the conditions in Corollary 4.3; hence, e_1, e_2, e_3, e_4 and all orthogonal sums of subsequences thereof belong to $C(E)$. In particular, $a \wedge b = e_1 \in C(E)$.

If $a \perp b$, then $a = a \wedge b' = e_2$, $b = a' \wedge b = e_3$, so $a \oplus b = e_2 \oplus e_3 \in C(E)$, and it follows that $C(E)$ is a sub-effect algebra of E. Since every element of $C(E)$ is principal in E, it is *a fortiori* principal in $C(E)$; hence $C(E)$ is an orthomodular poset. Furthermore, if $a, b \in C(E)$, then $a \wedge b$ exists in E and belongs to $C(E)$; hence, it is also the infimum of a and b as calculated in $C(E)$. Therefore, $C(E)$ is an orthomodular lattice. By part (iii) of Lemma 5.2, disjoint elements of $C(E)$ are orthogonal, so $C(E)$ is a Boolean algebra. □

6. Effect Algebras Arising from Groups and Rings

The so-called *positive cone* $G^+ := \{a \in G \mid 0 \leqslant a\}$ in an additively-written partially ordered abelian group satisfies the two conditions $G^+ + G^+ \subseteq G^+$ and $G^+ \cap -G^+ = \{0\}$. Conversely, given a subset G^+ of an abelian group G that satisfies these two conditions, G can be organized into a partially ordered group for which G^+ is the positive cone by defining $a \leqslant b$ iff $b - a \in G^+$ for $a, b \in G$.

If G is a partially ordered abelian group and $0 \neq u \in G^+$, the interval $G^+[0, u] := \{a \in G \mid 0 \leqslant a \leqslant u\}$ is an effect algebra with unit u under the partially defined operation $\oplus$ obtained by restriction of $+$ to $G^+[0, u]$. Such an effect algebra, or one isomorphic to it, is called an *interval effect algebra* [2]. In many cases, it is possible to embed the group G as a subgroup of the additive group of a ring R with unit 1 in such a way that $1 = u$ and the partial order structure of G interacts nicely with the ring structure of R. We are thus led to our next definition.

6.1. DEFINITION. By an *effect ring* we mean a ring R with unit 1 and a distinguished subset R^+ of R such that R^+ is closed under addition, $R^+ \cap -R^+ = \{0\}$, and the following conditions hold for all $a, b \in R^+$:

(i) $ab = ba \Rightarrow ab \in R^+$
(ii) $aba \in R^+$
(iii) $aba = 0 \Rightarrow ab = ba = 0$
(iv) $a = a^2 \Rightarrow 1 - a \in R^+$.

For motivation, the following important example should be kept in mind.

6.2. EXAMPLE. A unital C^*-algebra R is organized into an effect ring by taking $R^+ := \{aa^* \mid a \in R\}$.

More generally, if $\mathcal{H}$ is a Hilbert space, $\mathcal{R}$ is any subring of the Banach $*$-algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, $\mathbf{1} \in \mathcal{R}$, and $\mathcal{R}$ is closed under the formation of adjoints, then $\mathcal{R}$ becomes an affect ring if we define $\mathcal{R}^+$ to be the set of all positive semi-definite operators in $\mathcal{R}$.

For the remainder of this section, we assume that R is an effect ring.

6.3. DEFINITION. Let $R^+[0,1] := \{a \in R^+ \mid 1 - a \in R^+\}$ and define the partial binary operation $\oplus$ on $R^+[0,1]$ as follows: For $a, b \in R^+[0,1]$, $a \oplus b$ is defined iff $a + b \in R^+[0,1]$, in which case $a \oplus b := a + b$.

It is easily verified that $R^+[0,1]$ is an effect algebra with unit 1 under the partial binary operation $\oplus$ in Definition 6.3. Furthermore, for $a, b \in R^+[0,1]$, $a \leqslant b$ iff $b - a \in R^+$, and the orthosupplement of $a \in R^+[0,1]$ is given by $a' = 1 - a$. By part (ii) of the following lemma, if $a \in R^+[0,1]$, then either $a = a^2$ or else $a - a^2$ is an isotropic element of $R^+[0,1]$.

