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The Involutory Dimension of Involution Posets

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Abstract. The *involutory dimension,* if it exists, of an involution poset $\mathbf{P} := (P, \leq, ')$ is the minimum cardinality of a family of linear extensions of $\leq$, involutory with respect to $'$, whose intersection is the ordering $\leq$. We show that the involutory dimension of an involution poset exists iff any pair of isotropic elements are orthogonal. Some characterizations of the involutory dimension of such posets are given. We study prime order ideals in involution posets and use them to generate involutory linear extensions of the partial ordering on orthoposets. We prove several of the standard results in the theory of the order dimension of posets for the involutory dimension of involution posets. For example, we show that the involutory dimension of a finite orthoposet does not exceed the cardinality of an antichain of maximal cardinality. We illustrate the fact that the order dimension of an orthoposet may be different from the involutory dimension.

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Introduction

Involution posets have appeared in studies of Ockham algebras and de Morgan algebras [2], as well as elsewhere. For example, in the approach to the mathematical foundations of quantum physics called quantum logic [6, 9], the models for the propositional logics are orthomodular lattices, orthomodular posets, orthoalgebras, or effect algebras – all of which are involution posets, refer to [5, 8] or [4]. This paper initiates the investigation of the involutory dimension of such posets, which involves representing the partial order in an involution poset as the intersection of involutory chains which extend that ordering and have the same involution.

It is currently not an easy task to calculate the order dimension of a poset, as it involves rather lengthy constructions. Usually this involves finding cycle-free sets that cover its critical pairs. In Theorem 3.9, the involutory dimension of a nonlinear finite regular involution poset is characterized by involutory cycle-free sets that cover its critical pairs. Because cycle-free sets are usually easier to find when they are involutory, the involutory dimension is easier to calculate than the order dimension. Moreover, the involutory dimension (if it exists) of an involutory poset is an upper bound for its order dimension.

We study prime order ideals in an involution poset and use them to generate involutory chains that extend the ordering and respect the involution on the poset, thereby developing the theory of involutory dimension of involution posets. These results are used to compare the order and the involutory dimensions of certain structures of interest in quantum logic.

1. Definitions and Notation

In this section we give the definitions and results that are required to study the involutory dimension. We recall the definition of a critical pair in a poset and the set of critical pairs in an orthomodular poset is characterized.

An *involution poset* $\mathbf{P} := (P, \leq, ')$ is a poset $(P, \leq)$ with a unary mapping $'\colon P \to P$ with (i) $x'' = x$ and (ii) if $x \leq y$ then $y' \leq x'$. (We use the symbol ":=" to mean equal by definition.) Such an involution poset $\mathbf{P}$ is said to be bounded if it contains a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$. *Throughout this paper, unless otherwise stated,* $\mathbf{P} := (P, \leq, ', \mathbf{0}, \mathbf{1})$ *will denote a bounded involution poset.* It is said to be an *orthoposet* (OP) if $x \wedge x' = \mathbf{0}$ for every $x \in P$; in this case x' is called an *orthocomplement* of x. For $x, y \in P$, write $x \perp y$ (and call x and y *orthogonal*) iff $x \leq y'$, otherwise write $x \not\perp y$ (and call them *non-orthogonal*). An *orthomodular poset* (OMP) is an OP such that for $x, y \in P$, $x \perp y$ implies $x \vee y$ exists in P, and $x \leq y$ implies $y = x \vee (y' \vee x)'$ (orthomodular identity, OMI).

Let $\mathbf{Q} := (Q, \leq)$ be any poset. The elements $x, y \in Q$ are said to be *incomparable*, denoted $x \| y$, if neither $x \leq y$ nor $y \leq x$ holds. Let $\mathrm{inc}(\mathbf{Q}) := \{(x, y)\colon x, y \in Q \text{ and } x \| y\}$. For $x \in Q$ and $Y \subseteq Q$, we write $x \| Y$ or $Y \| x$ to mean that $x \| y$ for every $y \in Y$; and we define $x{\downarrow} := \{y \in Q \mid y \leq x\}$, and for $X \subseteq Q$, $X{\downarrow} := \cup\{x{\downarrow} \mid x \in X\}$. The sets $x{\uparrow}$ and $X{\uparrow}$ are defined dually. An incomparable pair $(x, y) \in \mathrm{inc}(\mathbf{Q})$ is a *critical pair* if $x{\uparrow}\backslash\{x\} \subseteq y{\uparrow}$ and $y{\downarrow}\backslash\{y\} \subseteq x{\downarrow}$. Let $\mathrm{crit}(\mathbf{Q}) := \{(x, y) \mid (x, y) \text{ is a critical pair of } \mathbf{Q}\}$. In [12] it is shown that if $\mathbf{Q} := (Q, \leq)$ is a finite poset and $(a, b) \in \mathrm{inc}(\mathbf{Q})$, then there exists $(u, v) \in \mathrm{crit}(\mathbf{Q})$ such that $(a, b) \in \mathrm{tr}(\leq \cup\{(u, v)\})$, where $\mathrm{tr}(R)$ is the transitive closure of the relation R.

Let $A := \mathrm{atom}(\mathbf{P})$ be the set of *atoms* of $\mathbf{P}$, that is, the elements of P that cover $\mathbf{0}$. For $Q \subseteq P$, define $Q' := \{x' \mid x \in Q\}$, in particular, A' is the set of *co-atoms* of $\mathbf{P}$. It is easy to see that, in an involution poset $\mathbf{P} := (P, \leq, ')$, $(a, b) \in \mathrm{crit}(\mathbf{P})$ iff $(b', a') \in \mathrm{crit}(\mathbf{P})$.

The following lemma is a characterization for the critical pairs of orthomodular posets. This result will be useful later when relating the critical pairs of an orthomodular poset to its involutory dimension.

LEMMA 1.1. *If* $\mathbf{P}$ *is an orthomodular poset, then*

$$\mathrm{crit}(\mathbf{P}) = \big\{(a', b) \mid (a, b) \in A \times A \backslash \perp\big\}.$$

Proof. Suppose that $(u, v) \in \mathrm{crit}(\mathbf{P})$. We claim that $u' \in A$. If not, then there exists $x \in P$ such that $\mathbf{0} < x < u'$ and $u < x' < \mathbf{1}$. By the OMI, $u' = x \vee (u \vee x)'$.

Since $u \vee x > u$, $x' > u$, and $(u, v) \in \text{crit}(\mathbf{P})$, it follows that $u \vee x > v$ and $x' > v$. Hence $(u \vee x)' < v'$ and $x < v'$. Thus $u' = x \vee (u \vee x)' \leq v'$. Therefore $u' \leq v'$ or $v \leq u$ which contradicts the assumption that $(u, v) \in \text{crit}(\mathbf{P})$. Hence $u' \in A$. Similarly it follows $v \in A$ since $(v', u') \in \text{crit}(\mathbf{P})$. Thus $u', v \in A$. Since $(u, v) \in \text{crit}(\mathbf{P})$, it follows that $u \| v$ and hence $u' \not\leq v$. Therefore we have $\text{crit}(\mathbf{P}) \subseteq \{(a', b) \mid (a, b) \in A \times A \setminus \perp\}$. The reverse inclusion is immediate. □

There is an orthoalgebra for which crit(**P**) cannot be so characterized. In fact, the simplest example is the (14-element) so called Wright triangle (our $\mathbf{L}_3^3$ in Section 4) which is the smallest orthoalgebra that is not an OMP.

Let $\mathbf{Q} = (Q, \leq)$ be a poset. When $\preceq$ is another partial ordering on the set Q, we call $\preceq$ *an extension of* $\leq$ whenever $x \leq y$ implies $x \preceq y$ for every $x, y \in Q$. Since any ordering on Q is viewed as a subset of $Q \times Q$, the set of all extensions of $\leq$ is partially ordered by inclusion. The maximal elements in this ordering are linear orderings [13], they are called *linear extensions of* $\leq$ or *chains on* **Q**. Let $\text{Ch}(\mathbf{Q}) := \{\mathbf{C} \mid \mathbf{C} \text{ is a chain extending } \leq\}$. A *partial extension of* $\leq$ is an extension of the restriction of $\leq$ to a subset of Q. It is proved in [12] that, for any partial extension $\preceq$ of $\leq$, $\text{tr}\,(\leq \cup \preceq)$ is an extension of $\leq$. When $\preceq$ is a partial extension of $\leq$ which itself is a chain, then we write $\mathbf{C} = [x_1, \ldots, x_n]$ for the chain $x_1 \preceq \cdots \preceq x_n$ and we define $\varphi(\mathbf{C}) := \{x_1, \ldots, x_n\}$. An *involutory chain is* a linearly ordered involution poset. If $\{C_i\}_{i=1}^n$ is a family of disjoint chains, then we define $[C_1, C_2, \ldots, C_n]$ to be the ordinal sum, that is, the chain respecting the orderings in C_i, and satisfying $x \leq y$ for every $x \in C_i$ and $y \in C_j$ with $i < j$. If some C_i is singleton set, then we drop the set bracket and simply write the element. It will sometimes be convenient to treat the empty set, at least in passing, as a chain. Therefore if $C_k = \phi$, then $[C_1, C_2, \ldots, C_n]$ designates $[C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_n]$. In discussing chains on an involution poset **P** we usually omit **0** and **1** since their place in any chain on **P** is fixed. In an involution poset we define $[x_1, \ldots, x_n]' := [x_n', \ldots, x_1']$. An *involutory linear extension of* $\leq$ (or *on* **P**) is a chain $\preceq$ that extends $\leq$, and if $x \preceq y$ then $y' \preceq x'$, where $'$ is the involution on P. Let $\text{ICh}(\mathbf{P}) := \{\preceq \mid \preceq \text{ is an involutory chain on } \mathbf{P}\}$. If $\mathbf{C} := (P, \leq, ')$ is an involutory chain on **P**, then the *base* of **C**, $\beta(\mathbf{C})$, is the chain induced on $\{x \mid x \leq x'\}$ by **C**.

