SEQUENTIAL PRODUCTS ON EFFECT ALGEBRAS

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Abstract

A sequential effect algebra (SEA) is an effect algebra on which a sequential product with natural properties is defined. The properties of sequential products on Hilbert space effect algebras are discussed. For a general SEA, relationships between sequential independence. coexistence and compatibility are given. It is shown that the sharp elements of a SEA form an orthomodular poset. The sequential center of a SEA is discussed and a characterization of when the sequential center is isomorphic to a fuzzy set system is presented. It is shown that the existence of a sequential product is a strong restriction that eliminates many effect algebras from being SEA's. For example, there are no finite nonboolean SEA's. A measure of sharpness called he sharpness index is studied. The existence of horizontal sums of SEA's is characterized and examples of horizontal sums and tensor products are presented.

1 Introduction

Two measurements a and b cannot be performed simultaneously in general, so they are frequently executed sequentially. We denote by $a \circ b$ a sequential measurement in which a is performed first and b second. We call $a \circ b$

the sequential product of a and b. We shall restrict our attention to yes-no measurements. called effects, which have only two possible results. For generality, we do not assume that effects are perfectly accurate measurements. That is, they may be fuzzy or unsharp. As we shall see, the sharp effects are those that satisfy $a \circ a = a$.

A paradigm situation is an optical bench in which a beam of particles prepared in a certain state is injected at the left and then impinge first upon a filter a and then upon a filter b. Particles that pass through both filters enter a detection device at the right of b. Because of quantum interference. the order of placement of a and b usually makes a difference and we have $a \circ b \neq b \circ a$. If it happens that $a \circ b = b \circ a$ we say that a and b are sequentially independent and write $a \mid b$.

In recent years quantum effects have been studied within a general algebraic framework called an effect algebra. In Section 2 we shall summarize the basic definitions concerning effect algebras and the properties of sequential products on Hilbert space effect algebras. The simplest of these properties are employed as axioms in Section 3 for a sequential effect algebra (SEA). A SEA is an effect algebra on which a sequential product with natural properties is defined. We believe that the axioms for a SEA are physically motivated and can be tested, for example, in the optical bench situation. Various properties of a SEA are proved in Section 3. For instance, relationships between sequential independence, coexistence and compatibility are given. It is also shown that the sharp elements form an orthomodular poset.

The sequential center $C(E)$ of a SEA E is the set of elements $a \in E$ such that $a \mid b$ for every $b \in E$. In Section 4 it is shown that $C(E)$ coincides with the set of sharp central elements which has previously been studied. Moreover. a characterization is given for when $C(E)$ is isomorphic to a fuzzy set system. Section 5 shows that the existence of a sequential product is a strong restriction that eliminates many effect algebras from being SEA's. It is shown that a Boolean algebra admits a unique sequential product and that certain effect algebras admit a sequential product only if they are Boolean. Moreover. it is proved that if a map preserves the sequential product then it completely preserves the effect algebra structure of the sharp elements.

Section 6 defines the sharpness index of an effect. It is demonstrated that if E is isotropically finite, then every unsharp element of E has sharpness index ∞. It is shown that a σ-SEA is sharply dominating. An example is presented in Section 7 that illustrates various points of the previous sections. Finally, horizontal sums and tensor products of SEA's are considered in Sections 8 and 9. The existence of horizontal sums is characterized and some examples of horizontal sums and tensor products are given.

2 Hilbert Space Sequential Products

This section summarizes the basic definitions concerning effect algebras [1, 6, 7, 8, 14, 15] and the properties of sequential products on Hilbert space effect algebras [2, 3, 10, 12, 13]. If $\oplus$ is a partial binary operation, we write $a \perp b$ if $a \oplus b$ is defined. An **effect algebra** is a system $(E, 0, 1, \oplus)$ where $0, 1$ are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

(E1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.

(E2) If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$, $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \perp a'$ and $a \oplus a' = 1$.

(E4) If $a \perp 1$, then $a = 0$.

In the sequel, whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$. We define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. If such a $c \in E$ exists then it is unique and we write $c = b \ominus a$. It can be shown that $(E, \leq, ')$ is a partially ordered set with $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$, and $a \leq b$ implies $b' \leq a'$. Moreover, we have $a \perp b$ if and only if $a \leq b'$. If $a \perp a$ we call a an **isotropic** element and when 0 is the only isotropic element of E, then we call E an **orthoalgebra**. If the n-fold orthosum $a \oplus a \oplus \cdots \oplus a$ is defined in E we denote this element of E by na. If there is a largest $n \in \mathbb{N}$ such that na is defined, then n is the **isotropic index** of a and if no such n exists, then n has isotropic index ∞. An element $a \in E$ is **sharp** if $a \wedge a' = 0$. Notice that if $a \neq 0$ is sharp then a has isotropic index 1. We say that E is **isotropically finite** if every $a \neq 0$ in E has finite isotropic index.

We now give some standard examples of effect algebras. For a Boolean algebra $\mathcal{B}$, define $a \perp b$ if $a \wedge b = 0$ and in this case $a \oplus b = a \vee b$. Then $(\mathcal{B}, 0, 1, \oplus)$ is an effect algebra that happens to be an orthoalgebra. In particular if X is a nonempty set, then $(2^X, \emptyset, X, \oplus)$ is an effect algebra. These effect algebras correspond to classical logic and set theory. For the function space $[0, 1]^X$

on the interval $[0,1] \subseteq \mathbb{R}$ define the functions f_0, f_1 by $f_0(x) = 0$. $f_1(x) = 1$ for all $x \in X$. For $f, g \subseteq [0,1]^X$, we define $f \perp g$ if $f(x) + g(x) \leq 1$ for all $x \in X$ and in this case $(f \oplus g)(x) = f(x) + g(x)$. Then $\left([0,1]^X, f_0, f_1, \oplus\right)$ is the effect algebra of fuzzy subsets of X. A particularly simple effect algebra is the interval $[0,1] \subseteq \mathbb{R}$. For $a, b \in [0,1]$ we define $a \perp b$ if $a + b \leq 1$ and in this case $a \oplus b = a + b$.

In this section we are mainly concerned with the set $\mathcal{E}(H)$ of all self-adjoint operators on a Hilbert space H that satisfy $0 \leq \langle Ax, x\rangle \leq \langle x, x\rangle$ for all $x \in H$. For $A, B \in \mathcal{E}(H)$ we define $A \perp B$ if $A + B \in \mathcal{E}(H)$ and in this case $A \oplus B = A + B$. Then $(\mathcal{E}(H), 0, I, \oplus)$ is an effect algebra that we call a **Hilbert space effect algebra.** This effect algebra is important in studies of the foundations of quantum physics and quantum measurement theory [2, 3. 4, 16, 17]. The **quantum effects** $A \in \mathcal{E}(H)$ correspond to yes-no measurements that may be unsharp. The set of projection operators $\mathcal{P}(H)$ on H form an orthoalgebra that is a sub-effect algebra of $\mathcal{E}(H)$. The elements of $\mathcal{P}(H)$ correspond to sharp quantum effects.

If E and F are effect algebras, we say that $\phi\colon E \to F$ is **additive** if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $\phi\colon E \to F$ is additive and $\phi(1) = 1$, then ϕ is a **morphism.** If $\phi\colon E \to F$ is a morphism and $\phi(a) \perp \phi(b)$ implies that $a \perp b$, then ϕ is a **monomorphism.** A surjective monomorphism is an **isomorphism.** It is easy to see that a morphism ϕ is an isomorphism if and only if ϕ is bijective and ϕ^{-1} is a morphism. A **state** on E is a morphism $s\colon E \to [0,1]$. We interpret $s(a)$ as the probability that the effect a is observed (has answer yes) when the system is in the state s. We denote the set of states on E by $\Omega(E)$. A set of states $S \subseteq \Omega(E)$ is **order determining** if $s(a) \leq s(b)$ for all $s \in S$ implies that $a \leq b$.

The **sequential product** on $\mathcal{E}(H)$ is defined by $A \circ B = A^{1/2}BA^{1/2}$ where $A^{1/2}$ is the unique positive square root of A [2, 3, 10, 12, 13]. We have that $A \circ B \in \mathcal{E}(H)$ because

$$\begin{aligned} 0 &\leq \left\langle A^{1/2}BA^{1/2}x, x\right\rangle = \left\langle BA^{1/2}x, A^{1/2}x\right\rangle \leq \left\langle A^{1/2}x, A^{1/2}x\right\rangle \\ &= \langle Ax, x\rangle \leq \langle x, x\rangle \end{aligned}$$

for all $x \in H$. Notice that $B \mapsto A \circ B$ is additive on $\mathcal{E}(H)$ and that $I \circ B = B$ for all $B \in \mathcal{E}(H)$. We now present some of the important properties of the sequential product on $\mathcal{E}(H)$. If $A \circ B = B \circ A$ we say that A and B are **sequentially independent** and write $A \mid B$. The following result is proved in [12. 13].

Theorem 2.1. (i) *For* $A, B \in \mathcal{E}(H)$, *if* $A \circ B \in \mathcal{P}(H)$ *then* $AB = BA$. (ii) *For* $A, B \in \mathcal{E}(H)$, $A \mid B$ *if and only if* $AB = BA$.

Applying Theorem 2.1 we obtain the following properties of the sequential product $A \circ B$.

Corollary 2.2. (i) *If* $A \circ B = 0$, *then* $B \circ A = 0$. (ii) *If* $A \mid B$, *then* $A \mid B'$ *and* $A \circ (B \circ C) = (A \circ B) \circ C$ *for all* $C \in \mathcal{E}(H)$. (iii) *If* $C \mid A$ *and* $C \mid B$ *then* $C \mid A \circ B$ *and* $C \mid (A \oplus B)$.

The next three results are also proved in [13]. Notice that Theorem 2.3 gives the converse of the second part of Corollary 2.2(ii).