6.4. LEMMA. *Let $a, b \in R^+[0,1]$. Then*:

(i) $ab = ba \Rightarrow ab,\ a + b - ab \in R^+[0,1]$ *and* $ab \leqslant a,\ b \leqslant a + b - ab$.
(ii) $a^2, a - a^2 \in R^+[0,1]$ *and* $a - a^2 \leqslant a, a'$.
(iii) $aba \in R^+[0,1]$ *and* $aba \leqslant a^2 \leqslant a$.

Proof. Suppose that $ab = ba$. Then $ab' = b'a$, so $ab, ab' \in R^+$ by part (i) of Definition 6.1. Since $R^+ + R^+ \subseteq R^+$, it follows that $(ab)' = a' + ab' \in R^+$, so $ab \in R^+[0,1]$. By the same argument applied to a and b', we have $ab' \in R^+[0,1]$. Likewise, $a'b, a'b' \in R^+[0,1]$. Part (i) now follows from the facts that $ab' = a - ab$, $a'b = b - ab$, and $(a'b')' = a + b - ab$. Part (ii) follows from part (i) upon first setting $b = a$ to obtain $a^2 \in R^+[0,1]$, then setting $b = a'$ and noting that $aa' = a - a^2$. To prove part (iii), we observe that $aba, ab'a \in R^+$ by part (ii) of Definition 6.1, that $ab'a = a^2 - aba$, and that $(aba)' = (a^2)' + ab'a$. □

6.5. DEFINITION. An idempotent element of R is called a *projection* iff it belongs to R^+. We denote by P the set of all projections in R.

In Example 6.2, the projections are precisely the self-adjoint idempotents in the C^*-algebra R. By part (iv) of Definition 6.1, $P \subseteq R^+[0,1]$.

6.6. THEOREM. *Let $e, f \in P$, $a \in R^+[0,1]$. Then*:

(i) $e \perp f$ *iff* $ef = fe = 0$ *iff* $e + f \in P$.
(ii) *P is a sub-effect algebra of $R^+[0,1]$.*
(iii) $a \leqslant e$ *iff* $a = ae = ea$ *iff* $a = eae$.

Proof. Suppose $e \perp f$, so that $e + f \in R^+[0,1]$. By part (iii) of Lemma 6.4, $e(e+f)e \in R^+[0,1]$ with $e(e+f)e \leqslant e$. Therefore,

$$-efe = e - e(e+f)e \in R^+[0,1] \subseteq R^+.$$

Since $efe \in R^+$, it follows that $efe = 0$; hence that $ef = fe = 0$ by part (iii) of Definition 6.1. If $ef = fe = 0$, then $e + f$ is idempotent and belongs to R^+, so it is a projection. Finally, if $e + f$ is a projection, then it belongs to $R^+[0, 1]$, so $e \perp f$. This proves part (i).

To prove part (ii), we note that $0, 1 \in P$ and P is closed under orthosupplementation. If $e, f \in P$ and $e \perp f$, then $e \oplus f = e + f \in P$ by part (i), so P is a sub-effect algebra of $R^+[0, 1]$.

Suppose $a \leqslant e$. Then $\exists b \in R^+[0, 1]$ with $a+b = e$, and it follows that $e'ae' + e'be' = e'ee' = 0$. Since both $e'ae'$ and $e'be'$ belong to R^+, it follows that $e'ae' = 0$; hence that $e'a = ae' = 0$ by part (iii) of Definition 6.1. Therefore, $a = ae = ea$. That $a = ae = ea$ iff $a = eae$ is clear. Finally, if $a = ae = ea$, then $a \leqslant e$ by part (i) of Lemma 6.4. □

6.7. COROLLARY. *As an effect algebra in its own right, P forms an orthomodular poset.*

Proof. If $e, f, g \in P$ with $e \perp f$, $e \perp g$, and $f \perp g$, then $e \oplus f \perp g$ by part (i) of Theorem 6.6. Therefore, P is an orthomodular poset by Theorem 5.3 in [13]. □

6.8. THEOREM. *If $e \in R^+[0, 1]$, then the following conditions are mutually equivalent*:

(i) *e is principal,*
(ii) $e \wedge e' = 0$,
(iii) $e \vee e' = 1$,
(iv) $ee' = 0$,
(v) $e \in P$,
(vi) *e' is principal.*