2. Isotropic Elements and Prime Order Ideals in Involution Posets

In this section we discuss isotropic elements and prime order ideals in involution posets. These ideals will be used to generate involutory chains on such posets. We generalize to regular involution posets the well-known result which states that, for any nonzero element in a Boolean algebra there exists a prime order ideal that does not contain this element. Some weaker forms of the corresponding result for involution posets are given.

A non-empty subset I of any poset is an *order ideal* in case $x\!\downarrow\, \subseteq I$ whenever $x \in I$. We say that I is a *prime order ideal* of a bounded involution poset **P** whenever I is an order ideal, and $x \in I$ iff $x' \notin I$. Let $\text{Pr}(\mathbf{P}) := \{I \mid I \text{ is a prime}$

order ideal of $\mathbf{P}$}. An element $x \in \mathbf{P}\backslash\{\mathbf{0}\}$ is *isotropic* if $x \le x'$; and x is a *half-element* if $x = x'$. We define $\mathrm{Iso}(\mathbf{P}) := \{x \in P \mid \mathbf{0} < x \le x'\}$, and $P_{1/2} := \{x \in P \mid x = x'\}$. It is easy to see that distinct half-elements are incomparable and if $P_{1/2} = \phi$ and $\mathbf{C} \in \mathrm{ICh}(\mathbf{P})$ then $\beta(\mathbf{C}) \in \mathrm{Pr}(\mathbf{P})$. An involution poset $\mathbf{P}$ is *regular* if every pair of the isotropic elements in $\mathbf{P}$ are orthogonal. The symbol $|X|$ indicates the cardinality of the set X. Note that if $\mathbf{P}$ is regular, then $|P_{1/2}| \le 1$; hence, for any involutory chain $\mathbf{C}$ on such $\mathbf{P}$, $|\beta(\mathbf{C}) \cap \beta(\mathbf{C})'| \le 1$, and $\varphi(\beta(\mathbf{C})) = \mathrm{Iso}(\mathbf{C})$.

Let $\mathbf{B} := (B, \le, ', \mathbf{0}, \mathbf{1})$ be a Boolean algebra with a possibly empty set of atoms A. For every nonzero element $x \in B$, there exists a prime order ideal I such that $x \notin I$ [10]. Hence we can use the set $\mathrm{Pr}(\mathbf{B})$ to generate bases for a set of involutory linear extentions $\{\preceq_i\}_{i\in I}$ of $\le$ on $\mathbf{B}$ with $\bigcap_{i\in I} \preceq_i = \le$ as follows: For every $I \in \mathrm{Pr}(\mathbf{B})$, let $\le_I$ be a chain on I. Define a chain $\preceq_I$ on $\mathbf{B}$ by $\preceq_I := [\le_I, \le_I']$. Then $\preceq_I$ is an involutory linear extension of $\le$ on $\mathbf{B}$. Since, in $\mathbf{B}$, prime order ideals distinguish the nonzero elements in the following sense: for every nonzero element $x \in B$, there exists $I \in \mathrm{Pr}(\mathbf{B})$ such that $x' \in I$, we see that $x' \preceq_I x$. Since $\mathrm{crit}(\mathbf{B}) = \{(x', x) \mid x \in A\}$, for every $(x', x) \in \mathrm{crit}(\mathbf{B})$ there exists involutory chain, say $\preceq$, such that $x' \preceq x$. Note that $\varphi(\beta(\preceq)) = x'\!\downarrow$. This is the basic construction which we develop in the general case of an involution poset.

Note that if $x \in P_{1/2}$, then there does not exist $I \in \mathrm{Pr}(\mathbf{P})$ with $x \in I$, on the other hand, if $x \in \mathrm{Iso}(\mathbf{P})\backslash P_{1/2}$, then $x \in I$ for every $I \in \mathrm{Pr}(\mathbf{P})$ and $x' \notin \mathrm{Iso}(\mathbf{P})$. In addition, it is easy to see that $\mathbf{P}$ is an orthoposet iff $\mathrm{Iso}(\mathbf{P}) = \phi$. It follows from the definition of $\mathrm{Iso}(\mathbf{P})$ that if $x \in \mathrm{Iso}(\mathbf{P})$, then $x\!\downarrow\!\backslash\{0\} \subseteq \mathrm{Iso}(\mathbf{P})$. Finally, if $y, y' \le z \in P\backslash\{\mathbf{1}\}$, then $z' \in \mathrm{Iso}(\mathbf{P})$. Note that if $P_{1/2} \ne \phi$, then there exist no prime order ideals in $\mathbf{P}$. Hence, in the following theorem, the assumption that $P_{1/2} = \phi$ is necessary.

THEOREM 2.1. *Let* $\mathbf{P}$ *be a bounded involution poset such that* $P_{1/2} = \phi$ *and let* $a, b \in P$. *Then* $a \not\le b$ *and* $a, b' \notin \mathrm{Iso}(\mathbf{P})$ *iff there exists* $I \in \mathrm{Pr}(\mathbf{P})$ *such that* $a', b \in I$.

Proof. Let $a, b' \in P\backslash \mathrm{Iso}(\mathbf{P})$, $a \not\le b$. Define an order ideal $I_0 := \cup\{x\!\downarrow \mid x \in \mathrm{Iso}(\mathbf{P}) \cup \{a', b\}\}$. Since $\mathbf{1} \notin I_0$, it follows that I_0 is a proper order ideal of $\mathbf{P}$. Define

$$\mathcal{J} := \big\{J \mid J \text{ is a proper order ideal of } \mathbf{P} \text{ containing } \mathrm{Iso}(\mathbf{P}) \cup \{a', b\} \text{ and } \{z, z'\} \not\subseteq J \text{ for all } z \in P\big\}.$$

Then it is easy to see that $I_0 \in \mathcal{J}$ and, using Zorn's Lemma, that $\mathcal{J}$ has a maximal element, say I. Now we show that $I \in \mathrm{Pr}(\mathbf{P})$. Since $I \in \mathcal{J}$, we need only show that for every $z \in P$, either z or $z' \in I$. Suppose that there exists $z \in P$ such that $z, z' \notin I$. Define $I_1 := z\!\downarrow \cup I$. Since $\mathrm{Iso}(\mathbf{P}) \subseteq J$ for every $J \in \mathcal{J}$, it follows $z, z' \notin \mathrm{Iso}(\mathbf{P})$; it is therefore not difficult to show that $I_1 \in \mathcal{J}$. Thus $z \in I_1\backslash I$ contradicting the maximality of I. Since $\{z, z'\} \not\subseteq I$, either $z \in I$ or $z' \in I$ but not both. Hence $I \in \mathrm{Pr}(\mathbf{P})$ with $a', b \in I$.

Conversely, if $I \in \mathrm{Pr}(\mathbf{P})$ with $a', b \in I$ and, then $a \not\le b$. If $a \in \mathrm{Iso}(\mathbf{P})$, then $a \le a' \in I$ so that $a, a' \in I$, which is a contradiction. Thus $a \notin \mathrm{Iso}(\mathbf{P})$. Similarly $b' \notin \mathrm{Iso}(\mathbf{P})$. □

The following corollary is now immediate and the observation follows from the definitions.

COROLLARY 2.2. *If* $P_{1/2} = \phi$, $x \in P \backslash \{\text{Iso}(\mathbf{P}) \cup \{\mathbf{0}\}\}$, *then there exists* $I \in \text{Pr}(\mathbf{P})$ *with* $x \notin I$ *and therefore* $\text{Pr}(\mathbf{P}) \neq \phi$.

OBSERVATION 2.3. If $P_{1/2} = \phi$, then $\text{Iso}(\mathbf{P}) \subseteq \bigcap\{I \mid I \in \text{Pr}(\mathbf{P})\}$.

3. Involutory Linear Extensions and Involutory Realizers

In this section, we define and study involutory cycle-free sets, involutory realizers and the involutory dimension of involution posets. Two of the most important results in this section are Theorem 3.3, which insures the existence of involutory linear extensions on a regular involution poset, and Corollary 3.6, which insures the existence of the involutory dimension of a regular involution poset. A characterization of the involutory dimension of finite regular involution posets that are not chains is given. We show that the involutory dimension of a finite orthoposet does not exceed its width. We also compare the $\text{idim}(\mathbf{P})$ to $\text{idim}(\mathbf{Q})$, where $Q = P \backslash \{x, x'\}$ for a non-isotropic atom x of P.

