

*Dedicated to the memory
of Professor Milan Kolibiar*

EFFECT ALGEBRA COUNTEREXAMPLES

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ABSTRACT. Two effect algebra counterexamples are presented. The first shows that the standard effect algebra of operators on a Hilbert space is not a lattice and the second shows that the tensor product of two effect algebras need not exist.

1. Introduction

Effect algebras (or D-posets) have been recently introduced as an axiomatic model for the foundations of quantum mechanics ([2], [9], [10], [17]). The most important effect algebra is the set $\mathcal{E}(H)$ of all self-adjoint operators \mathbf{A} on a Hilbert space H satisfying $\mathbf{0} \leq \mathbf{A} \leq \mathbf{I}$. The partial order on $\mathcal{E}(H)$ is defined by setting $\mathbf{A} \leq \mathbf{B}$ if $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \leq \langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle$ for all $\mathbf{x} \in H$. This effect algebra is the basis for a widely employed approach to quantum mechanics called the operational approach ([3], [4], [12], [16], [20]). In this note, it is shown that if $\dim H \geq 2$, then $\mathcal{E}(H)$ is not a lattice. This substantiates a long held opinion in the folklore of the subject ([2], [3], [4], [10], [16]).

Tensor products of effect algebras are important because they are used to describe coupled physical systems [1], [7], [8], [15] and various results concerning the existence of these tensor products have been obtained ([5], [6], [8], [19], [21], [22]). However, whether the tensor product of two arbitrary effect algebras exists has remained an open question ([5], [9]). This note also presents a counterexample that answers this question negatively.

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An *effect algebra* is a system $(L, 0, 1, \oplus)$, where $0, 1$ are distinct elements in L and \oplus is a partial binary operation on L satisfying the following conditions:

- (1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
- (2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
- (3) For any $a \in L$, there is a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.
- (4) If $a \oplus 1$ is defined, then $a = 0$.

It is easy to check that $\mathcal{E}(H)$ is an effect algebra, where $\mathbf{A} \oplus \mathbf{B}$ is defined for $\mathbf{A}, \mathbf{B} \in \mathcal{E}(H)$ if $\mathbf{A} + \mathbf{B} \leq \mathbf{1}$ and, in this case, $\mathbf{A} \oplus \mathbf{B} = \mathbf{A} + \mathbf{B}$.

Let P, Q and R be effect algebras. A mapping $\beta: P \times Q \rightarrow R$ is called a *bimorphism* if the following conditions are satisfied:

- (1) $\beta(1, 1) = 1$.
- (2) If $a \oplus b$ is defined, then $\beta(a, c) \oplus \beta(b, c)$ is defined for all $c \in Q$ and $\beta(a, c) \oplus \beta(b, c) = \beta(a \oplus b, c)$.
- (3) If $c \oplus d$ is defined, then $\beta(a, c) \oplus \beta(a, d)$ is defined for all $a \in P$ and $\beta(a, c) \oplus \beta(a, d) = \beta(a, c \oplus d)$.

The precise definition of the tensor product is not needed in this note (cf. [5], [6], [9]). Roughly speaking, the tensor product of P and Q is a pair (T, τ) , where T is an effect algebra and $\tau: P \times Q \rightarrow T$ is a bimorphism satisfying a universality condition. We shall show that P and Q need not admit a bimorphism in which case their tensor product does not exist.

2. $\mathcal{E}(H)$ is not a lattice

It has often been stated in the literature that the standard effect algebra $\mathcal{E}(H)$ is not a lattice for $\dim H \geq 2$ ([2], [3], [4], [10], [16]). However, until very recently [18], no explicit counterexample seems to have been given. Instead, the authors have referred to previous sources such as [13], [14]. For example, let $\mathcal{S}(H)$ denote the set of self-adjoint operators on H . It is shown in [13] that for $\mathbf{E}, \mathbf{F} \in \mathcal{E}(H)$, $\mathbf{E} \wedge_{\mathcal{S}} \mathbf{F}$ exists if and only if \mathbf{E} and \mathbf{F} are comparable. However, it cannot be concluded from this that $\mathbf{E} \wedge_{\mathcal{E}} \mathbf{F}$ does not exist when \mathbf{E} and \mathbf{F} are not comparable. For instance, it is also shown in [13] that $\mathbf{E} \wedge_{\mathcal{E}} \mathbf{F}$ exists for any two projection operators \mathbf{E} and \mathbf{F} . We now show that $\mathcal{E}(H)$ is not a lattice by characterizing those pairs $\mathbf{A}, \mathbf{B} \in \mathcal{E}(\mathbb{C}^2)$ such that $\mathbf{A} \wedge \mathbf{B}$ exists, where $\mathbf{A} \wedge \mathbf{B}$ means $\mathbf{A} \wedge_{\mathcal{E}} \mathbf{B}$ in the sequel.