Proof. (i) $\Rightarrow$ (ii) by Lemma 3.3 and (ii) $\iff$ (iii) is obvious. By part (ii) of Lemma 6.4, $0 \leqslant ee' = e - e^2 \leqslant e, e'$, so the implication (ii) $\Rightarrow$ (iv) holds. That (iv) $\iff$ (v) is obvious. To prove (v) $\Rightarrow$ (i), suppose $e \in P$ and $a, b \in R^+[0, 1]$ with $a \perp b$ and $a, b \leqslant e$. By part (iii) of Theorem 6.4, $a = ae = ea$ and $b = be = eb$; whence, $a + b = (a + b)e = e(a + b)$ so $a \oplus b \leqslant e$. Thus $e \in P$ iff e is principal. Since $e \in P$ iff $e' \in P$, we have (v) $\iff$ (vi). □

6.9. COROLLARY. *If $e, f \in P$ and $e \wedge f$ (respectively, $e \vee f$) exists in $R^+[0, 1]$, then $e \wedge f \in P$ (respectively, $e \vee f \in P$).*

Proof. Evidently the infimum of principal elements, if it exists, is again a principal element. □

6.10. LEMMA. $C(R^+[0, 1]) \subseteq C(P)$.

Proof. Suppose $e \in C(R^+[0, 1])$. By Definition 5.1 and Theorem 6.8, $e \in P$. Let $f \in P$. Then $f = (e \wedge f) \oplus (e' \wedge f)$ by part (ii) of Lemma 5.2 and $e \wedge f, e' \wedge f \in P$ by Corollary 6.9. By Corollary 6.7, P is an orthomodular poset, so $f = (e \wedge f) \vee (e' \wedge f)$, and it follows that $e \in C(P)$. □

If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ is the Banach $*$-algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})^+$ is the set of all positive semi-definite self-adjoint operators on $\mathcal{H}$, then $\mathcal{E}(\mathcal{H}) := \mathcal{B}(\mathcal{H})^+[\mathbf{0}, \mathbf{1}]$ is called the *standard effect algebra* over $\mathcal{H}$ [13, Example 4.8]. Elements of $\mathcal{E}(\mathcal{H})$ are called *effect operators* on $\mathcal{H}$. Let $\mathbb{P}(\mathcal{H})$ be the complete, irreducible, orthomodular lattice of all projection operators on $\mathcal{H}$. By Theorem 6.6, $\mathbb{P}(\mathcal{H})$ is a sub-effect algebra of $\mathcal{E}(\mathcal{H})$ and, for $A \in \mathcal{E}(\mathcal{H})$, $G \in \mathbb{P}(\mathcal{H})$, we have $A \leqslant G$ iff $A = AG = GA$. The latter result, a proof of which may be found in Giuntini and Grueling [15, p. 937], is thus generalized by Theorem 6.6. A. Dvurečenskij and S. Gudder have independently shown that an effect operator G is a projection iff $G \wedge G' = \mathbf{0}$ (or, equivalently, $G \vee G' = \mathbf{1}$), and this is generalized by Theorem 6.8. Finally, since $\mathbb{P}(\mathcal{H})$ is irreducible, Lemma 6.10 shows that $C(\mathcal{E}(\mathcal{H})) = \{\mathbf{0}, \mathbf{1}\}$.

The "non spatial" formulation of our results in this section (avoiding the use of spectral theory) has been influenced by conversations with R. Burckel, A. Dvurečenskij, S. Gudder, and K. Ravindran, for which we are grateful. Definition 6.1 is too weak to capture some of the essential structure of the standard effect algebra $\mathcal{E}(\mathcal{H})$ and its C^* generalizations. An obvious definition of a BZ-ring, obtained by combining Definition 6.1 with the definition of a BZ-poset [4] suggests itself, but we refrain from pursuing this idea in the present paper.

7. Distributive Effect Algebras

In this section we launch a study of distributive effect algebras and give a complete characterization of finite distributive effect algebras.

7.1. DEFINITION. The effect algebra E is a *scale algebra* iff, as a poset, it is totally ordered; it is *distributive* iff, as a poset, it forms a distributive lattice.

Obviously, every scale algebra is distributive, and a cartesian product of distributive effect algebras is again distributive. The following lemma may be regarded as a strengthened version of Corollary 3.6 for the case of a distributive effect algebra E. Note that the result holds automatically if E is an orthomodular poset.