An *alternating cycle* of length k in $\mathbf{P}$ is a sequence (x_i, y_i), $1 \leq i \leq k$, of ordered pairs of $\text{inc}(\mathbf{P})$ such that $y_i \leq x_{i+1}$ (mod k) in $\mathbf{P}$. A subset $S \subseteq \text{inc}(\mathbf{P})$ is *cycle-free* if it has no alternating cycle. An *involutory cycle-free set* is a cycle-free set S such that if $(x, y) \in S$ then $(y', x') \in S$. It is not difficult to show that if $\mathbf{P}$ is not a chain and $\preceq \in \text{ICh}(\mathbf{P})$, then $\preceq \cap \text{inc}(\mathbf{P})$ is an involutory cycle-free set.

LEMMA 3.1. *Let* $S \subseteq \text{inc}(\mathbf{P})$ *and* $\preceq := \text{tr}(\leq \cup S)$. *If S is an involutory cycle-free set, then $\preceq$ is a partial order on* $\mathbf{P}$ *such that $x \preceq y$ implies $y' \preceq x'$ for every $x, y \in P$.*

Proof. Let $S \subseteq \text{inc}(\mathbf{P})$ be an involutory cycle-free set. Clearly $\preceq$ is a partial order on $\mathbf{P}$. If $x \preceq y$, then there exists a sequence $x_0, x_1, \ldots, x_n$ with $x = x_0 R_1 x_1 R_2 \cdots R_n x_n = y$, where $R_i \in \{\leq, S\}$, $i \in \{1, 2, \ldots, n\}$. If $R_i = \leq$, then $x_i \leq x_{i+1}$ so $x'_{i+1} \leq x'_i$. If $R_i = S$, then $(x_i, x_{i+1}) \in S$ so $(x'_{i+1}, x'_i) \in S$ because S is an involutory cycle-free set. Consequently, $y' = x'_n R_n x'_{n-1} R_{n-1} \cdots R_2 x'_1 R_1 x'_0 = x'$. So $y' \preceq x'$. □

The converse of Lemma 3.1 does not hold. A counter example can be found in an involutory subposet of the eight-element Boolean algebra $\mathbf{2^3}$ by deleting the comparabilities (a, b') and (b, a') from the usual ordering, where a and b are distinct atoms of $\mathbf{2^3}$ [1].

LEMMA 3.2. *Let* $P_{1/2} = \phi$, *and* $I \in \text{Pr}(\mathbf{P})$. *Then there exists* $\preceq \in \text{ICh}(\mathbf{P})$ *such that $b \preceq a$ whenever one of the following holds:*

(i) $a, b \in I$ *with* $a \nleq b$.

(ii) $a', b \in I$ *with* $a' \neq b$.

Proof. If $a, b \in I$ with $a \not\leq b$ then choose a chain $\preceq_I$ on I with $b \preceq_I a$. Define a chain $\preceq$ on $\mathbf{P}$ by $\preceq := [\preceq_I, \preceq_I']$. Since $I \in \mathrm{Pr}(\mathbf{P})$, and $P_{1/2} = \phi$, $P = I \cup I'$ so that $\preceq \in \mathrm{ICh}(\mathbf{P})$ and $b \preceq a$.

Next, if $a', b \in I$ with $a' \neq b$ then $a \notin I$. Let $\preceq_I$ be a chain on I, and define a chain $\preceq$ on $\mathbf{P}$ by $\preceq := [\preceq_I, \preceq_I']$. Since $I \in \mathrm{Pr}(\mathbf{P})$ and $P = I \cup I'$, it follows that $\preceq \in \mathrm{ICh}(\mathbf{P})$. Moreover, $a \in \varphi(\beta(\preceq)')$ since $a \notin I$. Hence $b \preceq a$. □

Let $\phi \neq Q \subseteq P$. Then the involution subposet of P determined by Q is $\mathbf{Q} := (Q, \leq_Q, {'^Q})$, $\leq_Q := (Q \times Q) \cap \leq$ and ${'^Q}$ is the restriction of $'$ to Q. If $\mathbf{Q}$ is linearly ordered, we call it a *chain in* $\mathbf{P}$. Note that every involution subposet of a regular involution poset is also regular, and every involutory chain is a regular involution poset.

THEOREM 3.3. *Let* $\mathbf{P}$ *be a bounded regular involution poset, and let* $a, b \in P$ *with* $a \not\leq b$. *Then there exists an involutory linear extension* $\preceq$ *on* $\mathbf{P}$ *such that* $b \preceq a$.

Proof. Suppose that $P_{1/2} = \phi$. Hence by Corollary 2.2 $\mathrm{Pr}(\mathbf{P}) \neq \phi$. Let $a, b \in P$ with $a \not\leq b$. Note that, since $P_{1/2} = \phi$, if $x \in \mathrm{Iso}(\mathbf{P})$ then $x' \notin \mathrm{Iso}(\mathbf{P})$. Now suppose that $a = \mathbf{1}$. Then $a \in P \setminus \mathrm{Iso}(\mathbf{P})$. By Corollary 2.2, there exists $J \in \mathrm{Pr}(\mathbf{P})$ with $b' \in J$ or $b \in J$. If $b' \in J$, then applying Lemma 3.2(i) to a' and b', the result follows. If $b \in J$ then the result follows by Lemma 3.2(ii). A similar argument yields the result when $b = \mathbf{0}$. Thus we may assume that neither $a = \mathbf{1}$ nor $b = \mathbf{0}$. Suppose $b \in \mathrm{Iso}(\mathbf{P})$; then there exists $I \in \mathrm{Pr}(\mathbf{P})$ such that either $a, b \in I$ if $a \in \mathrm{Iso}(\mathbf{P})$ or $a', b \in I$ if $a \notin \mathrm{Iso}(\mathbf{P})$; in either case, by Lemma 3.2, there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ with $b \preceq a$. Thus we may assume that $b \notin \mathrm{Iso}(\mathbf{P})$. Then either $b' \in \mathrm{Iso}(\mathbf{P})$ or $b' \notin \mathrm{Iso}(\mathbf{P})$. Suppose that $b' \in \mathrm{Iso}(\mathbf{P})$. We may assume that $a \notin \mathrm{Iso}(\mathbf{P})$ or else $a \leq b$ by regularity of $\mathbf{P}$. Then there exists $I \in \mathrm{Pr}(\mathbf{P})$ with $b' \in I$ and $a \notin I$. So $a', b' \in I$ with $b' \not\leq a'$. Using Lemma 3.2 (with $b' = a$), it follows that there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ such that $a' \preceq b'$, and hence $b \preceq a$. Now, suppose $b' \notin \mathrm{Iso}(\mathbf{P})$. If $a \notin \mathrm{Iso}(\mathbf{P})$, then by Theorem 2.1, there exists $K \in \mathrm{Pr}(\mathbf{P})$ with $a \notin K$ and $b \in K$. Hence $a', b \in K$. By Lemma 3.2, there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ such that $b \preceq a$. Finally, if $a \in \mathrm{Iso}(\mathbf{P})$, then $a \in I$ for every $I \in \mathrm{Pr}(\mathbf{P})$, and there exists $J \in \mathrm{Pr}(\mathbf{P})$ with $b' \in J$. Thus $a, b \in J$ with $a \not\leq b$. By Lemma 3.2, there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ with $b \preceq a$.

Now suppose that $P_{1/2} \neq \phi$. Since $\mathbf{P}$ is regular, $|P_{1/2}| = 1$. Let $P_{1/2} = \{x\}$, and let $Q = P \setminus \{x\}$. Then $\mathbf{Q} := (Q, \leq_Q, {'^Q})$ is regular involution poset and $Q_{1/2} = \phi$. Then by the above argument if $a, b \in Q$ with $a \not\leq b$, then there exists $\preceq_Q \in \mathrm{ICh}(\mathbf{Q})$ with $b \preceq_Q a$. Now suppose that $a, b \in P$ with $a \not\leq b$. If $x \notin \{a, b\}$, then $a, b \in Q$ and there exists $\preceq_Q \in \mathrm{ICh}(\mathbf{Q})$ with $b \preceq_Q a$. Define $\preceq$ on P by $\preceq := [\beta(\preceq_Q), x, \beta(\preceq_Q)']$. Then $\preceq \in \mathrm{ICh}(\mathbf{P})$ and $b \preceq a$. So suppose that $a = x$. If $b \in \mathrm{Iso}(\mathbf{P})$, then $b \in I$ for every $I \in \mathrm{Pr}(\mathbf{Q})$. Hence $b \in \beta(\preceq_Q)$ for every $\preceq_Q \in \mathrm{ICh}(\mathbf{Q})$. Extend $\preceq_Q$ to an involutory linear extension $\preceq$ on $\mathbf{P}$ as follows: $\preceq := [\beta(\preceq_Q), a, \beta(\preceq_Q')]$. Then $b \preceq a$. Now suppose that $b \notin \mathrm{Iso}(\mathbf{P})$, then we may assume $b' \notin \mathrm{Iso}(\mathbf{P})$ or else $a \leq b$ by regularity of $\mathbf{P}$. Hence there exists $I \in \mathrm{Pr}(\mathbf{P})$ with $b \in I$. Let $\preceq_I$ be a chain on I, and define $\preceq \in \mathrm{ICh}(\mathbf{P})$ by $\preceq := [\preceq_I, a, \preceq_I']$. It follows that $b \preceq a$. Finally, suppose $b = x$. We may assume that $a \notin \mathrm{Iso}(\mathbf{P})$ or else $a \leq b$ by regularity of $\mathbf{P}$. Then there exists $I \in \mathrm{Pr}(\mathbf{P})$ with $a' \in I$. As in the previous case, we define $\preceq \in \mathrm{ICh}(\mathbf{P})$ with $a' \preceq b'$. Hence $b \preceq a$. Therefore, in all cases, there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ such that $b \preceq a$. □

COROLLARY 3.4. *Let* $\mathbf{P}$ *be a bounded involution poset. Then* $\mathbf{P}$ *is regular iff for all* $a, b \in \mathbf{P}$ *with* $a \nleq b$ *there exists* $\preceq \in \mathrm{ICh}(\mathbf{P})$ *such that* $b \preceq a$.