Let $\mathcal{E} = \mathcal{E}(\mathbb{C}^2)$, and let $\mathbf{A} \in \mathcal{E}(\mathbb{C}^2)$ with

$$\mathbf{A} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}, \quad a, c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

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It is easy to show that $\mathbf{A} \geq \mathbf{O}$ if and only if $a, c \geq 0$ and $ac \geq |b|^2$ ([13]). It follows that $\mathbf{A} \in \mathcal{E}$ if and only if $0 \leq a \leq 1, 0 \leq c \leq 1, ac \geq |b|^2$ and $(1-a)(1-c) \geq |b|^2$. If $\mathbf{B} \in \mathcal{E}$ has the form

$$\mathbf{B} = \begin{bmatrix} d & e \\ \bar{e} & f \end{bmatrix}, \quad d, f \in \mathbb{R}, \quad e \in \mathbb{C},$$

we conclude that $\mathbf{B} \leq \mathbf{A}$ if and only if $d \leq a, f \leq c$ and $(a-d)(c-f) \geq |b-e|^2$. For $a, b \in \mathbb{R}$ we use the notation $a \wedge b = \min(a, b)$.

LEMMA 1. *If $\mathbf{A} \in \mathcal{E}$ is a multiple of a 1-dimensional projection, then $\mathbf{A} \wedge \mathbf{B}$ exists for every $\mathbf{B} \in \mathcal{E}$.*

Proof. We can assume that \mathbf{A} is diagonal, so \mathbf{A} and \mathbf{B} have the form

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b & c \\ \bar{c} & d \end{bmatrix}.$$

Suppose that $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, where $\mathbf{C} \in \mathcal{E}$. Then \mathbf{C} has the form

$$\mathbf{C} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \quad 0 \leq e \leq a, b; \quad (b-e)d \geq |c|^2.$$

Now define

$$f = \begin{cases} b & \text{if } d = 0, \\ b - |c|^2/d & \text{if } d \neq 0. \end{cases}$$

It follows that

$$\begin{bmatrix} a \wedge f & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{A} \wedge \mathbf{B}.$$

□

THEOREM 2. *For $\mathbf{A}, \mathbf{B} \in \mathcal{E}$, $\mathbf{A} \wedge \mathbf{B}$ exists if and only if \mathbf{A} and \mathbf{B} are comparable, or either \mathbf{A} or \mathbf{B} is a multiple of a 1-dimensional projection.*

Proof. Sufficiency follows from Lemma 1. For necessity, suppose that $\mathbf{A}, \mathbf{B} \in \mathcal{E}$ are incomparable and neither is a multiple of a 1-dimensional projection. First assume that \mathbf{A} and \mathbf{B} commute, and hence, they can be simultaneously diagonalized

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix},$$

where $0 < a, b, c, d \leq 1$. Since \mathbf{A} and \mathbf{B} are incomparable, we can assume without loss of generality that $a < c, b > d$. Let $\varepsilon = ad/(a+b)$ and let $\beta = (c-a) \wedge (b-d)$. Then $\varepsilon > 0, \beta > 0$, and we let $\delta = (\beta\varepsilon + \varepsilon^2)^{1/2}$ so $\delta > \varepsilon$. Suppose that $\mathbf{A} \wedge \mathbf{B}$ exists, and let

$$\mathbf{C} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Since $\mathbf{C} \in \mathcal{L}$ and $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, we have $\mathbf{C} \leq \mathbf{A} \wedge \mathbf{B} \leq \mathbf{A}, \mathbf{B}$. It follows that $\mathbf{A} \wedge \mathbf{B} = \mathbf{C}$. Now let

$$\mathbf{D} = \begin{bmatrix} a - \varepsilon & \delta \\ \delta & d - \varepsilon \end{bmatrix}.$$