7.2. LEMMA. *Let E be distributive and suppose $e_1, e_2, \ldots, e_n$ is a pairwise disjoint and pairwise orthogonal sequence in E. Then $e_1, e_2, \ldots, e_n$ is an orthogonal sequence in E and $e_1 \oplus e_2 \oplus \cdots \oplus e_n = e_1 \vee e_2 \vee \cdots \vee e_n$.*

Proof. The proof is by induction on n. For $n = 1$, there is nothing to prove. Let $e_1, e_2, \ldots, e_n, e_{n+1}$ be a pairwise disjoint and pairwise orthogonal sequence in E. By the induction hypothesis, $e := e_1 \oplus e_2 \oplus \cdots \oplus e_n = e_1 \vee e_2 \vee \cdots \vee e_n$. Since $e_1, e_2, \ldots, e_n \leqslant e'_{n+1}$, we have $e \leqslant e'_{n+1}$. Since $e_j \wedge e_{n+1} = 0$ for $j = 1, 2, \ldots, n$, $e \wedge e_{n+1} = 0$ by the distributive law. Thus, $e \vee e_{n+1} = e \oplus e_{n+1}$ by Corollary 3.6. □

7.3. DEFINITION. An atom $a \in E$ is *type* 1 iff whenever m is a positive integer and ma is defined, the interval $E[0, ma]$ consists only of 0 and positive integer multiples of a; otherwise it is *type* 2.

An atom with isotropic index 1 is of type 1; in particular, any atom in an orthoalgebra is of type 1. If a is a type 1 atom in E and the isotropic index of a is n, it is clear that $\text{smt}(a)$ is principal and that $E[0, \text{smt}(a)]$ is a scale algebra with $n+1$ elements. Up to isomorphism, there is only one such scale algebra; it is called the *n-chain*, and denoted by C_n [2, Example 5.2].

7.4. EXAMPLE. The four-element effect algebra D, called the *diamond*, consists of 0, two atoms a, a^* with $a \neq a^*$, $a = a'$, $a^* = a^{*\prime}$, and the unit $2a = 2a^* = \text{smt}(a) = \text{smt}(a^*) = a \vee a^*$. The diamond D is a distributive effect algebra, and both a and a^* are type 2 atoms in D.

If a and a^* are the two atoms in the diamond D, then $a \not\perp a^*$, else $a \oplus a^* = 2a$, so $a^* = a$ by the cancellation law, contradicting $a \neq a^*$.

7.5. THEOREM. *Let E be distributive and let a, a^* be atoms in E with $a \neq a^*$ and $a \not\perp a^*$. Then a and a^* have isotropic index* 2, $\text{smt}(a) = \text{smt}(a^*) = 2a = 2a^* = a \vee a^*$, *$a$ and a^* are type 2 atoms, $a \vee a^*$ is principal, and $E[0, a \vee a^*]$ is the diamond.*

Proof. By the distributive law and the fact that a and a^* are atoms, the only elements in $E[0, a \vee a^*]$ are $0, a, a^*$, and $a \vee a^*$. Because $a \not\perp a^*$, we have $a \vee a^* \not\perp a$ and $a \vee a^* \not\perp a^*$. Since $a < a \vee a^*$, $\exists r \in E$ with $r \neq 0$, $a \perp r$ and $a \oplus r = a \vee a^*$. Thus $0 \neq r \in E[0, a \vee a^*]$, so $r = a$ or $r = a^*$ or $r = a \vee a^*$. Because $a \perp r$, $a \not\perp a^*$, and $a \not\perp a \vee a^*$, it follows that $r = a$, so $2a = a \vee a^*$. By symmetry, $2a^* = a \vee a^* = 2a$. If the isotropic index of a exceeds 2, then $3a = 2a \oplus a = 2a^* \oplus a$, contradicting the fact that $a \not\perp a^*$. Therefore, $\text{smt}(a) = 2a = a \vee a^*$ and, by symmetry, $\text{smt}(a^*) = 2a^* = a \vee a^*$. Because $E[0, a \vee a^*] = \{0, a, a^*, a \vee a^*\}$ and the only orthogonalities in $E[0, a \vee a^*]$ that do not involve 0 are $a \perp a$ and $a^* \perp a^*$, it follows that $a \vee a^*$ is principal in E and $E[0, a \vee a^*]$ is the diamond. □

7.6. THEOREM. *Let E be distributive and atomic, and let a be an atom in E with finite isotropic index n. Then a is type* 1 *iff $E[0, \text{smt}(a)]$ is the n-chain C_n, and a is type* 2 *iff $n = 2$ and $E[0, \text{smt}(a)]$ is the diamond D. In either case, $\text{smt}(a)$ is principal.*

Proof. Clearly, a is type 1 iff $E[0, \text{smt}(a)]$ is the n-chain C_n, in which case $\text{smt}(a)$ is principal. If $E[0, \text{smt}(a)]$ is the diamond D, then a is obviously type 2 and again $\text{smt}(a)$ is principal. Conversely, suppose a is type 2. If there is an atom $b \in E$ such that $b \neq a$ and $b \not\perp a$, the proof is complete by Theorem 7.5.