Proof. If $\mathbf{P}$ is regular then the result follows from Theorem 3.3. For the converse, let $x, y \in \mathrm{Iso}(\mathbf{P})$. We need to show that $x \leq y'$. Assume to the contrary that $x \nleq y'$. By hypothesis, there exists $\preceq \in \mathrm{ICh}(\mathbf{P})$ with $y' \preceq x$. Since $x, y \in \mathrm{Iso}(\mathbf{P})$, it follows $x, y \in \varphi(\beta(\preceq^*))$ for every $\preceq^* \in \mathrm{ICh}(\mathbf{P})$. Thus $x', y' \in \varphi(\beta(\preceq^*)')$ for every $\preceq^* \in \mathrm{ICh}(\mathbf{P})$. Hence $x \preceq^* y'$ for every $\preceq^* \in \mathrm{ICh}(\mathbf{P})$ contradicting $y' \preceq x$ and $x \neq y'$. So $x \leq y'$, and therefore $\mathbf{P}$ is regular. □

Let $\mathbf{P} := (P, \leq)$ be a poset, and let $\mathbf{R} := \{\preceq_\alpha\}_{\alpha \in J}$, where $\preceq_\alpha \in \mathrm{Ch}(\mathbf{P})$ for every $\alpha \in J$. Then $\mathbf{R}$ is called a *realizer* of $\leq$ whenever $\bigcap_{\alpha \in J} \preceq_\alpha = \leq$. Let

$$\mathfrak{R} := \{\mathbf{R} \mid \mathbf{R} \text{ is a realizer of } \leq \text{ of minimal cardinality}\}.$$

If $\mathbf{P} := (P, \leq, ')$ is an involution poset then we define an involutory realizer of $\leq$ to be a realizer of $\leq$ in which each chain is an involutory chain on $\mathbf{P}$. Let

$$\mathfrak{R}_I := \{\mathbf{R} \mid \mathbf{R} \text{ is an involutory realizer of } \leq \text{ of minimal cardinality}\}.$$

The *order dimension* or simply *dimension* of a poset $\mathbf{P} := (P, \leq)$, denoted $\dim(\mathbf{P})$, is defined in [3] by $\dim(\mathbf{P}) := |\mathbf{R}|$ for any $\mathbf{R} \in \mathfrak{R}$. We define the *involutory dimension* of an involution poset $\mathbf{P} := (P, \leq, ')$, denoted $\mathrm{idim}(\mathbf{P})$, by $\mathrm{idim}(\mathbf{P}) := |\mathbf{R}|$ for any $\mathbf{R} \in \mathfrak{R}_I$.

The proof of the following lemma follows that of [13] and hence is omitted.

LEMMA 3.5. *Let* $\mathbf{P}$ *be a regular involution poset, and let* $\{\preceq_\alpha\}_{\alpha \in I}$ *be a family of linear extensions of* $\leq$. *Then the following are equivalent.*

(i) $\{\preceq_\alpha\}_{\alpha \in I}$ *is an involutory realizer of* $\leq$.
(ii) *For every* $a, b \in P$ *with* $a \| b$ *there exists* $\alpha \in I$ *such that* $a \preceq_\alpha b$.
(iii) $\mathrm{inc}(\mathbf{P}) \subseteq \bigcup_{\alpha \in I} \preceq_\alpha$.

COROLLARY 3.6. *The involutory dimension of* $\mathbf{P}$ *exists iff* $\mathbf{P}$ *is a regular involution poset.*

Proof. Let $\mathbf{P} := (P, \leq, ')$ be an involution poset and suppose that $\mathrm{idim}(\mathbf{P})$ exists. Then there exists an involutory realizer $\{\preceq_\alpha\}_{\alpha \in I} \in \mathfrak{R}_I$. If $\mathbf{P}$ is not regular, then there exists $a, b \in \mathrm{Iso}(\mathbf{P})$ such that $a \nleq b'$. Since $\{\preceq_\alpha\}_{\alpha \in I} \in \mathfrak{R}_I$, there exists $\beta \in I$ such that $b' \preceq_\beta a$. Now $a, b \in \mathrm{Iso}(\mathbf{P})$ and $\preceq_\beta$ is an extension of $\leq$ implies that

$b \preceq_\beta b' \preceq_\beta a \preceq_\beta a'$. Hence $b' \preceq a$ and $a \preceq b'$, so $a = b'$ contradicting $a \not\leq b'$. Hence **P** is regular.

Conversely, if **P** is a regular involution poset, then by Corollary 3.4, for every $a, b \in P$, $a \not\leq b$ there exists $\preceq_{(a,b)} \in \mathrm{ICh}(\mathbf{P})$ with $b \preceq_{(a,b)} a$. Since

$$\bigcap \{\preceq_{(a,b)} \mid (a, b) \in (P \times P) \setminus \leq\} = \leq,$$

it follows $\{\preceq_{(a,b)} \mid (a, b) \in (P \times P) \setminus \leq\}$ is an involutory realizer of $\leq$. So $\mathrm{idim}(\mathbf{P})$ exists. □

Let **P** be a regular involution poset, and $\mathbf{Q} := (Q, \leq_Q, '^Q)$ be an involution subposet of **P**. If $\preceq \in \mathrm{ICh}(\mathbf{Q})$ then, using the fact that $\preceq \cap \mathrm{inc}(\mathbf{P})$ is an involutory cycle-free set, there exists $\preceq^* \in \mathrm{ICh}(\mathbf{P})$ with $\preceq^* |_Q = \preceq$; this is called the *interpolation property* for regular involution posets. Since $\mathrm{ICh}(\mathbf{P}) \subseteq \mathrm{Ch}(\mathbf{P})$, $\dim(\mathbf{P}) \leq \mathrm{idim}(\mathbf{P})$ and $\mathrm{idim}(\mathbf{Q}) \leq \mathrm{idim}(\mathbf{P})$.

The *product* of a family of regular involution posets is the Cartesian product of the underlying sets with the ordering, involution and orthogonality are defined componentwise. If there is no confusion, write $\leq$ and $'$ for $\leq_\mathbf{P}$ and $'^P$, respectively. It is easily seen that the product of regular involution posets is again a regular involution poset.

The proofs of the next three results, given in [1], follow the corresponding proofs for order dimension given in [13] and hence are omitted.

THEOREM 3.7 (A characterization of the involutory dimension). *The involutory dimension of a regular involution poset* $\mathbf{P} := (P, \leq, ')$ *is the minimum cardinality of a family* $\{(C_i, \leq_i, ') \mid i \in I\}$ *of involutory chains such that* **P** *is isomorphic to a subposet of the direct product* $\prod_{i\in I}(C_i, \leq_i, ')$.

COROLLARY 3.8. *If* **P** *and* **Q** *are regular involution posets, then*

$$\mathrm{idim}(\mathbf{P} \times \mathbf{Q}) = \mathrm{idim}(\mathbf{P}) + \mathrm{idim}(\mathbf{Q}).$$

THEOREM 3.9. *Let* **P** *be a finite regular involution poset that is not a chain, and let* $\mathrm{idim}(\mathbf{P}) = n \in \mathbf{N}$. *Then the following are equivalent.*

(i) $n = |\mathbf{R}|$ *for any* $\mathbf{R} \in \mathfrak{R}_I$.

(ii) $n = \min |\{\{S_i\}_{i\in I}$: *for every* $i \in I$, S_i *is an involutory cycle-free set such that* $\mathrm{crit}(\mathbf{P}) \subseteq \bigcup_{i\in I} S_i\}|$.

(iii) $n = \min |\{\{\leq_i\}_{i\in I}$: *for every* $i \in I$, $\leq_i$ *is a partial extension of* $\leq$ *such that if* $x \leq_i y$ *then* $y' \leq_i x'$ *and* $\mathrm{crit}(\mathbf{P}) \subseteq \bigcup_{i\in I} \leq_i\}|$.