Theorem 2.3. *For* $A, B \in \mathcal{E}(H)$, $A \circ (B \circ C) = (A \circ B) \circ C$ *for every* $C \in \mathcal{E}(H)$ *if and only if* $A \mid B$.

Theorem 2.4. *For* $A, B \in \mathcal{E}(H)$ *the following statements are equivalent.* (i) $A \circ B = B$. (ii) $B \circ A = B$. (iii) $AB = BA = B$.

Theorem 2.5. *For* $A, B \in \mathcal{E}(H)$ *we have* $B = A \circ B \oplus A' \circ B$ *if and only if* $A \mid B$.

We denote the set of positive trace class operators with trace 1 on H by $\mathcal{D}(H)$. The **normal states** on $\mathcal{E}(H)$ have the form $P_W(A) = \mathrm{tr}(WA)$ for some $W \in \mathcal{D}(H)$. We say that $A, B \in \mathcal{E}(H)$ are **stochastically independent** relative to $W \in \mathcal{D}(H)$ if $P_W(A \circ B) = P_W(A)P_W(B)$. The next result is proved in [13].

Theorem 2.6. *For* $A, B \in \mathcal{E}(H)$ *the following statements are equivalent.* (i) $A \circ (C \circ B) = (A \circ C) \circ B$ *for every* $C \in \mathcal{E}(H)$. (ii) $C \circ (A \circ B) = (C \circ A) \circ B$ *for every* $C \in \mathcal{E}(H)$. (iii) $P_W(A \circ B) = P_W(A)P_W(B)$ *for every* $W \in \mathcal{D}(H)$. (iv) $A = cI$ *or* $B = cI$ *for some* $0 \le c \le 1$.

We close this section with an application of the interesting work in [18]. This result shows that the sequential product determines the effect algebra structure of $\mathcal{E}(H)$ when $\dim H \ge 3$.

Theorem 2.7. *Suppose that* $\dim H \ge 3$. *If* $\phi\colon \mathcal{E}(H) \to \mathcal{E}(H)$ *is a bijection satisfying* $\phi(A \circ B) = \phi(A) \circ \phi(B)$ *for all* $A, B \in \mathcal{E}(H)$, *then* ϕ *is an effect algebra isomorphism.*

Proof. For $A \in \mathcal{E}(H)$ we have

$$\phi(A^2) = \phi(A \circ A) = \phi(A) \circ \phi(A) = \phi(A)^2$$

Hence, for $A, B \in \mathcal{E}(H)$ we have

$$\phi(ABA) = \phi(A^2 \circ B) = \phi(A^2) \circ \phi(B) = \phi(A)^2 \circ \phi(B) = \phi(A)\phi(B)\phi(A)$$

Applying Theorem 2 [18], ϕ has the form $\phi(A) = UAU^*$ where U is either a unitary or an anti-unitary operator on H. It easily follows that ϕ is an effect algebra isomorphism. □

3 Sequential Effect Algebras

This section summarizes the basic definitions and results for sequential effect algebras. For a binary operation $\circ$, if $a \circ b = b \circ a$ we write $a \mid b$. A **sequential effect algebra** (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ \colon E \times E \to E$ is a binary operation that satisfies the following conditions.

(S1) $b \mapsto a \circ b$ is additive for all $a \in E$.

(S2) $1 \circ a = a$ for all $a \in E$.

(S3) If $a \circ b = 0$, then $a \mid b$.

(S4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in E$.

(S5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$.

We call an operation that satisfies (S1)–(S5) a **sequential product** on E. If $a \mid b$ for all $a, b \in E$ we call E a **commutative** SEA.

The effect algebra $[0, 1] \subseteq \mathbb{R}$ is a SEA with sequential product $a \circ b = ab$. Corollary 2.2 shows that $\mathcal{E}(H)$ is a SEA under the operation $A \circ B = A^{1/2}BA^{1/2}$. More generally, this operation makes any von Neumann algebra a SEA. A Boolean algebra is a SEA under the operation $a \circ b = a \wedge b$. Let X be a nonempty set and let $\mathcal{F} \subseteq [0, 1]^X$. We call $\mathcal{F}$ a **fuzzy set system** on X if $f_0, f_1 \in \mathcal{F}$, if $f \in \mathcal{F}$ then $f_1 - f \in \mathcal{F}$, if $f, g \in \mathcal{F}$ with $f + g \leq 1$ then $f + g \in \mathcal{F}$ and if $f, g \in \mathcal{F}$ then $fg \in \mathcal{F}$. Then $\mathcal{F}$ becomes a SEA when

$f \oplus g = f + g$ for $f + g \leq 1$ and $f \circ g = fg$. Except for $\mathcal{E}(H)$, all of these examples are commutative. The following lemma summarizes some of the properties of a SEA E.

Lemma 3.1. (i) *$a \circ 0 = 0 \circ a = 0$ and $a \circ 1 = 1 \circ a = a$ for all $a \in E$.* (ii) *$a \circ b \leq a$ for all $a, b \in E$.* (iii) *If $a \leq b$, then $c \circ a \leq c \circ b$ for all $c \in E$.* (iv) *If $a \leq b$, then $c \circ (b \ominus a) = c \circ b \ominus c \circ a$.* (v) *If $a \leq b$, $c \mid a$ and $c \mid b$, then $c \mid (b \ominus a)$.*

Proof. (i) By additivity we have

$$a \circ 0 \oplus 0 = a \circ 0 = a \circ (0 \oplus 0) = a \circ 0 \oplus a \circ 0$$

and by cancellation we have $a \circ 0 = 0$. Applying (S3) gives $0 \circ a = 0$. Since $0 \mid a$, applying (S2) and (S4) gives $a \circ 1 = 1 \circ a = a$.
(ii) Applying (S1) gives

$$a = a \circ a = a \circ (b \oplus b') = a \circ b \oplus a \circ b' \geq a \circ b$$

(iii) If $a \leq b$ there exists a $d \in E$ such that $a \oplus d = b$. Hence,

$$c \circ b = c \circ a \oplus c \circ d \geq c \circ a$$

(iv) If $a \leq b$, then by (iii) $c \circ a \leq c \circ b$. Since $a \oplus (b \ominus a) = b$ we have that $c \circ a \oplus c \circ (b \ominus a) = c \circ b$. Hence, $c \circ (b \ominus a) = c \circ b \ominus c \circ a$.
(v) This follows from (S4), (S5) and the identity $b \ominus a = (a \oplus b')'$. □

We denote the set of sharp elements of E by E_S. It is clear that $0, 1 \in E_S$ and that $a' \in E_S$ whenever $a \in E_S$.

Lemma 3.2. *The following statements are equivalent.* (i) $a \in E_S$. (ii) $a \circ a' = 0$. (iii) $a \circ a = a$.

Proof. To show that (i) implies (ii) suppose that $a \wedge a' = 0$. By Lemma 3.1(ii) we have

$$a \circ a' = a' \circ a \leq a, a'$$

Hence, $a \circ a' = 0$. To show that (ii) implies (iii) suppose that $a \circ a' = 0$. Then

$$a = a \circ a \oplus a \circ a' = a \circ a$$

To show that (iii) implies (i) suppose that $a \circ a = a$. Then

$$a = a \circ a \oplus a \circ a' = a \oplus a \circ a'$$

so by cancellation $a \circ a' = 0$. If $b \leq a, a'$, then by Lemma 3.1(iii) we have that $a \circ b \leq a \circ a' = 0$. Hence, $a \circ b = 0$ and similarly $a' \circ b = 0$. Hence, $b \circ a = b \circ a' = 0$ so that

$$b = b \circ a \oplus b \circ a' = 0$$

Hence. $a \wedge a' = 0$. □

Lemma 3.3. (i) *If $a \circ b = 0$, then $a \perp b$.* (ii) *For $a \in E$, $b \in E_S$, $a \circ b = 0$ if and only if $a \perp b$.* (iii) *For $a, b \in E_S$, with $a \perp b$ we have $a \oplus b \in E_S$.* (iv) *For $a, b \in E_S$ with $a \leq b$ we have $b \ominus a \in E_S$.* (v) *For $a, b \in E_S$ with $a \mid b$ we have $a \circ b \in E_S$.*

Proof. (i) If $a \circ b = 0$, then $b \circ a = a \circ b = 0$. Hence,

$$a = a \circ b \oplus a \circ b' = a \circ b' = b' \circ a \leq b'$$

(ii) If $a \leq b'$ then $b \circ a \leq b \circ b' = 0$. Hence, $a \circ b = b \circ a = 0$.
(iii) Since $a \circ b = 0$ we have

$$\begin{aligned}(a \oplus b) \circ (a \oplus b) &= (a \oplus b) \circ a \oplus (a \oplus b) \circ b \\ &= a \circ (a \oplus b) \oplus b \circ (a \oplus b) = a \oplus b\end{aligned}$$

(iv) This follows from iii) and the identity $b \ominus a = (a \oplus b')'$.
(v) Since $a \mid b$ we have

$$\begin{aligned}(a \circ b) \circ (a \circ b) &= a \circ [b \circ (a \circ b)] = a \circ [b \circ (b \circ a)] \\ &= a \circ [(b \circ b) \circ a] = a \circ (b \circ a) = a \circ (a \circ b) \\ &= (a \circ a) \circ b = a \circ b\end{aligned}$$

□

It follows from Lemma 3.3(iii) that E_S is a sub-effect algebra of E that is an orthoalgebra. In general, if $a, b \in E_S$ then $a \circ b \notin E_S$ so E_S is not a sub-SEA of E.