Thus, we can assume that every atom $b \neq a$ satisfies $b \perp a$. Since a is type 2, $\exists q \neq 0$, $q < \text{smt}(a) = na$, but q is not a positive integer multiple of a. By Theorem 3.9, $q \not\perp a$. Since E is atomic and $q \leqslant na$, there is an atom b with $b \perp q$ and $q \oplus b \leqslant na$.

Since $q \not\leqslant a$, it follows that $b \neq a$, so $b \perp a$. Also, $b \leqslant q \oplus b \leqslant na$; hence, $a \leqslant b$ by Corollary 3.8. But b is an atom, so $a = b$, contradicting $a \neq b$. □

7.7. LEMMA. *Let E be distributive and atomic, suppose a and b are atoms in E both of which have finite isotropic index, a is type* 1, *and $a \neq b$. Then* $\mathrm{smt}(a) \wedge \mathrm{smt}(b) = 0$ *and* $\mathrm{smt}(a) \perp \mathrm{smt}(b)$.

Proof. If $\mathrm{smt}(a) \wedge \mathrm{smt}(b) \neq 0$, there is an atom $c \leqslant \mathrm{smt}(a), \mathrm{smt}(b)$. Since a is type 1, $c = a$, so $a \leqslant \mathrm{smt}(b)$. Because $b \neq a \leqslant \mathrm{smt}(b)$, b is type 2, so $E[0, \mathrm{smt}(b)] = \{0, a, b, a \vee b\}$, contradicting the fact that a is type 1. Thus, $\mathrm{smt}(a) \wedge \mathrm{smt}(b) = 0$.

Since a is type 1 and $b \neq a$, we have $b \not\leqslant \mathrm{smt}(a)$, so $\mathrm{smt}(a) < \mathrm{smt}(a) \vee b$. Therefore, there is an atom c in E with $\mathrm{smt}(a) \perp c$ and $\mathrm{smt}(a) \oplus c \leqslant \mathrm{smt}(a) \vee b$. By the definition of $\mathrm{smt}(a)$, we cannot have $c = a$. By the distributive law, $c \leqslant \mathrm{smt}(a)$ or else $c = b$, the first possibility is ruled out because a is type 1 and $c \neq a$, and therefore $c = b$. Thus, $b \perp \mathrm{smt}(a)$.

Let the positive integer n be maximal with respect to the properties that nb is defined and $nb \perp \mathrm{smt}(a)$. To complete the proof, it is sufficient to show that n is the isotropic index of b. Assume the contrary, so that $nb < (n+1)b \leqslant \mathrm{smt}(b)$. Since $\mathrm{smt}(b)$ is disjoint from $\mathrm{smt}(a)$, it follows that nb and $(n+1)b$ are disjoint from $\mathrm{smt}(a)$, so $nb \vee \mathrm{smt}(a) = nb \oplus \mathrm{smt}(a)$ by Corollary 3.6; moreover, $(n+1)b \not\leqslant nb \vee \mathrm{smt}(a)$ by the distributive law. Thus, $nb \oplus \mathrm{smt}(a) < (n+1)b \vee \mathrm{smt}(a)$, so there is an atom c with $c \perp (nb \oplus \mathrm{smt}(a))$ and $nb \oplus \mathrm{smt}(a) \oplus c \leqslant (n+1)b \vee \mathrm{smt}(a)$. Note that $c \perp nb$, so $c \perp b$. Also, $c \perp \mathrm{smt}(a)$, so $c \neq a$ by the definition of $\mathrm{smt}(a)$. If $c = b$, then $(n+1)b = nb \oplus b \perp \mathrm{smt}(a)$, contradicting the maximality of n, so $c \neq b$. By the distributive law, $c \leqslant (n+1)b$ or else $c \leqslant \mathrm{smt}(a)$. Since a is type 1 and $c \neq a$, $c \leqslant (n+1)b$ is the only possibility; hence, since $c \neq b$, b is type 2, $n = 1$, $c \leqslant 2b$, and $c \not\leqslant b$, contradicting $c \perp b$. □