(iv) $n = \min |\{\{\preceq_i\}_{i\in I}$: *for every* $i \in I$, $\preceq_i \in \mathrm{ICh}(\mathbf{P})$ *and* $\mathrm{crit}(\mathbf{P}) \subseteq \bigcup_{i\in I} \preceq_i\}|$.

THEOREM 3.10. *Let* **P** *be a bounded regular involution poset, and let* $C \subset P$ *be a chain with* $C \cap (\{\mathbf{0}, \mathbf{1}\} \cup \mathrm{Iso}(\mathbf{P}) \cup \mathrm{Iso}(\mathbf{P})') = \phi$. *If* $Y := \{y \in P : y \| C\}$, *then there exist* $\preceq_1, \preceq_2 \in \mathrm{ICh}(\mathbf{P})$ *such that* $C \preceq_1 y$ *and* $y \preceq_2 C$ *for every* $y \in Y$.

Proof. Suppose that $P_{1/2} = \phi$. Let $I_0 := \bigcup_{x\in C} x\downarrow \cup \text{Iso}(\mathbf{P})$. Then it is easily seen that I_0 is a proper order ideal of $\mathbf{P}$ not containing any set of the form $\{x, x'\}$. Let

$$\mathcal{J} := \{J \mid J \text{ is a proper order ideal of } \mathbf{P} \text{ containing } C \text{ such that } \{x, x'\} \nsubseteq J \text{ for all } x \in P\}.$$

Since, $P_{1/2} = \phi$, $I_0 \in \mathcal{J}$ and, as in the proof of Theorem 2.1, there exists $I \in \text{Pr}(\mathbf{P})$ with $I_0 \subseteq I$. Let $K := I\backslash I_0$. We may assume that $K \neq \phi$. Note that, since I_0 is an ideal of $\mathbf{P}$, for every $x \in I_0$ there does not exists $y \in K$ with $y \leq x$. Hence either (i) there exist some $x \in I_0\backslash\{\mathbf{0}\}$ with $x < z$ for some $z \in K$ or (ii) $x \| K$ for every $x \in I_0\backslash\{\mathbf{0}\}$. In either case we can choose linear extensions $\preceq_{C\downarrow}$, $\preceq_{\text{Iso}(\mathbf{P})\backslash C\downarrow}$, and $\preceq_K$ restricted to $C\downarrow$, $\text{Iso}(\mathbf{P})\backslash C\downarrow$, and K, respectively, with $[\preceq_{C\downarrow}, \preceq_{\text{Iso}(\mathbf{P})\backslash C\downarrow}, \preceq_K]$ a chain on I. Define an involutory linear extension $\preceq_1$ on $\mathbf{P}$ by $\beta(\preceq_1) := [\preceq_{C\downarrow}, \preceq_{\text{Iso}(\mathbf{P})\backslash C\downarrow}, \preceq_K]$. Since I_0 is an ideal of $\mathbf{P}$, $C \preceq_1 y$ for every $y \in Y$. Since $\mathbf{P}$ is an involution poset, C' is a chain in $\mathbf{P}$ with $C' \cap (\{\mathbf{0}, \mathbf{1}\} \cup \text{Iso}(\mathbf{P}) \cup \text{Iso}(\mathbf{P})') = \phi$. Also $y\|C$ iff $y'\|C'$. By the above argument, there exists an involutory linear extension $\preceq_2$ on $\mathbf{P}$ with $C' \preceq_2 y'$ Hence $y \preceq_2 C$ for every $y \in Y$, and we are done in this case.

Next, suppose that $P_{1/2} \neq \phi$. Since $\mathbf{P}$ is regular, $|P_{1/2}| = 1$. Let $P_{1/2} = \{x\}$ and define $\mathbf{Q} := (Q, \leq_Q, {}'^Q)$, where $Q := P\backslash\{x\}$, and $\leq_Q$ and ${}'^Q$ are the restrictions of $\leq$ and $'$ to Q, respectively. As in the previous case there exits $\preceq_1^*, \preceq_2^* \in \text{ICh}(\mathbf{Q})$ such that $C \preceq_1^* y$ and $y \preceq_2^* C$ for every $y \in Q$ with $y\|C$. Let $\preceq_1 := [\beta(\preceq_1^*), x, \beta(\preceq_1^*)']$ and $\preceq_2 := [\beta(\preceq_2^*), x, \beta(\preceq_2^*)']$. Then $\preceq_1, \preceq_2 \in \text{ICh}(\mathbf{P})$ with $C \preceq_1 y$ and $y \preceq_2 C$ for every $y \in P$ with $y\|C$. □

Since orthoposets contain no isotropic elements and are therefore regular, the following corollary is immediate.

COROLLARY 3.11. *Let* $\mathbf{P}$ *be an orthoposet, and let* $C \subseteq P\backslash\{\mathbf{0}, \mathbf{1}\}$ *be a chain in* $\mathbf{P}$. *Then there exist* $\preceq_1, \preceq_2 \in \text{ICh}(\mathbf{P})$ *such that* $C \preceq_1 y$ *and* $y \preceq_2 C$ *for every* $y\|C$.

An ordered set $S \subseteq \mathbf{P}$ is an *antichain* if $x \leq y$ in S only if $x = y$. The set of all chains on $\mathbf{P}$ is partially ordered by set theoretic inclusion and the maximal elements in this set are called *maximal chains*. *Maximal antichains* are defined analogously. The *width* of a poset $\mathbf{Q} := (Q, \leq)$ is the supremum of the cardinalities of all antichains in $\mathbf{P}$. Dilworth proved that if $\mathbf{Q}$ is a poset of width n, then there exists a partition $Q = \bigcup_{i=1}^n C_i$, where each C_i is a chain in Q, see [13]. This is known as *Dilworth's Chain Covering Theorem*.

THEOREM 3.12. *Let* $\mathbf{P} := (P, \leq, ')$ *be a finite orthoposet. Then* $\text{idim}(\mathbf{P}) \leq \text{width}(\mathbf{P})$.

Proof. We may assume that $\mathbf{P}$ is not a chain, otherwise $\mathrm{idim}(\mathbf{P}) = 1 = \mathrm{width}(\mathbf{P})$. In particular, $|\mathbf{P}| > 2$. $\mathbf{P}$ induces an involutory ordering on $P^* := P\backslash\{\mathbf{0}, \mathbf{1}\}$. Since $\mathbf{P}$ is finite, by Dilworth's Chain Covering Theorem, there exists a partition $\mathbf{P}^* = \bigcup_{i=1}^{n} C_i$, where each $C_i := [c_{i_1}, c_{i_2}, \ldots, c_{i_{s_i}}]$ is a chain in $\mathbf{P}^*$. Let $D_{i_k} := \{y \leq c_{i_k} \mid y \not\leq c_{i_{k-1}}\}$, and let $\preceq_{D_{i_k}}$ be a linear extension of $\leq$ restricted to D_{i_k}. Then $\preceq_{C_i} := [\preceq_{D_{i_1}}, \preceq_{D_{i_2}}, \ldots, \preceq_{D_{i_{s_i}}}]$ is a chain on $c_{i_{s_i}}\!\downarrow := C_i\!\downarrow$. Let

$$\mathcal{J} := \big\{J \mid J \text{ is a proper order ideal of } \mathbf{P} \text{ containing } C_i \text{ such that } \{x, x'\} \not\subseteq J \text{ for all } x \in P\big\}.$$

Since $\mathbf{P}$ is an orthoposet, $\{x, x'\} \not\subseteq C_i$ for any i. Thus $C_i\!\downarrow \in \mathcal{J}$ and, as in the proof of Theorem 2.1, maximal elements of $\mathcal{J}$ are prime order ideals and there exists $I \in \mathrm{Pr}(\mathbf{P})$ with $C_i\!\downarrow \subseteq I$. Let $K := I\backslash C_i\!\downarrow$. We may assume that $K \neq \phi$. Note that, there does not exist $y \in K$ with $y \leq x$ for any $x \in C_i\!\downarrow$. Hence $[\preceq_{C_i\downarrow}, \preceq_K]$, with $\preceq_K$ a linear extension of $\leq$ restricted to K, is a chain on I. Now extend $[\preceq_{C_i\downarrow}, \preceq_K]$ to an involutory linear extension $\preceq_i$ on $\mathbf{P}$ by $\beta(\preceq_i) := [\preceq_{C_i\downarrow}, \preceq_K]$. We claim that $\{\preceq_i\}_{i=1}^{n}$ is an involutory realizer of $\leq$. To see this, it suffices to show that for every $(x, y) \in \mathrm{inc}(\mathbf{P})$, there exists $j \in \{1, 2, \ldots, n\}$ with $x \preceq_j y$. The desired value of j is determined by choosing a chain C_j for which $x \in C_j$. □

We conjecture that, for regular involution posets $\mathbf{P}$, $\mathrm{idim}(\mathbf{P}) \leq |\mathbf{P}|/2$ and $\mathrm{idim}(\mathbf{P}) \leq \mathrm{width}(\mathbf{P})$.