To show that $\mathbf{D} \in \mathcal{L}$, we have $0 \leq a - \varepsilon \leq 1$, $0 \leq d - \varepsilon \leq 1$. Also, $ad = (a + b)\varepsilon$, so

$$ad - (a + d)\varepsilon + \varepsilon^2 = (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2$$

and $(1 - a)(1 - d) \geq (a + b - 2)\varepsilon$, so

$$(1 - a)(1 - d) + (2 - a - d)\varepsilon + \varepsilon^2 \geq (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2.$$

Hence, $(1 - a + \varepsilon)(1 - d + \varepsilon) \geq \delta^2$, so $\mathbf{D} \in \mathcal{L}$. Now

$$\mathbf{A} - \mathbf{D} = \begin{bmatrix} \varepsilon & -\delta \\ -\delta & b - d + \varepsilon \end{bmatrix}.$$

Since $\varepsilon \geq 0$, $b - d + \varepsilon \geq 0$ and

$$\varepsilon(b - d + \varepsilon) = (b - d)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2,$$

we have $\mathbf{A} - \mathbf{D} \geq \mathbf{0}$, so $\mathbf{D} \leq \mathbf{A}$. Moreover,

$$\mathbf{B} - \mathbf{D} = \begin{bmatrix} c - a + \varepsilon & -\delta \\ -\delta & \varepsilon \end{bmatrix}.$$

Since $c - a + \varepsilon$, $\varepsilon \geq 0$ and

$$(c - a + \varepsilon)\varepsilon = (c - a)\varepsilon + \varepsilon^2 \geq \beta\varepsilon + \varepsilon^2 = \delta^2,$$

we have $\mathbf{B} - \mathbf{D} \geq \mathbf{0}$, so $\mathbf{D} \leq \mathbf{B}$. We conclude that $\mathbf{D} \leq \mathbf{C}$. But

$$\mathbf{C} - \mathbf{D} = \begin{bmatrix} \varepsilon & -\delta \\ -\delta & \varepsilon \end{bmatrix}$$

and $\varepsilon^2 < \delta^2$, which is a contradiction.

Now consider the case in which \mathbf{A} and \mathbf{B} do not necessarily commute. Assume that $\mathbf{C} = \mathbf{A} \wedge \mathbf{B}$ exists and, without loss of generality, that \mathbf{C} is diagonal. Then $\mathbf{A}, \mathbf{B}, \mathbf{C}$ have the forms

$$\mathbf{A} = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} d & e \\ \bar{e} & f \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}.$$

By assumption, we have $\mathbf{C} \neq \mathbf{A}, \mathbf{B}$, $ac > |b|^2$ and $df > |e|^2$. Since $\mathbf{C} \leq \mathbf{A}, \mathbf{B}$, we have $g \leq a$, $h \leq c$, $(a - g)(c - h) \geq |b|^2$, $g \leq d$, $h \leq f$, $(d - g)(f - h) \geq |e|^2$. Now at most one of the following equalities holds: $g = a$, $h = c$, $g = d$, $h = f$. Indeed, assume that $g = a$, so $b = 0$. Then, if $g = d$, we have $e = 0$, so \mathbf{A} and \mathbf{B} are comparable. If $h = c$, then $\mathbf{C} = \mathbf{A}$. If $h = f$, then $e = 0$, so \mathbf{A} and \mathbf{B} are

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diagonal, and this case was treated earlier. The other cases are similar. Hence, we can assume without loss of generality that $h < c$ and $h < f$. If $g \neq 0$, let

$$0 < \delta < \min\left(1 - h, \frac{g(c - h)}{a}, \frac{g(f - h)}{d}\right),$$

and let

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & h + \delta \end{bmatrix}.$$

Then $\mathbf{D} \not\leq \mathbf{C}$. Moreover, $\mathbf{D} \leq \mathbf{A}$ because

$$h + \delta \leq h + g(c - h)/a \leq h + c - h = c$$

and

$$a(c - h - \delta) = (a - g)(c - h) + g(c - h) - a\delta \geq |b|^2 + g(c - h) - a\delta \geq |b|^2.$$