Theorem 3.4. *Let $a \in E$ and $b \in E_S$.* (i) *$a \leq b$ if and only if $a \circ b = b \circ a = a$ and $b \leq a$ if and only if $a \circ b = b \circ a = b$.* (ii) *If $a \mid b$, then $a \wedge b = a \circ b$.* (iii) *If $a \perp b$, then $a \oplus b = a \vee b = (a' \circ b')'$.*

Proof. (i) If $b \circ a = a$, then $a = b \circ a \leq b$. Similarly, if $a \circ b = b$ then $b \leq a$. Conversely, suppose that $a \leq b$. Then $a \perp b'$ so by Lemma 3.3(ii) $a \circ b' = b' \circ a = 0$. Hence, $a \mid b$ and we have

$$a = a \circ b \oplus a \circ b' = a \circ b$$

If $b \leq a$ then $a' \leq b'$ so from our previous work, $a' \circ b' = b' \circ a' = a'$. Hence,

$$b \circ a' = b \circ (b' \circ a') = (b \circ b') \circ a' = 0$$

and we have

$$b = b \circ a \oplus b \circ a' = b \circ a = a \circ b$$

(ii) We have $a \circ b = b \circ a \leq a, b$. Suppose that $c \leq a, b$. Then there exists a $d \in E$ such that $c \oplus d = a$ and by (i) we have $b \circ c = c$. Hence,

$$b \circ a = b \circ (c \oplus d) = b \circ c \oplus b \circ d = c \oplus b \circ d \geq c$$

We conclude that $a \wedge b = a \circ b$.
(iii) It is clear that $a, b \leq a \oplus b$. Suppose that $a, b \leq c$. Then there exists a $d \in E$ such that $a \oplus d = c$ and by (i) we have $c \circ b = b \circ c = b$. By Lemma 3.3(ii), $a \circ b = b \circ a = 0$ so that

$$b = b \circ c = b \circ (a \oplus d) = b \circ a \oplus b \circ d = b \circ d$$

Applying Lemma 3.1(v) we have $b \mid (c \ominus a)$. Hence, $b \mid d$ and $b = d \circ b \leq d$. It follows that $a \oplus b \leq a \oplus d = c$. Hence, by (ii) we have that

$$a \oplus b = a \vee b = (a' \wedge b')' = (a' \circ b')' \qquad \square$$

Corollary 3.5. *E_A is a sub-effect algebra of E that is an orthomodular poset.*

Proof. This follows from Theorem 3.4(iii). $\square$

We say that $a, b \in E$ **coexist** if there exist $c, d, e \in E$ such that $c \oplus d \oplus e$ is defined and $a = c \oplus d$, $b = c \oplus e$ [16, 17].

Theorem 3.6. (i) *If $a \mid b$ then a and b coexist.* (ii) *For $a \in E$, $b \in E_S$, $a \mid b$ if and only if a and b coexist.*

Proof. (i) If $a \mid b$, then $a = a \circ b \oplus a \circ b'$ and

$$b = b \circ a \oplus b \circ a' = a \circ b \oplus a' \circ b$$

Now

$$1 = a \oplus a' = a \oplus (a' \circ b \oplus a' \circ b') = (a \oplus a' \circ b) \oplus a' \circ b'$$

Hence,

$$(a' \circ b')' = a \oplus a' \circ b = a \circ b \oplus a \circ b' \oplus a' \circ b$$

It follows that a and b coexist. (ii) If a and b coexist, then $a = c \oplus d$, $b = c \oplus e$ for some $c, d, e \in E$ where $c \oplus d \oplus e$ is defined. Since $c \leq b$ we have $b \mid c$. Since $d \perp b$ we have $d \leq b'$. Hence, $b \mid d$ and it follows that $b \mid a$. □

It is well-known that the converse of Theorem 3.6(i) does not hold [3]. We say that $a, b \in E_S$ are **compatible** if there exist mutually orthogonal elements $c, d, e \in E_S$ such that $a = c \vee d$ and $b = c \vee e$.

Corollary 3.7. *For $a, b \in E_S$, $a \mid b$ if and only if a and b are compatible.*

Proof. If $a \mid b$, then by the proof of Theorem 3.6 and Lemma 3.3(v), $a \circ b$, $a \circ b'$ and $a' \circ b$ are mutually orthogonal elements of E_S. By Theorem 3.4(iii) we have

$$a = a \circ b \oplus a \circ b' = a \circ b \vee a \circ b'$$

and

$$b = a \circ b \oplus a' \circ b = a \circ b \vee a' \circ b$$

Hence, a and b are compatible. Conversely, suppose that a and b are compatible and $a = c \vee d$, $b = c \vee e$ where $c, d, e \in E_S$ are mutually orthogonal. By Theorem 3.4(iii), $a = c \oplus d$, $b = c \oplus e$. Since $e \perp c$, $e \perp d$, we have $e \perp a$. Hence, $c \oplus d \oplus e$ is defined. Thus, a and b coexist so by Theorem 3.6(ii) we have that $a \mid b$. □

Corollary 3.8. *A SEA is a commutative orthoalgebra if and only if it is a Boolean algebra. In a commutative SEA E, E_S is a Boolean algebra.*

The next result shows that for certain special cases, $\circ$ and $\oplus$ are closely related.

Corollary 3.9. (i) *If $a \mid b$ and $a \perp b$, then $a \oplus b = a \circ b \oplus (a' \circ b')'$.* (ii) *If $a, b \in E_S$ and $a \perp b$, then $a \oplus b = (a' \circ b')'$.* (iii) *If $a, b \in E_S$ and $a' \perp b'$, then $a \circ b = (a' \oplus b')'$.*

4 The Sequential Center of a SEA

We define the **sequential center** of a SEA E as

$$C(E) = \{a \in E : a \mid b \text{ for all } b \in E\}$$

The next result follows from our previous work.

Theorem 4.1. (i) *$C(E)$ is a sub-SEA of E which is a commutative monoid under $\circ$.* (ii) *$C(E) \cap E_S$ is a Boolean algebra.*

An element $a \in E$ is **principal** if $b, c \leq a$ and $b \perp c$ imply that $b \oplus c \leq a$. It is shown in [9] that principal elements are sharp. We now prove that the converse holds in a SEA.

Lemma 4.2. *An element $a \in E$ is principal if and only if $a \in E_S$.*

Proof. Suppose a is principal and $b \leq a, a'$. Then $b \perp a$ and $b, a \leq a$. Hence,

$$a \oplus b \leq a = a \oplus 0$$

By cancellation $b = 0$ so that $a \wedge a' = 0$. Conversely, suppose that $a \in E_S$ and $b, c \leq a$ with $b \perp c$. By Theorem 3.4(i) $a \circ b = b$ and $a \circ c = c$. Hence,

$$a \geq a \circ (b \oplus c) = a \circ b \oplus a \circ c = b \oplus c \qquad \square$$

The next result holds for an arbitrary effect algebra [9]. However, the proof is much simpler for a SEA.

Corollary 4.3. (i) *If $a, b \in E_S$ and $a \wedge b$ exists, then $a \wedge b \in E_S$.* (ii) *If $a, b \in E_S$ and $a \vee b$ exists, then $a \vee b \in E_S$.*

Proof. (i) By Lemma 4.2, a and b are principal. Suppose that $c, d \leq a \wedge b$ and $c \perp d$. Then $c \leq a, b$ and $d \leq a, b$ so that $c \oplus d \leq a, b$. Hence, $c \oplus d \leq a \wedge b$. Thus, $a \wedge b$ is principal so by Lemma 4.2, $a \wedge b \in E_S$. (ii) Since $a \vee b$ exists, $a' \wedge b' = (a \vee b)'$ exists and is sharp. Hence, $a \vee b \in E_S$. $\square$

According to [9], an element $a \in E$ is **central** if a, a' are principal and for every $p \in E$ there exist $q, r \in E$ such that $q \leq a$, $r \leq a'$ and $p = q \oplus r$. In [9] the center $\widetilde{C}(E)$ is defined to be the set of all central elements.

Theorem 4.4. (i) *If $a \in E_S$, then $a \mid p$ if and only if there exist $q, r \in E$ such that $q \leq a$, $r \leq a'$ and $p = q \oplus r$.* (ii) *$\widetilde{C}(E) = C(E) \cap E_S$.*

Proof. (i) Suppose $p = q \oplus r$ where $q \leq a$, $r \leq a'$. Since $a \mid q$ and $a \mid r$ we have that $a \mid p$. Conversely, suppose that $a \mid p$. Then $p = p \circ a \oplus p \circ a'$ and we have that $p \circ a = a \circ p \leq a$ and $p \circ a' = a' \circ p \leq a'$. (ii) If $a \in \widetilde{C}(E)$, then by Lemma 4.2, $a \in E_S$. Moreover, by (i) we have that $a \in C(E)$. Hence, $\widetilde{C}(E) \subseteq C(E) \cap E_S$. Conversely, if $a \in C(E) \cap E_S$, then by (i) we have that $a \in \widetilde{C}(E)$. Hence, $C(E) \cap E_S \subseteq \widetilde{C}(E)$. □

Example. Let $\mathcal{F}$ be the set of differentiable functions $f\colon [0,1] \to [0,1]$. Then $\mathcal{F}$ is a fuzzy set system under the previously defined partial operation $f \oplus g = f + g$ if $f + g \leq 1$ and the operation $f \circ g = fg$. Hence, $\mathcal{F}$ is a commutative SEA so that $C(\mathcal{F}) = \mathcal{F}$. However, $\mathcal{F}$ is not lattice order because $f \wedge g$ does not exist in $\mathcal{F}$ in general. This shows that a commutative SEA need not be an MV-algebra. □

This example showed that $C(E)$ need not be lattice ordered. If E and F are SEA's, a map $\phi\colon E \to F$ is an **isomorphism** if ϕ is an effect algebra isomorphism satisfying $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all $a, b \in E$. We shall give an example later which shows that $C(E)$ need not be isomorphic to a fuzzy set system. Our next result characterizes when $C(E)$ is indeed isomorphic to a fuzzy set system. In contrast to $C(E)$, it is interesting to note that $\widetilde{C}(E)$ is always a Boolean algebra even for an arbitrary effect algebra [9]. A state s on a is called **multiplicative** if $s(a \circ b) = s(a)s(b)$ for all a, b.