An element $x \in P$ is called *irreducible* if, for any $S \subseteq \mathbf{P}$, $x = \bigvee S$ or $x = \bigwedge S$ implies $x \in S$. Denote the irreducible elements of $\mathbf{P}$ by $\mathrm{irred}(\mathbf{P})$. For $n \geq 3$, let $S_n := \mathrm{irred}(\mathbf{2^n})$. With the ordering induced from $\mathbf{2^n}$, $\mathbf{S_n}$ is a poset with $|S_n| = 2n$ and $\dim(\mathbf{S_n}) = n$, it is called the *standard* (*involution*) *poset of dimension* n. It is not difficult to show that $\mathrm{idim}(\mathbf{S_n}) = \dim(\mathbf{S_n}) = n$. Clearly, $\mathrm{crit}(\mathbf{S_n}) = \{(a', a) \mid a \in \mathrm{atom}(\mathbf{2^n})\}$. Hence if $(u', u) \in \mathrm{crit}(\mathbf{2^n})$ and $S^\circ := 2^n\backslash\{u', u\}$, then $\mathrm{irred}(\mathbf{S}^\circ) = S_{n-1}$. So $|\mathrm{crit}(\mathbf{S}^\circ)| = |\mathrm{crit}(\mathbf{S_{n-1}})|$. By Theorem 3.9, $\mathrm{idim}(\mathbf{S}^\circ) = \mathrm{idim}(\mathbf{S_{n-1}}) = n-1$. Hence removing a critical pair from the Boolean algebra $\mathbf{2^n}$ decreases the involutory dimension of the remaining involutory subposet by one. We conclude this section by showing that removing a non-isotropic atom and its prime from a regular involution poset decreases the involutory dimension of the remaining involution subposet by at most one.

THEOREM 3.13. *Let* $\mathbf{P}$ *be a bounded regular involution poset with* $|P| > 2$. *If* $x \in P\backslash\mathrm{Iso}(\mathbf{P})$ *is an atom,* $Q := P\backslash\{x, x'\}$, *and* $\mathbf{Q} := (Q, \leq_Q, {}'^Q)$, *then* $\mathrm{idim}(\mathbf{P}) \leq 1 + \mathrm{idim}(\mathbf{Q})$.

Proof. We may work with $Q^* := Q\backslash\{\mathbf{0}, \mathbf{1}\}$. Suppose that $\mathrm{idim}(\mathbf{Q}^*) = t$. Let $\{\preceq_{Q_i^*}\}_{i\in I}$ be an involutory realizer of $\leq_Q$ of cardinality t. For each $i \in I$, let $\preceq_i := [x, \preceq_{Q_i}, x']$. Define $Z := (\mathbf{0}, x') := \{z \in P \mid \mathbf{0} < z < x'\}$ and $Y := \{y : y\|x, y\|x'\}$. We may assume that Z and Y are not both empty since, if they were, $\mathbf{P}$ would be the 4-element Boolean algebra and the inequality would trivially obtain. Observe that Z and Y are disjoint, also $y \in Y$ iff $y' \in Y$, and that $Q^* = Z \cup Z' \cup Y$.

For $i \in \{1, 2\}$, let $\leq_Z$ and $\leq_Y$, be chains on Z and Y, respectively, such that $\leq_Y$ is involutory. Since $x \notin \mathrm{Iso}(\mathbf{P})$, we define $\preceq_o$ by $\preceq_o := [[\leq_Z, x', \leq_Y], x, \leq'_Z]$ (we may assume $o \notin I$). Then $\preceq_o$ is an involutory linear extension of $\leq$. We claim that $\{\preceq_i\}_{i \in I \cup \{o\}}$ is an involutory realizer of $\leq$ of cardinality $t + 1$. To see this, let $(u, v) \in \mathrm{inc}(\mathbf{P})$. We need to show that there exist some $k \in I \cup \{o\}$ such that $u \preceq_k v$. Now we have the following cases: $u = x$, $v = x$, $u = x'$, $v = x'$ or $u, v \notin \{x, x'\}$. Now we analyze each case: If $u = x$, then $u \preceq_k v$ for all $k \in I$. If $v = x$, then $u \notin Z' := (x, \mathbf{1})$, and so $u \preceq_o v$. If $u = x'$, then $v \notin Z$, and it follows that $u \preceq_o v$. If $v = x'$, then $u \preceq_k v$ for all $k \in I$. Finally, if $u, v \notin \{x, x'\}$, then $u \preceq_k v$ for some $k \in I$ since $\{\preceq_{Q_i^*}\}_{i \in I}$ is an involutory realizer of $\leq_Q^*$. Therefore $\{\preceq_i\}_{i \in I \cup \{o\}}$ is an involutory realizer of $\leq$ of cardinality $t + 1$. This proves the claim and completes the proof of the theorem. □

4. The Involutory Dimension of Some Atomic Amalgams

In this section, we define the horizontal sum of bounded involution posets and an amalgamation of Boolean algebras. We calculate the involutory dimension of the horizontal sum of a finite family of bounded involution posets (if it exists), and the involutory dimension of some atomic amalgams. We also show that the involutory dimension might not equal the order dimension.

We say that $\mathbf{H}$ is the *horizontal sum* of the bounded involution posets $\mathbf{P}_\alpha$, $\alpha \in I$, with $|I| \geq 2$ and $|\mathbf{P}_\alpha| > 2$ for all $\alpha \in I$, denoted $\mathbf{H} := \bigcirc_{\alpha \in I} \mathbf{P}_\alpha$, whenever $H = \bigcup_\alpha P_\alpha$, the ordering and involution on H are induced by those from the $\mathbf{P}_\alpha$'s, and $P_\alpha \cap P_\beta = \{\mathbf{0}, \mathbf{1}\}$ for every $\alpha, \beta \in I$ with $\alpha \neq \beta$.

We now calculate the involutory dimension of a regular horizontal sum of a family of bounded involution posets.

THEOREM 4.1. *For $\alpha \in I$, let $\mathbf{P}_\alpha := (P_\alpha, \leq_\alpha, '^\alpha)$ be a bounded regular involution poset, not a chain, with $|P_\alpha| > 2$ and let $\mathbf{H} = \bigcirc_{\alpha \in I} \mathbf{P}_\alpha$ be the horizontal sum of $\mathbf{P}_\alpha$. Then*

(i) $\mathbf{H}$ *is regular iff* $|\{\alpha\colon \mathrm{Iso}(\mathbf{P}_\alpha) \neq \phi\}| \leq 1$,

(ii) *if* $\mathbf{H}$ *is regular, then* $\mathrm{idim}(\mathbf{H}) = \sup \mathrm{idim}(\mathbf{P}_\alpha)$.

Proof. (i) Suppose that $\mathbf{H}$ is regular but $|\{\alpha\colon \mathrm{Iso}(\mathbf{P}_\alpha) \neq \phi\}| > 1$. Then there exists $x \in \mathrm{Iso}(\mathbf{P}_\alpha)$, $y \in \mathrm{Iso}(\mathbf{P}_\beta)$, with $\alpha \neq \beta$. Since x and y are nonzero elements of different summands, $x \not\leq y$ which contradicts the assumption that $\mathbf{H}$ is regular. Therefore it follows $|\{\alpha\colon \mathrm{Iso}(\mathbf{P}_\alpha) \neq \phi\}| \leq 1$. We now prove the converse. If $|\{\alpha\colon \mathrm{Iso}(\mathbf{P}_\alpha) \neq \phi\}| < 1$, then $\mathrm{Iso}(\mathbf{H}) = \phi$ and so $\mathbf{H}$ is regular; if $|\{\alpha\colon \mathrm{Iso}(\mathbf{P}_\alpha) \neq \phi\}| = 1$, then there exists a unique $\beta \in I$ with $\mathrm{Iso}(\mathbf{P}_\beta) \neq \phi$. Thus $\mathrm{Iso}(\mathbf{H}) = \mathrm{Iso}(\mathbf{P}_\beta)$ and $\mathbf{H}$ is regular.

(ii) Let $\mathbf{H} = \bigcirc_{\alpha \in I} \mathbf{P}_\alpha$ be the horizontal sum of $\mathbf{P}_\alpha$, $\alpha \in I$, where $(I, \ll)$ is a linear ordering with a bottom element θ. Suppose $\mathrm{idim}(\mathbf{P}_\alpha) = |J_\alpha|$, and pick a J with $\sup |J_\alpha| = |J|$. We may assume that $J_\alpha \subseteq J$. Let $\Re_\alpha := \{\preceq_{\alpha,\delta} \mid \delta \in J_\alpha\}$ be an involutory realizer of $\leq_\alpha$. Delete the 0's from $\beta(\leq_{\alpha,\delta})$ for all $\delta \in J_\alpha$ with $\delta \neq \theta$.