Also, $\mathbf{D} \leq \mathbf{B}$ because

$$h + \delta \leq h + g(f - h)/d \leq h + f - h = f$$

and

$$d(f - h - \delta) = (d - g)(f - h) + g(f - h) - d\delta \geq |e|^2 + g(f - h) - d\delta \geq |e|^2.$$

If $g = 0$, let

$$0 < \varepsilon < \min\left(a - \frac{|b|^2}{c}, d - \frac{|e|^2}{f}\right),$$

and let

$$\mathbf{D} = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\mathbf{D} \not\leq \mathbf{C}$. Moreover, $\mathbf{D} \leq \mathbf{A}$ because

$$\varepsilon \leq a - |b|^2/c \leq a$$

and

$$(a - \varepsilon)c = ac - \varepsilon c \geq |b|^2,$$

also, $\mathbf{D} \leq \mathbf{B}$ because

$$\varepsilon \leq d - |e|^2/f \leq f$$

and

$$(d - \varepsilon)f = df - \varepsilon f \geq |e|^2.$$

□

Theorem 2 also holds for $\mathcal{E}(\mathbb{R}^2)$ with essentially the same proof. We conclude that there are many pairs \mathbf{A}, \mathbf{B} in $\mathcal{E}(\mathbb{C}^2)$ and $\mathcal{E}(\mathbb{R}^2)$ such that $\mathbf{A} \wedge \mathbf{B}$ does not exist. A simple concrete example is

$$\mathbf{A} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Related results are given in [18]. For example, letting $\mathbf{A}' = \mathbf{I} - \mathbf{A}$, it is shown in [18] that there exists an $\mathbf{A} \in \mathcal{E}(\mathbb{C}^2)$ such that $\mathbf{A} \wedge \mathbf{A}'$ does not exist. But the verification of such counterexamples follow directly from Theorem 2. For instance, letting \mathbf{B} be defined as above, we see that \mathbf{B} and \mathbf{B}' are incomparable and neither is a multiple of a 1-dimensional projection. Applying Theorem 2, we conclude that $\mathbf{B} \wedge \mathbf{B}'$ does not exist. Now let H be a real or complex Hilbert space with $\dim H \geq 2$, and let ϕ and ψ be orthogonal unit vectors in H . Define $\mathbf{A}, \mathbf{B} \in \mathcal{E}(H)$ by $\mathbf{A}\phi = \frac{1}{2}\phi$, $\mathbf{A}\psi = \frac{1}{2}\psi$, $\mathbf{B}\phi = \frac{3}{4}\phi$, $\mathbf{B}\psi = \frac{1}{4}\psi$, and $\mathbf{A}\gamma = \mathbf{B}\gamma = \mathbf{O}$ for all γ in the orthogonal complement of the span of $\{\phi, \psi\}$. It follows from Theorem 2 that $\mathbf{A} \wedge \mathbf{B}$ does not exist, so $\mathcal{E}(H)$ is not a lattice. Theorem 2 might be useful in solving the following.

OPEN PROBLEM. Characterize the pairs of elements $\mathbf{A}, \mathbf{B} \in \mathcal{E}(H)$ such that $\mathbf{A} \wedge \mathbf{B}$ exists.

Applying Theorem 2 and De Morgan's laws, we conclude that for $\mathbf{A}, \mathbf{B} \in \mathcal{E}(\mathbb{C}^2)$, $\mathbf{A} \vee \mathbf{B}$ exists if and only if \mathbf{A} and \mathbf{B} are comparable, or either $\mathbf{I} - \mathbf{A}$ or $\mathbf{I} - \mathbf{B}$ is a multiple of a 1-dimensional projection. Thus, $\mathbf{A} \wedge \mathbf{B}$ can exist while $\mathbf{A} \vee \mathbf{B}$ does not exist and vice versa. For example, letting

$$\mathbf{A} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$$

we conclude that $\mathbf{A} \wedge \mathbf{B}$ exists, but $\mathbf{A} \vee \mathbf{B}$ does not exist. Moreover, even if \mathbf{A} and \mathbf{B} are incomparable, it is possible that $\mathbf{A} \wedge \mathbf{B}$ and $\mathbf{A} \vee \mathbf{B}$ both exist. For example, this happens for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad a, b \in \mathbb{R}, \quad a \neq 1, \quad b \neq 0.$$

Finally, we have the converse of Lemma 1.