Theorem 4.5. *The sequential center $C(E)$ is isomorphic to a fuzzy set system if and only if $C(E)$ admits an order determining set of multiplicative states.*

Proof. Suppose $\mathcal{F}$ is a fuzzy set system on Ω and $\phi\colon C(E) \to \mathcal{F}$ is an isomorphism. For $\omega \in \Omega$, $a \in C(E)$ define $\omega(a) = \phi(a)(\omega)$. Then $\omega(1) = \phi(1)(\omega) = 1$. If $a \perp b$ we have

$$\begin{aligned}\omega(a \oplus b) &= \phi(a \oplus b)(\omega) = [\phi(a) \oplus \phi(b)](\omega) = \phi(a)(\omega) + \phi(b)(\omega)\\ &= \omega(a) + \omega(b)\end{aligned}$$

Hence, $\{\omega\colon \omega \in \Omega\}$ forms a set of states on $C(E)$. These states are multiplicative because

$$\begin{aligned}\omega(a \circ b) &= \phi(a \circ b)(\omega) = [\phi(a)\phi(b)](\omega) = \phi(a)(\omega)\phi(b)(\omega)\\ &= \omega(a)\omega(b)\end{aligned}$$

To show that this set of states is order determining, suppose that $\omega(a) \leq \omega(b)$ for every $\omega \in \Omega$. Then $\phi(a)(\omega) \leq \phi(b)(\omega)$ for every $\omega \in \Omega$ so that $\phi(a) \leq \phi(b)$. Since ϕ is an isomorphism, we have that $a \leq b$.

Conversely, suppose Ω is an order determining set of multiplicative states on $C(E)$. Define $\phi\colon C(E) \to [0,1]^{\Omega}$ by $\phi(a)(\omega) = \omega(a)$ and let $\mathcal{F} \subseteq [0,1]^{\Omega}$ be the range of ϕ. Since $\phi(1) = f_1$ and $\phi(0) = f_0$, $f_1, f_0 \in \mathcal{F}$. For $\phi(a) \in \mathcal{F}$ we have

$$[f_1 - \phi(a)](\omega) = 1 - \omega(a) = \omega(a') = \phi(a')(\omega)$$

so that $f_1 - \phi(a) \in \mathcal{F}$ and $\phi(a') = f_1 - \phi(a)$. If $a \perp b$, then

$$\phi(a)(\omega) = \omega(a) \leq \omega(b') = 1 - \phi(b)(\omega)$$

Hence, $\phi(a) \perp \phi(b)$ and we have

$$\phi(a \oplus b)(\omega) = \omega(a \oplus b) = \omega(a) + \omega(b) = \phi(a)(\omega) + \phi(b)(\omega)$$

for every $\omega \in \Omega$. We conclude that $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. For $\phi(a), \phi(b) \in \mathcal{F}$ with $\phi(a) + \phi(b) \leq 1$ we have $\omega(a) \leq \omega(b')$. Since Ω is order determining, we have that $a \perp b$. Hence, ϕ is a monomorphism and $\phi(a) + \phi(b) \in \mathcal{F}$. For $\phi(a), \phi(b) \in \mathcal{F}$ we have

$$[\phi(a)\phi(b)](\omega) = \phi(a)(\omega)\phi(b)(\omega) = \omega(a)\omega(b) = \omega(a \circ b) = \phi(a \circ b)(\omega)$$

for ever $\omega \in \Omega$. Hence, $\phi(a)\phi(b) \in \mathcal{F}$ and $\phi(a \circ b) = \phi(a)\phi(b)$. We conclude that $\mathcal{F}$ is a fuzzy set system and $\phi\colon C(E) \to \mathcal{F}$ is an isomorphism. □

Corollary 4.6. *A SEA E admits an order determining set of multiplicative states Ω if and only if E is isomorphic to a fuzzy set system Ω. Under this isomorphism, E_S is a Boolean algebra of subsets of Ω.*

Proof. For $a \in E_S$ we have $\phi(a)^2 = \phi(a \circ a) = \phi(a)$. Hence, $\phi(a)$ is a characteristic function which can be considered to be a subset of Ω. The result now follows. □

5 Existence of Sequential Products

This section shows that the existence of a sequential product is a strong restriction that eliminates many effect algebvras from being SEA's.

Lemma 5.1. *For an effect algebra E that is a Boolean algebra there is a unique sequential product $a \circ b = a \wedge b$.*

Proof. Since $E = E_S$ and any two elements of E are compatible, by Corollary 3.7 we have that $a \mid b$ for every $a, b \in E$ and any sequential product $\circ$. It follows from Theorem 3.4(ii) that $a \circ b = a \wedge b$. □

Lemma 5.2. *Let E be a SEA.* (i) *If $a \in E$ is an atom then for every $b \in E$ either $a \leq b$ or $a \leq b'$.* (ii) *If $a, b \in E$ are distinct atoms, then $a \perp b$.*

Proof. (i) Since $a \circ b \leq a$ we have $a \circ b = 0$ or $a \circ b = a$. By Lemma 3.3(i), if $a \circ b = 0$, then $a \leq b'$. If $a = a \circ b$ then $a \circ b' = 0$ so again by Lemma 3.3(i) we have that $a \leq b$. (ii) By (i) we have that $a \perp b$ or $a \leq b$. In the second case, $a = b$ which is a contradiction. □

Theorem 5.3. *An atomistic orthoalgebra E admits a sequential product if and only if E is Boolean.*

Proof. If E is Boolean, we have seen that it admits a sequential product. Conversely, suppose that E admits a sequential product. Since $E = E_S$, by Corollary 3.5, E is an orthomodular poset. Let $c, d \in E$. By Lemma 5.2 we have that $c = (\vee c_i) \vee (\vee e_i)$, $d = (\vee c_i) \vee (\vee d_i)$ where c_i, d_i, e_i are distinct mutually orthogonal atoms. Since $(\vee e_i) \perp (\vee d_i)$, c and d are compatible. It follows that E is Boolean. □

Corollary 5.4. *There does not exist a sequential product on $\mathcal{P}(H)$,* $\dim H \geq 2$.

Proof. For $\dim H \geq 2$, $\mathcal{P}(H)$ is a nonboolean, atomistic orthoalgebra. □

Theorem 5.5. (i) *An orthoalgebra E is a commutative SEA if and only if E is Boolean.* (ii) *If E is a chain finite SEA, then E is Boolean.*

Proof. (i) It is clear that a Boolean algebra is a commutative SEA. Conversely, if an orthoalgebra E is commutative then by Corollary 3.7 any two elements of E are compatible. It follows that E is Boolean.
(ii) If $a \in E$ is an atom, then $a = a \circ a \oplus a \circ a'$ implies that $a \circ a = 0$ or $a \circ a' = 0$. Suppose that $a \circ a = 0$. By Lemma 3.3(i) $a \perp a$ so that $2a$ exists. Now

$$a \circ (2a) = a \circ (a \oplus a) = a \circ a \oplus a \circ a = 0$$

Hence, $2a \perp a$ so that $3a$ exists. Continuing by induction, na exists for all $n \in \mathbb{N}$. Since $a < 2a < 3a < \cdots$ forms an infinite chain, this is a contradiction. Hence, $a \circ a' = 0$ so by Lemma 3.2 a is sharp. Since the orthosum of sharp elements is sharp, we have $E = E_S$. Hence, E is an atomistic orthoalgebra and the result follows from Theorem 5.3. □

Lemma 5.6. *Let E be a SEA and let $\phi\colon E \to E$ be an additive function satisfying* (i) $a = \phi(1) \in C(E)$, (ii) *if $b \in E$ with $b \leq a$, then $\phi(b) = b$. Then $a \in E_S$ and $\phi(b) = a \circ b$ for every $b \in E$.*

Proof. By (ii) $\phi(a) = a$. Hence,

$$a = \phi(1) = \phi(a) \oplus \phi(a') = a \oplus \phi(a')$$

so by cancellation we have that $\phi(a') = 0$. If $b \leq a'$, then $b \oplus c = a'$ for some $c \in E$. Hence,

$$\phi(b) \oplus \phi(c) = \phi(a') = 0$$

so that $\phi(b) = 0$. If $d \in E$ then $a' \circ d \leq a'$ so that $\phi(a' \circ d) = 0$. Also, $a \circ d \leq a$ so that $\phi(a \circ d) = a \circ d$. Thus, for every $d \in E$ we have that

$$\phi(d) = \phi(d \circ a \oplus d \circ a') = \phi(a \circ d) \oplus \phi(a' \circ d) = a \circ d$$

Since $a = \phi(a) = a \circ a$ we conclude that $a \in E_S$. □

Lemma 5.6 can be used to give another proof of Lemma 5.1. Indeed, let $a \circ b = a \wedge b$ be the sequential product on E. For $a \in E = C(E)$ define $\phi_a\colon E \to E$ by $\phi_a(b) = a \bullet b$ where $a \bullet b$ is another sequential product on E. Then ϕ_a satisfies the conditions of Lemma 5.6 so that $a \bullet b = \phi_a(b) = a \circ b$.

Let X be a finite nonempty set and let $\mathcal{F}(X)$ be the fuzzy set system $\mathcal{F}(X) = [0,1]^X$. A sequential product $\circ$ on $\mathcal{F}(X)$ is **homogeneous** if $\lambda f_1 \circ f = f \circ \lambda f_1 = \lambda f$ for all $f \in \mathcal{F}(X)$, $\lambda \in [0,1]$.

Lemma 5.7. *$\mathcal{F}(X)$ admits a unique homogeneous sequential product $f \circ g = fg$.*

Proof. It is clear that $f \circ g = fg$ is a homogeneous sequential product on $\mathcal{F}(X)$. Let $f \bullet g$ be another homogeneous sequential product on $\mathcal{F}(X)$. Since any two elements of $\mathcal{F}(X)_S$ are compatible, it follows from Corollary 3.7 and Lemma 3.3(v) that $\mathcal{F}(X)_S$ is a commutative sub-SEA of $(\mathcal{F}(X), \circ)$.