Now if there exists $\tau \in I$ such that $\mathbf{P}_\tau$ has a half element, then there is at most one such τ and, if it exists, then it contains precisely one half element. We may assume that $\theta \neq \tau$. Also, we may reorder I so that $\theta \ll \alpha \ll \tau$ for every $\alpha \in I$. For every $\delta \in J$, define an involutory chain $\preceq_\delta^*$ on $\bigcup_{\alpha\in I} \mathbf{P}_\alpha$ by $\beta(\preceq_\delta^*)$:= $[(\beta(\preceq_{\alpha,\delta}))_{\alpha\in I}]$ with the lexicographical ordering and $\beta(\leq_{\beta,\delta}) = \beta(\leq_{\theta,\delta})$ for every $\beta \in J\backslash J_\alpha$. Note that if x is the half element of $\mathbf{P}_\tau$, then $x \preceq_\delta^* y$ for every $y \in \beta(\preceq_\delta^*)$. We claim that $\{\preceq_\delta^*\}$ is an involutory realizer of the ordering in $\mathbf{H}$. To see this, let $x, y \in \operatorname{inc}(\mathbf{H})$. If $x, y \in P_\alpha$, $\alpha \in I$, then $x \preceq_{\alpha,\delta} y$ for some $\preceq_{\alpha,\delta} \in \Re_\alpha$. It follows that $x \preceq_\delta^* y$. Suppose $x \in P_\alpha$, $y \in P_\beta$, $\alpha, \beta \in I$ with $\alpha \neq \beta$. If $\alpha \ll \beta$, then there exist $\preceq_{\alpha,\delta}$ and $\preceq_{\beta,\delta}$ with $\delta \in J_\alpha$, and $\gamma \in J_\beta$ such that $x \in \varphi(\beta(\preceq_{\alpha,\delta}))$ and $y \in \varphi(\beta(\preceq_{\beta,\delta}))$ or $y' \in \varphi(\beta(\preceq_{\beta,\delta}))$. Since $\alpha \ll \beta$, it follows $x \preceq_\delta^* y$ or $x \preceq_\delta^* y' \preceq_\delta^* y$, respectively. Hence $x \preceq_\delta^* y$. Similarly, if $\beta \ll \alpha$, then $(y', x') \in \operatorname{inc}(\mathbf{H})$ and, as above, there exists $\preceq_\delta^*$ such that $y' \preceq_\delta^* x'$ from which it follows that $x \preceq_\delta^* y$. This proves the claim and completes the proof of the theorem. □

Let $I := \{1, 2, \ldots, n\}$, $n \geq 3$, and let $L := \bigcup\{B_i \mid i \in I\}$ with (i) $(B_i, \leq_i, {}'^i)$ is a Boolean lattice for every $i \in I$, (ii) if $x \in B_i \cap B_j$ with $i, j \in I$, then $x'^i = x'^j$, (iii) if $i \neq j$, then $B_i \cap B_j = \{\mathbf{0}, \mathbf{1}\}$ or $\{\mathbf{0}, \mathbf{1}, a, a'\}$, where a is an atom of both B_i and B_j, $a' = a'^i = a'^j$, and (iv) $|B_i| \geq 2^3$ for every $i \in I$. As in [7], put the induced ordering and orthocomplementation on L to obtain an orthoposet $\mathbf{L} := (L, \leq, ')$. If , for $1 \leq k < j < i \leq n$, $B_i \cap B_j \cap B_k = \{\mathbf{0}, \mathbf{1}\}$, and

$$B_i \cap B_j = \begin{cases} \{\mathbf{0}, \mathbf{1}, a_{ij}, a'_{ij}\} & \text{if } |i-j| \in J, \\ \{\mathbf{0}, \mathbf{1}\} & \text{otherwise,} \end{cases}$$

where $a_{ij} \in \operatorname{atom}(\mathbf{L})$, then $\mathbf{L}$ is an *atomic loop* of order n when $J = \{1, n\}$, and is a *picket fence of order n* when $J = \{1\}$. By [7] atomic loops of order 4 or more (respectively, 5 or more) are orthomodular posets (respectively, orthomodular lattices). If $|B_i| = 2^m$ for every $i \in \{1, 2, \ldots, n\}$, then we denote the atomic loop $\{\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n\}$ by $\mathbf{L}_n^m$.

For any subset $S \subseteq P \times P$, we define $S^* := \{(y', x') \mid (x, y) \in S\}$.

THEOREM 4.2. *If* $\mathbf{L}$ *is a picket fence with finite blocks* $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$, *then*

$$\operatorname{idim}(\mathbf{L}) = \max \operatorname{idim}(\mathbf{B}_i).$$

Proof. Let $A = \operatorname{atom}(\mathbf{L}) := \{x_j \mid j = 1, 2, \ldots, N\}$, and for each $\mathbf{B}_i$, let $\operatorname{atom}(\mathbf{B}_i) := \{x_{i1}, x_{i2}, \ldots, x_{im_i}\}$, such that $x_{11} = x_1$, $x_{im_i} = x_{(i+1)1}$ and $x_{nm_n} = x_N$. Suppose that $m = \max \operatorname{idim}(\mathbf{B}_i)$, and thus, for some $\mathbf{B}_j$, $m = \operatorname{idim}(\mathbf{B}_j) \leq \operatorname{idim}(\mathbf{L})$. For $k \in \{1, 2, \ldots, m\}$, define $S_k := \{(x_i', x_j) \in A' \times A \mid i \equiv k \pmod m,\ x_i \not\perp x_j$ and $i \leq j \leq N\}$. Note that if $\{(a', b), (c', d)\} \subseteq S_k$, then $a \not\perp c$. Let $\overline{S}_k := S_k \cup S_k^*$. Note that each $\overline{S}_k$ is involutory and that $(x_k', x_k) \in S_k \neq \phi$. Suppose that there exists $(x_s', x_t) \in \overline{S}_P \cap \overline{S}_q$; then we may assume that $s \leq t$ so $(x_s', x_t) \in S_p \cap S_q$ and

$s \equiv p \pmod m$ and $s \equiv q \pmod m$, so $p = q$. By Lemma 1.1, $\{\overline{S}_k \mid 1 \leq k \leq m\}$ is a partition of $\mathrm{crit}(\mathbf{L})$.

We now show that $\overline{S}_k$ is a cycle-free set. If it is not then it contains an alternating cycle $S := \{(u_i', v_i) \mid i = 1, 2, \ldots, t\} \subseteq \overline{S}_k$. We may assume that $(u_1', v_1) \in S_k^*$, $(u_2', v_2) \in S_k$, and $v_1 \perp v_2$. By definition of S_k^*, it follows $\{(v_1', u_1), (u_2', v_2)\} \subseteq S_k$, with $v_1 \perp u_2$ contradicting the construction of S_k. Thus there exist m involutory cycle-free sets $\{\overline{S}_k \mid 1 \leq k \leq m\}$ satisfying $\mathrm{crit}(\mathbf{L}) \subseteq \bigcup_{k=1}^{m} \overline{S}_k$. Therefore $\mathrm{idim}(\mathbf{L}) \leq m$. So $\mathrm{idim}(\mathbf{L}) = m$. □

For the next two results we use numbers instead of letters to denote atoms.

OBSERVATION 4.3. *The involutory dimension of the loop* $\mathbf{L}_3^3$ *is* 3.

Proof. Since $\mathbf{2^3}$ is a subalgebra of $\mathbf{L}_3^3$ and $\mathrm{idim}(\mathbf{2^3}) = 3$, it is clear that $\mathrm{idim}(\mathbf{L}_3^3) \geq 3$. Consider $\mathbf{L}_3^3$ with blocks $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$. Let $\mathrm{atom}(\mathbf{B}_0) = \{1, 2, 3\}$, $\mathrm{atom}(\mathbf{B}_1) = \{3, 4, 5\}$, and $\mathrm{atom}(\mathbf{B}_2) = \{5, 6, 1\}$. Let $\beta(\preceq_1) := [3, 5, 2, 6, 1', 4']$, $\beta(\preceq_2) := [1, 3, 2', 4, 6, 5']$ and $\beta(\preceq_3) := [1, 5, 6', 2, 4, 3']$. Then $\{\preceq_i\}_{i=1}^3$ is an involutory realizer of $\leq$ on $\mathbf{L}_3^3$. Thus $\mathrm{idim}(\mathbf{L}_3^3) = 3$. □

Note that if x, y, z, w are atoms such that $x \perp w$, $y \perp z$ and $(x', y) \in S$, where S is an involutory cycle-free set, then $(z', w) \notin S$, otherwise $\{(x', y), (z', w)\} \subseteq S$ is an alternating cycle. Note that z might equal w in this discussion.

In the following remark and its corollary we use the notation $\mathrm{crit}_{\preceq_i}(\mathbf{P})$ or $\mathrm{crit}_{C_i}(\mathbf{P})$ to denote $\{(x', y) \in \mathrm{crit}(\mathbf{P}) \mid x' \preceq_i y\}$, where $\preceq_i$ is the linear ordering in C_i.

REMARK 4.4. *The involutory dimension of the loop* $\mathbf{L}_6^3$ *is* 4.