7.8. LEMMA. *Let E be distributive and atomic and suppose a and b are atoms in E both of which have finite isotropic index in E. Then, either* $\mathrm{smt}(a) = \mathrm{smt}(b)$ *or else* $\mathrm{smt}(a)$ *and* $\mathrm{smt}(b)$ *are disjoint and orthogonal.*

Proof. Suppose $\mathrm{smt}(a) \neq \mathrm{smt}(b)$, so that $a \neq b$. By Lemma 7.7, we can assume that both a and b are type 2. By Theorem 7.5, $a \perp b$, both a and b have isotropic index 2, and both $E[0, 2a]$ and $E[0, 2b]$ are diamonds with $2a = \mathrm{smt}(a) \neq \mathrm{smt}(b) = 2b$. Thus, there are atoms $a^* \neq a$, $b^* \neq b$ in E such that $E[0, 2a] = \{0, a, a^*, 2a\}$, $2a = a \vee a^*$, $E[0, 2b] = \{0, b, b^*, 2b\}$, $2b = b \vee b^*$, $a \perp b, b^*$ and $b \perp a, a^*$. Obviously, $2a \wedge 2b = 0$. Also, $\mathrm{smt}(a) = a \vee a^* \perp b \vee b^* = \mathrm{smt}(b)$. □

7.9. THEOREM. *A finite distributive effect algebra E is a cartesian product of finite chains and diamonds.*

Proof. By Lemma 7.2, Theorem 7.6, and Lemma 7.8, the distinct isotropic summits of atoms in E form an orthogonal family $e_1, e_2, \ldots, e_n$ of nonzero principal elements. Since every nonzero element of E is a finite orthogonal sum of atoms, and each atom is dominated by its own isotropic summit, Theorem 4.2 implies that E is isomorphic to the cartesian product $\times_j E[0, e_j]$. By Theorem 7.6, each $E[0, e_j]$ is either a finite chain or a diamond. □

As a consequence of Theorem 7.9, the center $C(E)$ of a finite distributive effect algebra E is the sub-effect algebra of E generated by the isotropic summits of the atoms in E, and these isotropic summits are precisely the atoms in the Boolean algebra $C(E)$.

7.10. COROLLARY. *A finite distributive effect algebra is an interval algebra.*

Proof. By [2, Example 5.2], every finite chain is an interval algebra. By [13, Example 4.7], the diamond D is an interval algebra. By [13, Theorem 7.1] a cartesian product of interval algebras is again an interval algebra, so the result follows from Theorem 7.9. □

As a *poset* (but not as an effect algebra!) the diamond D is isomorphic to the 4-element Boolean algebra 2^2. However, there are finite distributive lattices – for instance the free distributive lattice on three generators – that cannot be factored into chains and copies of 2^2; hence, not every distributive lattice can be organized into an effect algebra.

It is hoped that some of the techniques developed in this paper will be useful in characterizing not only distributive effect algebras in general, but also *semimodular* and *modular* effect algebras.

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Unsolved Problems

A Problem List

Is every finite ordered set (with more than three elements) determined by its collection of unlabelled one-point deleted subsets? (Cf. *ORDER* (1985) **1**, 311–313.)

In a finited ordered set which is not a chain there are two noncomparable elements a and b such that

$$\frac{1}{3} \leq \frac{\textit{number of all linear extensions in which } a < b}{\textit{number of all linear extensions}} \leq \frac{2}{3}.$$

(Cf. *ORDER* (1985) **2**, 327–330.)

Characterize the finite ordered sets with the fixed point property. (Cf. *ORDER* (1985) **2**, 219–221.)

Characterize the undirected graphs which are covering graphs of ordered sets. (Cf. *ORDER* (1985) **2**, 101–104.)

Does every ordered set (with at least three elements) contain a pair of elements whose removal decreases the order dimension by at most one? (Cf. *ORDER* (1984) **1**, 217–218.)

What is the number of antichains in the power set of an n-element set? (Cf. *ORDER* (1986) **2**, 415–417.)