For $f \in \mathcal{F}(X)_S$ define $\phi_f\colon \mathcal{F}(X)_S \to \mathcal{F}(X)_S$ by $\phi_f(g) = f \bullet g$. Applying Lemma 5.6, we have that

$$f \bullet g = \phi_f(g) = f \circ g = fg$$

Hence, $f \bullet g = fg$ for every $f, g \in \mathcal{F}(X)_S$. Since $f \bullet g$ is homogeneous, we have $(\lambda f) \bullet (\mu g) = (\mu g) \bullet (\lambda f)$ for every $f, g \in \mathcal{F}(X)_S$, $\lambda, \mu \in [0,1]$. For any $u, v \in \mathcal{F}(X)$ we have $u = \sum_{i=1}^{n} \lambda_i f_i$, $v = \sum_{j=1}^{m} \mu_j g_j$ where $\lambda_i, \mu_j \in [0,1]$, $f_i, g_j \in \mathcal{F}(X)_S$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. It follows that $u \bullet g_j = g_j \bullet u$ and $v \bullet f_i = f_i \bullet v$ for every $i = 1, \ldots, n$, $j = 1, \ldots, m$. Hence,

$$\begin{aligned} u \bullet v &= \sum \mu_j u \bullet g_j = \sum \mu_j g_j \bullet u = \sum \mu_j \lambda_i g_j \bullet f_i \\ &= \sum \lambda_i \mu_j f_i \circ g_j = u \circ v \end{aligned}$$

□

The next result shows that if a map preserves the sequential product then it completely preserves the structure of the sharp elements.

Theorem 5.8. *Let E, F be SEA's and let $\phi\colon E \to F$ be a bijection satisfying $\phi(a \circ b) = \phi(a) \circ \phi(b)$. Then $\phi\colon E_S \to F_S$ is an isomorphism. Moreover, if $a \in E_S$ or $b \in E_S$ and $a \le b$ then $\phi(a) \le \phi(b)$.*

Proof. If $a \in E_S$, then $\phi(a) = \phi(a \circ a) = \phi(a) \circ \phi(a)$ so that $\phi(a) \in F_S$. If $b \in F_S$ then $\phi(a) = b$ for some $a \in E$. But then

$$\phi(a) = b \circ b = \phi(a) \circ \phi(a) = \phi(a \circ a)$$

Since ϕ is injective, we have that $a = a \circ a$ so $a \in E_S$. Hence, $\phi\colon E_S \to F_S$ is a bijection. Now $\phi(a) = 1$ for some $a \in E$. Thus,

$$\phi(1) = \phi(1) \circ 1 = \phi(1) \circ \phi(a) = \phi(1 \circ a) = \phi(a) = 1$$

Similarly, $\phi(b) = 0$ for some $b \in E$ and we have

$$\phi(0) = \phi(0 \circ b) = \phi(0) \circ \phi(b) = 0$$

If $a \in E_S$ or $b \in E_S$ with $a \le b$ then $a \circ b = b \circ a = a$. Hence,

$$\phi(a) = \phi(b \circ a) = \phi(b) \circ \phi(a) \le \phi(b)$$

Therefore, $\phi\colon E_S \to F_S$ preserves order. If $a \in E_S$,

$$\phi(a) \circ \phi(a') = \phi(a \circ a') = \phi(0) = 0$$

Applying Lemma 3.3(ii) we have that $\phi(a') \leq \phi(a)'$. Now $\phi(b) = \phi(a)'$ for some $b \in E_S$. Since

$$\phi(a \circ b) = \phi(a) \circ \phi(b) = \phi(a) \circ \phi(a)' = 0$$

we have that $a \circ b = 0$. By Lemma 3.3(ii) we have that $b \leq a'$ so that $\phi(a)' = \phi(b) \leq \phi(a')$. Hence, $\phi(a') = \phi(a)'$. If $a, b \in E_S$ and $a \perp b$ then applying Theorem 3.4(iii) gives $a \oplus b = (a' \circ b')'$. Since $\phi(a) \perp \phi(b)$ we have that

$$\begin{aligned}\phi(a \oplus b) &= \phi\left((a' \circ b')'\right) = [\phi(a' \circ b')]' = [\phi(a') \circ \phi(b')]' \\ &= [\phi(a)' \circ \phi(b)']' = \phi(a) \oplus \phi(b)\end{aligned}$$

Also, $\phi^{-1}\colon F_S \to E_S$ preserves order because $\phi(a) \leq \phi(b)$ implies that

$$\phi(b \circ a) = \phi(b) \circ \phi(a) = \phi(a)$$

Hence, $a = b \circ a \leq b$. Thus, $\phi\colon E_S \to F_S$ is a monomorphism and therefore is an isomorphism. □

The following example show that Theorem 5.8 cannot be strengthened. Let $E = [0,1]^X$ be a fuzzy set system. Define $\phi\colon E \to E$ by $\phi(f) = f^2$. Then ϕ is a bijection and $o(f \circ g) = \phi(f) \circ \phi(g)$. But if $f \notin E_S$ then

$$o(f') = (1-f)^2 \neq 1 - f^2 = \phi(f)'$$

Moreover, if $f, g \notin E_S$ with $f \perp g$, then $\phi(f \oplus g) \neq \phi(f) \oplus \phi(g)$.

6 Sharpness Index

For $n \in \mathbb{N}$, $a \in E$ we define $a^n = a \circ a \circ \cdots \circ a$ (n factors).

Lemma 6.1. (i) *We have that $a^n \in E_S$ if and only if $a^{n+1} = a^n$.* (ii) *Also, $a^n \in E_S$ if and only if $a^m = a^n$ for all $m \geq n$.*

Proof. (i) If $a^n \in E_S$, then $a^{2n} = a^n$. But $a^{2n} \leq a^{n+1} \leq a^n$ so that $a^{n+1} = a^n$. Conversely, if $a^{n+1} = a^n$ then we have that

$$a^n = a^{n+1} = \cdots = a^{2n} = (a^n)^2$$

so that $a^n \in E_S$. Moreover, (ii) follows from (i). □

If there exists an m such that $a^m \in E_S$, then the smallest $n \in \mathbb{N}$ such that $a^n \in E_S$ is the **sharpness index** of a. If no such m exists, then the sharpness index of a is ∞. We denote the sharpness index of a by $s(a)$. Of course, $a \in E_S$ if and only if $s(a) = 1$.

Lemma 6.2. *If $n = s(a) < \infty$, then a^n is the largest sharp element below a.*

Proof. Clearly, $a^n \in E_S$ and $a^n \leq a$. Suppose that $b \leq a$ and $b \in E_S$. Then by Theorem 3.4(ii) $b \circ a = a \circ b$. Hence,

$$b \circ a^2 = a^2 \circ b = a \circ b = b$$

and continuing by induction we have that $b \circ a^n = a^n \circ b = b$. Hence, $b \leq a^n$. □

A **σ-effect algebra** is an effect algebra such that $a_1 \geq a_2 \geq \cdots$ implies that $\wedge a_i$ exists. A σ-SEA is a SEA that is a σ-effect algebra E satisfying:

(1) If $a_1 \geq a_2 \geq \cdots$, then $b \circ (\wedge a_i) = \wedge(b \circ a_i)$ for every $b \in E$;

(2) If $a_1 \geq a_2 \geq \cdots$ and $b \mid a_i$, $i = 1, 2, \ldots$, then $b \mid \wedge a_i$.

It can be shown that $\mathcal{E}(H)$ is a σ-SEA.

Theorem 6.3. *Let E be a σ-SEA. If $a \in E$, then there exist $b, c \in E_S$ such that b is the largest sharp element below a and c is the smallest sharp element above a.*

Proof. If $n = s(a) < \infty$, then by Lemma 6.2, a^n is the largest sharp element below a. Suppose that $s(a) = \infty$. Since $a \geq a^2 \geq \cdots$, $b = \wedge a^n$ exists. Now by (1) we have that

$$a^m \circ b = a^m \circ (\bigwedge_n a^n) = \bigwedge_n a^{n+m} = b$$

Since $a^m \mid a^n$ for $n = 1, 2, \ldots$, applying (2) gives $a^m \mid b$. Hence, for all $m \in \mathbb{N}$ we have that

$$b = a^m \circ b = b \circ a^m$$

Therefore,

$$b^2 = b \circ (\wedge a^n) = \wedge(b \circ a^n) = b$$

so that $b \in E_S$. Clearly, $b \leq a$. Suppose that $d \leq a$ with $d \in E_S$. Then $d = d \circ a = a \circ d$ so that $d \leq a^n$ for all $n \in \mathbb{N}$. Hence, $d \leq \wedge a^n = b$. We conclude that b is the largest sharp element below a. Let e be the largest sharp element below a' and let $c = e'$. Then $a \leq e' = c$ and $c \in E_S$. If $a \leq f$ where $f \in E_S$, then $f' \leq a'$. Hence, $f' \leq e$ so that $c = e' \leq f$. Hence, c is the smallest sharp element above a. □

Theorem 6.3 shows that a σ-SEA is sharply dominating [6]. In particular, $\mathcal{E}(H)$ is sharply dominating. We denote the isotropic index of an element a by $i(a)$.