Proof. Note that $\mathrm{idim}(\mathbf{L}_6^3) \geq 3$. We will show that $\mathrm{idim}(\mathbf{L}_6^3) \neq 3$. Let $B_1 := \{1, 2, 3\}$, $B_2 := \{3, 4, 5\}, \ldots, B_6 := \{11, 12, 1\}$ be the blocks of $\mathbf{L}_6^3$, and assume that $\mathrm{idim}(\mathbf{L}_6^3) = 3$. Then there exist three involutory cycle-free sets $\{S_i\}_{i=1}^3$ such that $\mathrm{crit}(\mathbf{L}_6^3) \subseteq \bigcup_{i=1}^3 S_i$. For convenience, we define $\bar{x} := (x', x)$. We may assume that $\bar{i} \in S_i$ for $i = 1, 2, 3$. Then $\bar{4} \in S_1$ or $\bar{4} \in S_2$, in which case $\bar{5} \in S_2$ or $\bar{5} \in S_1$, respectively. Continuing the case analysis in this way, we may list all 11 possible ways of covering $\{\bar{i} \mid i = 1, 2, 3\}$ by 3 sets S_i. By a consideration of the symmetry group of $\mathbf{L}_6^3$, it becomes apparent that there are essentially only 4 cases. They are the following cases.

(i) $\bar{1}, \bar{4}, \bar{6}, \bar{9} \in S_1$, $\bar{2}, \bar{5}, \bar{8}, \overline{10}, \overline{12} \in S_2$, and $\bar{3}, \bar{7}, \overline{11} \in S_3$.
(ii) $\bar{1}, \bar{4}, \bar{6}, \bar{9} \in S_1$, $\bar{2}, \bar{5}, \bar{8}, \overline{11} \in S_2$, and $\bar{3}, \bar{7}, \overline{10}, \overline{12} \in S_3$.
(iii) $\bar{1}, \bar{4}, \bar{7}, \overline{10} \in S_1$, $\bar{2}, \bar{5}, \bar{8}, \overline{11} \in S_2$, and $\bar{3}, \bar{6}, \bar{9}, \overline{12} \in S_3$.
(iv) $\bar{1}, \bar{5}, \bar{9} \in S_1$, $\bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12} \in S_2$, and $\bar{3}, \bar{7}, \overline{11} \in S_3$.

In each verification, implications follow from the comments preceding the Remark. In each case we shall produce a critical pair not in $\bigcup_{i=1}^3 S_i$.

If (i) holds, then $(1', 5) \in S_1$ or $(1', 5) \in S_2$. Suppose $(1', 5) \in S_1$. Then we may conclude, in turn, that $(6', 3) \in S_3$, $(4', 7) \in S_1$, $(9', 5) \in S_2$, $(7', 10) \in S_3$.

It follows that $(11', 6) \notin \bigcup_{i=1}^{3} S_i$. Next, suppose that $(1', 5) \in S_2$. Then we may conclude, in turn, that $(3', 12) \in S_3$, $(11', 2) \in S_2$, $(1', 10) \in S_1$, $(9', 12) \in S_2$, $(11', 8) \in S_3$, $(7', 10) \in S_2$, and $(9', 5) \in S_1$. It follows that $(4', 11) \notin \bigcup_{i=1}^{3} S_i$.

If (ii) holds, then $(1', 5) \in S_1$ or $(1', 5) \in S_2$. Suppose $(1', 5) \in S_1$. Then, in turn, $(6', 3) \in S_3$, $(4', 7) \in S_1$, $(9', 5) \in S_2$, $(7', 11) \in S_3$, $(12', 9) \in S_1$, and $(10', 1) \in S_3$. It follows that $(3', 11) \notin \bigcup_{i=1}^{3} S_i$. Next, suppose that $(1', 5) \in S_2$, then, in turn, $(3', 11) \in S_3$, $(10', 1) \in S_1$, and $(12', 9) \in S_3$. It follows that $(7', 11) \notin \bigcup_{i=1}^{3} S_i$.

If (iii) holds, then $(1', 5) \in S_1$, or $(1', 5) \in S_2$. Suppose $(1', 5) \in S_1$. Then, in turn, $(7', 3) \in S_3$, and $(5', 9) \in S_2$. It follows that $(11', 7) \notin \bigcup_{i=1}^{3} S_i$. Next, suppose $(1', 5) \in S_2$. Then, in turn, $(3', 11) \in S_3$, and $(9', 1) \in S_1$. It follows that $(11', 7) \notin \bigcup_{i=1}^{3} S_i$.

Finally, if (iv) holds, then $(1', 4) \in S_1$ or $(1', 4) \in S_2$. Suppose $(1', 4) \in S_1$. Then, in turn, $(5', 2) \in S_2$, $(3', 6) \in S_3$, $(7', 4) \in S_2$, $(5', 8) \in S_1$, $(9', 6) \in S_2$, $(7', 10) \in S_3$, $(11', 8) \in S_2$, $(9', 12) \in S_1$, $(1', 10) \in S_2$, $(11', 2) \in S_3$, and $(3', 12) \in S_2$. It follows that $(1', 7) \notin \bigcup_{i=1}^{3} S_i$. Finally, suppose $(1', 4) \in S_2$. Then, in turn, $(3', 12) \in S_3$, $(11', 2) \in S_2$, $(1', 10) \in S_1$, $(9', 12) \in S_2$, $(11', 8) \in S_3$, $(9', 6) \in S_1$, $(7', 10) \in S_2$, $(9', 6) \in S_1$, $(5', 8) \in S_2$, $(7', 4) \in S_3$, $(3', 6) \in S_2$, and $(5', 2) \in S_1$. It follows that $(1', 7) \notin \bigcup_{i=1}^{3} S_i$. Therefore $\mathrm{idim}(\mathbf{L}_6^3) > 3$.

Next, we show that $\mathrm{idim}(\mathbf{L}_6^3) = 4$ by constructing four involutory chains $\{\preceq_i\}_{i=1}^{4}$ such that $\mathrm{crit}(\mathbf{L}_6^3) = \bigcup_{i=1}^{4} \mathrm{crit}_{\preceq_i}(\mathbf{L}_6^3)$ follows:

$$\begin{aligned}
\beta(\preceq_1) &:= [2, 3, 11, 12, 1', 4, 5, 6, 7, 8, 9, 10],\\
\beta(\preceq_2) &:= [1, 3, 2', 4, 6, 7, 5', 9, 8', 10, 12, 11'],\\
\beta(\preceq_3) &:= [1, 2, 4, 5, 3', 7, 6', 8, 10, 11, 9', 12'],\\
\beta(\preceq_4) &:= [3, 5, 4', 6, 8, 9, 7', 11, 10', 1, 2, 12].
\end{aligned}$$

One can check that $\mathrm{crit}(\mathbf{L}_6^3) = \bigcup_{i=1}^{4} \mathrm{crit}_{\preceq_i}(\mathbf{L}_6^3)$. Thus $\mathrm{idim}(\mathbf{L}_6^3) = 4$. □

COROLLARY 4.5. *The involutory dimension of a regular involution poset, even of an orthomodular lattice, may be strictly greater than its order dimension.*

Proof. From Remark 4.4 we have $\mathrm{idim}(\mathbf{L}_6^3) = 4$. In 1993, Haviar and Hrnčiar [11] proved that the order dimension of $\mathbf{L}_n^3$ is 3, for $n = 2k \geq 3$. The proof is long and involved. For simplicity and completeness, we now show that the order dimension $\dim(\mathbf{L}_6^3) = 3$ by defining three chains $\{C_i\}_{i=1}^{3}$ such that $\mathrm{crit}(\mathbf{L}_6^3) = \bigcup_{i=1}^{3} \mathrm{crit}_{C_i}(\mathbf{L}_6^3)$. (These chains can be gleaned from the proof in [11].) Let

$$C_1 := [11, 12, 2, 3, 1', 1, 2', 5, 4', 4, 3', 6, 12', 9, 10', 10, 11', 8, 7', 7, 5', 6', 8', 9'],$$

and let C_2 and C_3 be the chains obtained from C_1 by incrementing the elements of C_1 by 4 and 8, respectively, calculating modulo 12. The sets $\mathrm{crit}_{C_i}(\mathbf{L}_6^3)$ are pairwise disjoint. Each contains 36 critical pairs. Since the number of elements in $\mathrm{crit}(\mathbf{L}_6^3)$ is 108, it follows that $\bigcup_{i=1}^{3} \mathrm{crit}_{C_i}(\mathbf{L}_6^3)$ covers $\mathrm{crit}(\mathbf{L}_6^3)$. Thus $\dim(\mathbf{L}_6^3) = 3$. Hence $\mathrm{idim}(\mathbf{L}_6^3) = 4 > 3 = \dim(\mathbf{L}_6^3)$. □

Orthomodular lattices or posets (as well as more general structures) which admit full sets of states are called *quantum logics.* A state on L is a mapping $\sigma\colon L \to [0, 1]$ such that the restriction σ_B of σ to any Boolean subalgebra B of L is a probability measure. A set S of states on L is *full* in case $x \le y$ in L iff $\sigma(x) \le \sigma(y)$ for all σ in $\mathcal{S}$. A state is *dispersion-free* in case it takes only the values 0 and 1. It is easy to see that L_n^m, for $m \ge 3$ and $n \ge 4$, admits a full set of dispersion-free states. Thus L_6^3 provides an example of a lattice ordered quantum logic in which the involutory dimension exceeds the order dimension.

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