COROLLARY 3. For $\mathbf{A} \in \mathcal{E}(\mathbb{C}^2)$, $\mathbf{A} \wedge \mathbf{B}$ exists for every $\mathbf{B} \in \mathcal{E}(\mathbb{C}^2)$ if and only if \mathbf{A} equals \mathbf{I} or a multiple of a 1-dimensional projection.

3. Nonexistence of tensor products

Since \oplus is an associative partial operation, we can write $(a \oplus b) \oplus c$ as $a \oplus b \oplus c$ when this expression is defined. It follows by induction that we do not need parentheses in expressions of the form $a_1 \oplus \cdots \oplus a_n$. If $a_1 \oplus \cdots \oplus a_n$ is defined, where $a_i = a$, $i = 1, \dots, n$, we denote this element by na . An n -chain generated by a is an effect algebra with elements $0, a, 2a, \dots, na$, where $na = 1$.

Let P_1 be the horizontal sum of three 4-chains generated by a , b and c , respectively, with the additional requirement that $a \oplus b \oplus c = 1$. The \oplus table for P_1 is displayed below. In this table, a dash indicates that the sum is not defined and the trivial sums involving a 0 or 1 are not displayed. It is not hard to show that P_1 is indeed an effect algebra.

\oplus	a	$2a$	$3a$	b	$2b$	$3b$	c	$2c$	$3c$
a	$2a$	$3a$	1	$3c$	—	—	$3b$	—	—
$2a$	$3a$	1	—	—	—	—	—	—	—
$3a$	1	—	—	—	—	—	—	—	—
b	$3c$	—	—	$2b$	$3b$	1	$3a$	—	—
$2b$	—	—	—	$3b$	1	—	—	—	—
$3b$	—	—	—	1	—	—	—	—	—
c	$3b$	—	—	$3a$	—	—	$2c$	$3c$	1
$2c$	—	—	—	—	—	—	$3c$	1	—
$3c$	—	—	—	—	—	—	1	—	—

THEOREM 4. *If Q is the 4-chain generated by d , then P_1 and Q do not admit a bimorphism. Hence, the tensor product of P_1 and Q does not exist.*

Proof. Suppose a bimorphism $\beta: P_1 \times Q \rightarrow R$ exists. Then

$$\begin{aligned}
 1 &= \beta(1, 1) = \beta(1, 4d) = 4\beta(1, d) = 4\beta(a \oplus b \oplus c, d) \\
 &= 4[\beta(a, d) \oplus \beta(b, d) \oplus \beta(c, d)] \\
 &= 4\beta(a, d) \oplus 4\beta(b, d) \oplus 4\beta(c, d) \\
 &= \beta(4a, d) \oplus \beta(4b, d) \oplus \beta(4c, d) = 3\beta(1, d).
 \end{aligned}$$

Since $3\beta(1, d) \oplus \beta(1, d) = 1$, we conclude that $\beta(1, d) \oplus 1$ is defined. Hence, $\beta(1, d) = 0$. It follows that $0 = 3\beta(1, d) = 1$, which is a contradiction. \square

It is clear that there are many counterexamples of this type. There are also counterexamples in which one of the effect algebras is an orthomodular poset

or lattice. For example, let W be the 3×4 window ([11]), and let Q be a 4-chain. An argument similar to that given in Theorem 4 shows that W and Q do not admit a bimorphism. Moreover, if we replace d by $a \in P_1$ in the proof of Theorem 4, then we conclude that the tensor product of P_1 and P_1 does not exist. Nevertheless, we conjecture that the tensor product of W and W does exist. If this conjecture is true, it would give an example of a tensor product in which neither of its components possesses a state. A *state* on an effect algebra P is a map $\phi: P \rightarrow [0, 1] \subseteq \mathbb{R}$ such that $\phi(1) = 1$, and if $a \oplus b$ is define, then $\phi(a) + \phi(b) \leq 1$ and $\phi(a \oplus b) = \phi(a) + \phi(b)$. Such an example would be of interest because it is known that if two effect algebras P and Q each possess a state, then their tensor product exists ([6]).

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