Theorem 6.4. *If $1 < s(a) < \infty$, then there exists an $m \in \mathbb{N}$ such that $a^m \circ (a^m)' \neq 0$ and $i\left[a^m \circ (a^m)'\right] = \infty$.*

Proof. Suppose that $s(a) = 2$. Since $a \notin E_S$ we have that $a \circ a' \neq 0$. By Lemma 6.1(ii), $a^3 = a^2$. Then

$$a^2 = a^3 \oplus a^2 \circ a' = a^2 \oplus a^2 \circ a'$$

implies that $a^2 \circ a' = 0$. Now

$$a' = (a')^2 \oplus a' \circ a$$

implies that

$$a' \circ a = a \circ a' = a \circ (a')^2 \oplus a^2 \circ a' = a \circ (a')^2 = (a')^2 \circ a$$

Hence,

$$(a')^3 \circ a = (a')^2 \circ a = a' \circ a$$

Continuing by induction, we have that $(a')^n \circ a = a' \circ a$ for every $n \in \mathbb{N}$. Now

$$(a')^2 = (a')^3 \oplus (a')^2 \circ a = (a')^3 \oplus a' \circ a$$

It follows that

$$a' = \left[(a')^3 \ominus a' \circ a\right] \oplus a' \circ a$$

Hence, $2(a' \circ a)$ is defined and $a' = (a')^3 \oplus 2(a' \circ a)$. In a similar way

$$(a')^3 = (a')^4 \oplus (a')^3 \circ a = (a')^4 \oplus a' \circ a$$

so that

$$a' = \left[(a')^4 \ominus a' \circ a\right] \oplus 2(a' \circ a)$$

Hence, $3(a' \circ a)$ is defined and $a' = (a')^4 \oplus 3(a' \circ a)$. Continuing by induction, we conclude that $n(a' \circ a)$ is defined for all $n \in \mathbb{N}$. Hence, $i(a \circ a') = \infty$.

For the general case, assume that $s(a) = n$ where $1 < n < \infty$. Suppose that n is even and $n = 2m$. Then $(a^m)^2 = a^{2m} \in E_S$ and since $m \leq 2m - 1 = n - 1$ we have $a^m \notin E_S$. Hence, $s(a^m) = 2$. It follows from our previous work that $a^m \circ (a^m)' \neq 0$ and $i\left[a^m \circ (a^m)'\right] = \infty$. Finally, suppose that n is odd and $n = 2m - 1$. Then $(a^m)^2 = a^{2m} \in E_S$. Since $m \leq 2m - 2 = n - 1$, we have $a^m \notin E_S$. Hence, $s(a^m) = 2$ and again $a^m \circ (a^m)' \neq 0$ and $i\left[a^m \circ (a^m)'\right] = \infty$. ☐

Corollary 6.5. *If E is isotropically finite, then for every $a \in E \setminus E_S$ we have $s(a) = \infty$.*

It is easy to see that $\mathcal{E}(H)$ is isotropically finite. However, the nonstandard unit interval ${}^*[0,1]$ with its usual product is a SEA that is not isotropically finite. In fact, every infinitesimal in ${}^*[0,1]$ has infinite isotropic index. It follows from Corollary 6.5 that every unsharp element of $\mathcal{E}(H)$ has infinite sharpness index. The converse of Corollary 6.5 does not hold. For example, in ${}^*[0,1]$ every unsharp element has infinite sharpness index. In the next section we shall show that there exists a SEA with an element a satisfying $1 < s(a) < \infty$.

Corollary 6.6. *Let E be an isotropically finite SEA.* (i) *Every atom of E is sharp.* (ii) *If E is atomic then E is an orthoalgebra.*

Proof. (i) If a is an atom, since $a^2 \leq a$ we have $a^2 = a$ or $a^2 = 0$. In the latter case $s(a) = 2$ which contradicts Corollary 6.5. Hence, a is sharp.
(ii) Suppose that E is not an orthoalgebra. Then there exists a $b \in E$ with $b \wedge b' \neq 0$. Since E is atomic, there exists an atom a with $a \leq b, b'$. But then $a \leq b \leq a'$ so that $a \notin E_S$. This contradicts (i). ☐

Theorem 6.7. (i) *If $a \in E_S$ and $b \in E$ with $b \perp a$, then $(a \oplus b)^n = a \oplus b^n$.* (ii) *If E is a σ-SEA and $a, b \in E$ with $b \perp a$, then $\bigwedge_n (a \oplus b^n) = a \oplus \bigwedge_n b^n$.*

Proof. (i) Since $b \leq a'$ we have $b \circ a' = a' \circ b = b$. Hence. $a \circ b = 0$. We now prove the result by induction n. The result certainly holds for $n = 1$. Assuming the result holds for n we have that

$$\begin{aligned}(a \oplus b)^{n+1} &= (a \oplus b)^n \circ (a \oplus b) = (a \oplus b^n) \circ (a \oplus b) \\ &= (a \oplus b^n) \circ a \oplus (a \oplus b^n) \circ b = a \circ (a \oplus b^n) \oplus b \circ (a \oplus b^n) \\ &= a \oplus b^{n+1}\end{aligned}$$

(ii) Since $a \oplus \wedge b^n \leq a \oplus b^n$, we have that $a \oplus \wedge b^n \leq \wedge(a \oplus b^n)$. Since $a \oplus b^n \geq a$, we have that $\wedge(a \oplus b^n) \geq a$. From $\wedge(a \oplus b^n) \leq a \oplus b^n$ it follows that $\wedge(a \oplus b^n) \ominus a \leq b^n$. Hence, $\wedge(a \oplus b^n) \ominus a \leq \wedge b^n$ so that $\wedge(a \oplus b^n) \leq a \oplus \wedge b^n$. □

If E is a σ-SEA we have seen that for any $a \in E$ there exists a largest sharp element below a. We denote this element by $\lfloor a \rfloor$.

Corollary 6.8. *Let E be a σ-SEA with $a, b \in E_S$ and $c, d \in E$.* (i) *If $a \perp c$, $b \perp d$ and $a \oplus c = b \oplus d$, then*

$$\lfloor a \oplus c \rfloor = a \oplus \lfloor c \rfloor = b \oplus \lfloor d \rfloor = \lfloor b \oplus d \rfloor$$

(ii) *If $\lfloor c \rfloor = \lfloor d \rfloor = 0$, $a \perp c$, $b \perp d$ and $a \oplus c = b \oplus d$, then $a = b$ and $c = d$.*
(iii) *If $b \perp d$, $\lfloor d \rfloor = 0$ and $a \leq b \oplus d$, then $a \leq b$.*

Proof. (i) By Theorem 6.7(i) we have

$$a \oplus c^n = (a \oplus c)^n = (b \oplus d)^n = b \oplus d^n$$

By Theorem 6.7(ii) and the proof of Theorem 6.3 we have

$$a \oplus \lfloor c \rfloor = a \oplus \wedge c^n = \wedge(a \oplus c^n) = \wedge(b \oplus d^n) = b \oplus d^n = b \oplus \lfloor d \rfloor$$

Moreover.

$$\lfloor a \oplus c \rfloor = \wedge(a \oplus c)^n = \wedge(a \oplus c^n) = a \oplus \wedge c^n = a \ominus \lfloor c \rfloor$$

(ii) By (i) we have

$$a = a \oplus \lfloor c \rfloor = b \oplus \lfloor d \rfloor = b$$

Hence, $c = d$ by cancellation.
(iii) Since $a \leq b \oplus d$, there exists a $c \in E$ such that $a \oplus c = b \oplus d$. Applying (i) gives

$$a \leq a \oplus \lfloor c \rfloor = b \oplus \lfloor d \rfloor = b$$

□

Theorem 6.9. *If E is a σ-SEA then any $a \in E$ has a unique representation $a = b \oplus c$ where $b \in E_S$ and $\lfloor c \rfloor = 0$.*

Proof. It is clear that $a = \lfloor a \rfloor \oplus (a \ominus \lfloor a \rfloor)$. If $b \in E_S$ and $b \leq a \ominus \lfloor a \rfloor$, then $b \leq a$. Hence, $b \leq \lfloor a \rfloor$ and $b \leq \lfloor a \rfloor'$ so that $b = 0$. Therefore, $\lfloor a \ominus \lfloor a \rfloor \rfloor = 0$. This gives the desired representation. To show uniqueness, suppose we have two such representations $a = b \oplus c = b_1 \oplus c_1$, $b, b_1 \in E_S$, $\lfloor c \rfloor = \lfloor c_1 \rfloor = 0$. Then by Corollary 6.8(ii) we have $b = b_1$ and $c = c_1$. □

7 An Example

This section presents an example of a SEA that has an element a satisfying $s(a) = 2$. Let $E = \omega + \omega^*$ be the set with elements

$$\{0, 1, a, 2a, \ldots, a', (2a)', \ldots\}$$

By convention, we define $0a = 0$. Defining $\oplus$ on E by

$$(ma) \oplus (na) = (m + n)a$$

and when $n \leq m$

$$(ma)' \oplus (na) = (na) \oplus (ma)' = ((m - n)a)'$$

it is easy to see that $(E, 0, 1, \oplus)$ is an effect algebra [1]. Moreover, $ja \leq (ka)'$ for every $j, k \in \mathbb{N}$ because

$$(ja) \oplus ((k + j)a)' = (ka)'$$

Theorem 7.1. *There is a unique sequential product on the effect algebra $E = \omega + \omega^*$.*

Proof. For $x, y \in E$ define

$$x \circ y = \begin{cases} 0 & \text{if } x = ma, y = na \\ x \wedge y & \text{if } x = ma,\ y = (na)' \text{ or } x = (ma)',\ y = na \\ ((m+n)a)' & \text{if } x = (ma)',\ y = (na)' \end{cases}$$

It is clear that $1 \circ x = x$ for every $x \in E$. Since E is commutative under $\circ$, in order to show that $\circ$ is a sequential product we only need to verify

(D) $x \circ (y \oplus z) = (x \circ y) \oplus (x \circ z)$, $y \perp z$

(A) $x \circ (y \circ z) = (x \circ y) \circ z$

There are eight possibilities for x, y, z.

	x	y	z
(1)	ra	ma	na
(2)	ra	ma	$(na)'$
(3)	ra	$(ma)'$	na
(4)	ra	$(ma)'$	$(na)'$
(5)	$(ra)'$	ma	na
(6)	$(ra)'$	ma	$(na)'$
(7)	$(ra)'$	$(ma)'$	na
(8)	$(ra)'$	$(ma)'$	$(na)'$

To verify (D) we only need to consider cases (1), (2), (5) and (6) because (2) and (3) as well as (6) and (7) are symmetric and (4) and (8) never occur because $y \not\perp z$ in these cases. In Case (1) both the left and right hand sides are 0. In Case (2) both the left and right hand sides are x. In Case (5) both the left and right hand sides are $y \oplus z$. In Case (6) both the left and right hand sides are $((n - m + r)a)'$.

To verify (A), we check all eight cases.

(1) $ra \circ (ma \circ na) = ra \circ 0 = 0 = 0 \circ (na) = (ra \circ ma) \circ na$

(2) $ra \circ (ma \circ (na)') = ra \circ ma = 0 = 0 \circ (na)' = (ra \circ ma) \circ (na)'$

(3) $ra \circ ((ma)' \circ na) = ra \circ na = 0 = (ra \circ (ma)') \circ na$

(4) $ra \circ ((ma)' \circ (na)') = ra \circ ((m+n)a)' = ra = ra \circ (na)'$

$$= (ra \circ (ma)') \circ (na)'$$

(5) $(ra)' \circ (ma \circ na) = (ra)' \circ 0 = 0 = ma \circ na = ((ra)' \circ ma) \circ na$

(6) $(ra)' \circ (ma \circ (na)') = (ra)' \circ ma = ma = ma \circ (na)'$

$$= ((ra)' \circ ma) \circ (na)'$$

(7) $(ra)' \circ ((ma)' \circ na) = (ra)' \circ na = na = ((r+m)a)' \circ na$

$$= ((ra)' \circ (ma)') \circ na$$

(8) $(ra)' \circ ((ma)' \circ (na)') = (ra)' \circ ((m+n)a)' = ((r+m+n)a)'$

$$= ((r+m)a)' \circ (na)' = ((ra)' \circ (ma)') \circ (na)'$$

It follows that $(E, \circ)$ is a SEA.

For uniqueness, suppose that $\bullet\colon E \to E$ is another sequential product on E. Since a is an atom and $a \bullet a \leq a$, we have $a \bullet a = 0$ or $a \bullet a = a$. But $a \leq a'$ so that $a \notin E_S$. Hence, $a \bullet a = 0$. Therefore, $na \bullet ma = nma \bullet a = 0$ for every $n, m \in \mathbb{N}$. Since every $x \in E$ has the form na or $(na)'$, it is clear that E is commutative under $\bullet$. For $x = ma$, $y = (na)'$ we have that

$$x \bullet y = ma \bullet (na)' = (ma \bullet na) \oplus (ma \bullet (na)') = ma = x \wedge y$$

Applying Theorem 3.4(iii) we have that

$$((na)' \bullet (ma)')' = na \oplus ma = (n+m)a$$

Hence, $(na)' \bullet (ma)' = ((n+m)a))'$. We conclude that $x \bullet y = x \circ y$ for every $x, y \in E$. □

The elements $na \in \omega + \omega^*$, $n \neq 0$, satisfy $s(na) = 2$ while the elements $(na)'$, $n \neq 0$, satisfy $s\left((na)'\right) = \infty$. For this example, if $x \perp y$, then $s(x \oplus y) = \max\left(s(x), s(y)\right)$. Of course, this equation does not hold for an arbitrary SEA. Notice that $\omega + \omega^*$ is not isotropically finite and is not a σ-SEA. Since $\omega + \omega^*$ does not admit an order determining set of states, $\omega + \omega^*$ is a commutative SEA that is not isomorphic to a fuzzy set system.

8 Horizontal Sums

Let $(E_i, 0_i, 1_i, \oplus_i)$, $i \in I$, be a collection of effect algebras with card $(I) > 1$. Their **horizontal sum** $E = HS(E_i, i \in I)$ is defined as follows. Identify all the 0_i with a single element 0 and all the 1_i with a single element 1. Let $E_i' = E_i \smallsetminus \{0_i, 1_i\}$, form the disjoint union $\dot{\cup} E_i'$ and let $E = \{0,1\} \cup \dot{\cup} E_i'$. For $a, b \in E_i$ for some $i \in I$ define $a \oplus b = a \oplus_i b$ and no other orthosums are define on E. It is then easy to check that $(E, 0, 1, \oplus)$ is an effect algebra. If the E_i, $i \in I$, are SEA's we now investigate whether $HS(E_i, i \in I)$ admits a sequential product and is thus a SEA. For an arbitrary effect algebra E we use the notation $E' = E \smallsetminus \{0, 1\}$ and we denote the trivial effect algebra $\{0, 1\}$ by 2.

Theorem 8.1. *Let $E_1, E_2 \neq 2$ be effect algebras.* (i) *If E_1 has an atom, then $HS(E_1, E_2)$ does not admit a sequential product.* (ii) *If E_1 is an orthoalgebra, then $HS(E_1, E_2)$ does not admit a sequential product.*

Proof. (i) Suppose that $HS(E_1, E_2)$ admits a sequential product $\circ$. Let $a \in E_1$ be an atom and let $b \in E_2'$. Then $a \circ b = 0$ or $a \circ b' = 0$. If $a \circ b = 0$, then

$$a = a \circ b' = b' \circ a \leq b'$$

which is a contradiction. Similarly, if $a \circ b' = 0$, then

$$a = a \circ b = b \circ a \leq b$$

which is again a contradiction.
(ii) Suppose that $HS(E_1, E_2)$ admits a sequential product $\circ$. Let $a \in E_1'$, $b \in E_2'$. Then $a \circ b \neq 0$ because otherwise we would again obtain $a \leq b'$ which is a contradiction. Similarly, $a \circ b' \neq 0$ so that $0 < a \circ b, a \circ b' < 1$. Since E_1 is an orthoalgebra and $a \circ b \leq a$ we have that

$$a \circ (a \circ b) = (a \circ b) \circ a = a \circ b$$

and $a \mid (a \circ b)'$. Hence,

$$((a \circ b)' \circ a) \circ b = (a \circ b)' \circ (a \circ b) = 0$$

But

$$a = [a \circ (a \circ b)] \oplus [a \circ (a \circ b)'] = a \circ b \oplus [(a \circ b)' \circ a]$$

Since $a = (a \circ b) \oplus (a \circ b')$ and $a \circ b' \neq 0$, we have $a \neq a \circ b$. Hence, $0 < (a \circ b)' \circ a < 1$. It follows that $((a \circ b)' \circ a) \circ b \neq 0$ which is a contradiction. □

Applying Theorem 8.1, if $HS(E_1, E_2)$ is a SEA then neither E_1 nor E_2 can be a nontrivial Boolean algebra or $\omega + \omega^*$. The next result characterizes horizontal sums of SEA's that admit a sequential product and hence form a SEA. If E, F are effect algebras, an additive map $\phi\colon E \to F$ is **positive** if $\phi(a) = 0$ implies $a = 0$.

Theorem 8.2. *Let E_i, $i \in I$, be SEA's and let $E = HS(E_i, i \in I)$. Then E admits a sequential product if and only if for every $a \in E_i'$ there exists a positive additive map $\phi_{ji}^a\colon E_j \to E_i$ such that $\phi_{ji}^a(1) = a$ for every $i, j \in I$ with $i \neq j$ and if $a, b \in E_i'$, $c \in E_j$ with $a \circ b = b \circ a \neq 0$, then $\phi_{ji}^{a \circ b}(c) = a \circ \phi_{ji}^b(c)$.*

Proof. Suppose that E admits a sequential product $\circ$. For $a \in E_i'$, define $\phi_{ji}^a\colon E_j \to E_i$, $j \neq i$, by $\phi_{ji}^a(b) = a \circ b$. Notice that $a \circ b \in E_i$ because $a \circ b \leq a$. Now ϕ_{ji}^a is clearly additive and $\phi_{ji}^a(1) = a$. Suppose that $a \circ b = \phi_{ji}^a(b) = 0$ and $b \neq 0$. Then $a \mid b$ so that $a \mid b'$. Hence,

$$a = a \circ b' = b' \circ a \in E_i \cap E_j$$

But then $a \in \{0, 1\}$ which is a contradiction. Hence, ϕ_{ji}^a is positive. If $a, b \in E_i'$, $c \in E_j$ with $a \circ b = b \circ a \neq 0$, then

$$\phi_{ji}^{a \circ b}(c) = (a \circ b) \circ c = a \circ (b \circ c) = a \circ \phi_{ji}^b(c)$$

Conversely, suppose $\phi_{ji}^a\colon E_j \to E_i$, $i, j \in I$, $i \neq j$, satisfy the given conditions. Define the operation $\circ$ on E by

$$a \circ b = \begin{cases} a \circ b & \text{if } a, b \in E_i \text{ for some } i \in I \\ \phi_{ji}^a(b) & \text{if } a \in E_i',\ b \in E_j',\ i \neq j \in I \end{cases}$$

We now show that $\circ$ is a sequential product on E. If $a, b_1, b_2 \in E_i$ for some $i \in I$ with $b_1 \perp b_2$ then clearly $a \circ (b_1 \oplus b_2) = a \circ b_1 \oplus a \circ b_2$. Otherwise, $a \in E_i'$, $b_1, b_2 \in E_j$, $i \neq j$, and we have that

$$a \circ (b_1 \oplus b_2) = \phi_{ji}^a(b_1 \oplus b_2) = \phi_{ji}^a(b_1) \oplus \phi_{ji}^a(b_2) = a \circ b_1 \oplus a \circ b_2$$

Hence, (S1) holds. It is clear that (S2) holds. To show that (S3) holds, suppose that $a \circ b = 0$. If $a, b \in E_i$ for some $i \in I$, then $b \circ a = 0$ so suppose that $a \in E_i'$, $b \in E_j$, $i \neq j$. Then $\phi_{ji}^a(b) = 0$ so by positivity, $b = 0$. Hence, $b \circ a = 0$. To verify (S4) suppose that $a \circ b = b \circ a$ where $a \in E_i'$, $b \in E_j'$, $i \neq j$. Then

$$a \circ b = \phi_{ji}^a(b) = b \circ a = \phi_{ji}^b(a) \in E_i \cap E_j$$

Hence. $a \circ b \in \{0,1\}$. If $a \circ b = 0$, then $\phi_{ji}^{a}(b) = 0$ so that $b = 0$ which is a contradiction. If $a \circ b = 1$, then $\phi_{ji}^{a}(b) = 1$. Hence,

$$a = \phi_{ji}^{a}(1) \geq \phi_{ji}^{a}(b) = 1$$

so that $a = 1$ which is again a contradiction. We conclude that $a, b \in E_i$ for some $i \in I$. Hence, $a \mid b'$. Moreover, if $a, b \in E_i'$ and $c \in E_j$, $i \neq j$, then

$$a \circ (b \circ d) = a \circ \phi_{ji}^{b}(c) = \phi_{ji}^{a \circ b}(c) = (a \circ b) \circ c$$

To verify (S5), suppose that $c \mid a$ and $c \mid b$. As before, $a, b, c \in E_i$ or some $i \in I$ and the result follows. □

The next result gives a useful method for constructing a SEA from a horizontal sum of SEA's $E = HS(E_i, i \in I)$. Suppose there exist effect algebra morphisms $\phi_{ij}\colon E_i \to E_j$, $i \neq j \in I$. Define the operation $\circ$ on E by

$$a \circ b = \begin{cases} a \circ b & \text{if } a, b \in E_i \text{ for some } i \in I \\ a \circ \phi_{ji}(b) & \text{if } a \in E_i',\ b \in E_j',\ i \neq j \in I \end{cases}$$

Corollary 8.3. *We have that $(E, \circ)$ is a SEA if and only if for every $a \in E_i$, $b \in E_j$, $i \neq j \in I$, $a \circ b = 0$ implies that $a = 0$ or $b = 0$.*

Proof. As in the proof of Theorem 8.2, if $(E, \circ)$ is a SEA and $a \circ b = 0$, then

$$a = a \circ b' = b' \circ a \in E_i \cap E_j$$

Hence, $a \in \{0,1\}$. If $a = 1$, then $b = 0$.

Conversely, assume that $\circ$ satisfies the given condition. For $a \in E_i'$ define the map $\phi_{ji}^{a}\colon E_j \to E_i$, $i \neq j \in I$, by $\phi_{ji}^{a}(b) = a \circ \phi_{ji}(b)$. Then ϕ_{ji}^{a} is additive and $\phi_{ji}^{a}(1) = a$. If $\phi_{ji}^{a}(b) = 0$, then $a \circ b = 0$. Since $a \neq 0$, by assumption $b = 0$. Hence, v is positive. Suppose that $a, b \in E_i'$, $c \in E_j$ with $a \circ b = b \circ a \neq 0$. Then

$$\phi_{ji}^{a \circ b}(c) = (a \circ b) \circ \phi_{ji}(c) = a \circ (b \circ \phi_{ji}(c)) = a \circ \phi_{ji}^{b}(c)$$

It follows from the proof of Theorem 8.2 that $(E, \circ)$ is a SEA. □

Theorem 9.1. *If $X \neq \emptyset$ is a set and E is a SEA, then the SEA tensor product of 2^X and E exists.*

Proof. We call a function $f\colon X \to E$ **simple** if f has a finite number of values and we define

$$T = \{f \in E^X : f \text{ is simple}\}$$

On T define $f \perp g$ if $f(x) \perp g(x)$ for all $x \in X$ and if $f \perp g$ define $(f \oplus g)(x) = f(x) \oplus g(x)$. Defining $0(x) = 0$ and $1(x) = 1$ for all $x \in X$, it is easy to check that $(T, 0, 1, \oplus)$ is an effect algebra. For $f, g \in T$ define $f \circ g(x) = f(x) \circ g(x)$. Again, it is easy to check that $(T, 0, 1, \oplus, \circ)$ is a SEA. Define $\tau\colon 2^X \times E \to T$ by

$$\tau(A, a)(x) = \begin{cases} a & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

We use the notation $\tau(A, a) = \chi_A a$. It is clear that τ is an effect algebra bimorphism. Since

$$\tau(A, a) \circ \tau(B, b) = (\chi_A a) \circ (\chi_B b) = \chi_{A \cap B} a \circ b = \tau(A \circ B, a \circ b)$$

we see that τ is a SEA-bimorphism. Moreover, every $f \in T$ has the unique representation

$$f = \bigoplus_{i=1}^{n} \chi_{A_i} a_i = \bigoplus_{i=1}^{n} \tau(A_i, a_i)$$

where $a_i \neq a_j$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\cup A_i = X$. Let $\beta\colon 2^X \times E \to F$ be a SEA-bimorphism. Define $\phi\colon T \to F$ as follows. If $f = \oplus \chi_{A_i} a_i$ is the unique representation of f, then $\phi(f) = \oplus \beta(A_i, a_i)$. Notice that $\oplus \beta(A_i, a_i)$ is defined because $\oplus \beta(A_i, 1) = 1$ and $\beta(A_i, a_i) \leq \beta(A_i, 1)$. It is straightforward to show that ϕ is a SEA-morphism. Moreover,

$$\beta(A, a) = \phi(\chi_A a) = \phi \circ \tau(A, a)$$ □

A slightly more delicate argument than that used in Theorem 9.1 can be employed to show that the SEA tensor product of a Boolean algebra with an arbitrary SEA always exists.

Let $X \neq \emptyset$ be a set, Q be the rational numbers and define

$$\mathcal{F}(X) = \{u \colon X \to Q \cap [0,1]\}$$

Then $\mathcal{F}(X)$ is a fuzzy set system and thus forms a SEA.

Theorem 9.2. *The SEA tensor product of $\mathcal{F}(X)$ and $\mathcal{E}(H)$ exists.*

Proof. Let $E = \mathcal{E}(H)$ and define the SEA T as in the proof of Theorem 9.1. Define $\tau \colon \mathcal{F}(X) \times E \to T$ by $\tau(u,a)(x) = u(x)a$. Then τ is clearly an effect algebra bimorphism. Since

$$\tau(u,a) \circ \tau(v,b) = (ua) \circ (vb) = uva \circ b = \tau(u \circ v, a \circ b)$$

τ is a SEA-bimorphism. As in Theorem 9.1, every $f \in T$ has the unique representation $f = \oplus\tau(X_{A_i}, a_i)$. Let $\beta \colon \mathcal{F}(X) \times E \to F$ be a SEA-bimorphism. As in Theorem 9.1, define $\phi \colon T \to F$ by $\phi(f) = \oplus\beta(\chi_{A_i}, a_i)$. Again, ϕ is a SEA-morphism. Moreover, if $u \in \mathcal{F}(X)$ then u has the unique representation $u = \sum \lambda_i \chi_{A_i}$ where $\lambda_i \neq \lambda_j$, $A_i \cap A_j = \emptyset$, $i \neq j$ and $\cup A_i = X$. It is straightforward to show that $\beta(ru, a) = \beta(u, ra)$ for every $r \in Q \cap [0,1]$. We then have that

$$\begin{aligned}\beta(u,a) &= \beta(\oplus\lambda_i\chi_{A_i}, a) = \oplus\beta(\lambda_i\chi_{A_i}, a) = \oplus\beta(\chi_{A_i}, \lambda_i a)\\ &= \oplus\phi(\lambda_i\chi_{A_i}, a) = \phi\left(\sum \lambda_i\chi_{A_i}, a\right) = \phi \circ \tau(u,a)\end{aligned}$$

□

References

[1] M. K. Bennett and D. J. Foulis, Interval and scale effect algebras, *Adv. Appl. Math.* **91**, (1977), 200–215.

[2] P. Busch, P. J. Lahti and P. Middlestaedt, *The Quantum theory of Measurements*, Springer-Verlag, Berlin, 1991.

[3] P. Busch, M. Grabowski and P. J. Lahti, *Operation Quantum Physics* Springer-Verlag, Berlin, 1995.

[4] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, New York, 1976.

[5] A. Dvurečenskij, Tensor products of difference posets, *Trans. Amer. Math. Soc.*, **147** (1995), 1043–1057.

[6] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.

[7] D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24**, (1994), 1325–1346.

[8] R. Giuntini and H. Greuling, Toward a formal language for unsharp properties, *Found. Phys.* **19** (1989), 931–945.

[9] R. J. Greechie, D. J. Foulis and S. Pulmannová, The center of an effect algebra, *Order* **12** (1995), 91–106.

[10] S. Gudder, A histories approach to quantum mechanics, *J. Math. Phys.* **39** (1998), 5772–5788.

[11] S. Gudder, Morphisms, tensor products and σ-effect algebras, *Rep. Math. Phys.* **42** (1998), 321–346.

[12] S. Gudder and G. Nagy, Sequentially independent effects, *Proc. Amer. math. Soc.* (to appear).

[13] S. Gudder and G. Nagy, Sequentially quantum measurements, (to appear).

[14] F. Kôpka, D-Posets and fuzzy sets, *Tatra Mountains Publ.* **1** (1992), 83–87.

[15] F. Kôpka and F. Chovanec, D-Posets *Math. Slovaca* **44** (1994), 21–34.

[16] K. Kraus, *States, Effects, and Operations*, Springer-Verlag, Berlin, 1983.

[17] G. Ludwig, *Foundations of Quantum Mechanics*, Springer-Verlag, Berlin, 1983.

[18] L. Molnar, On some automorphisms of the set of effects on Hilbert space, *Letts. math. Phys.* **51** (2000) 37–45.