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States on Orthomodular Amalgamations Over Trees[1]

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An amalgamation of bounded involution posets over a strictly directed graph is introduced and states on this amalgamation are studied. We introduce conditions under which the amalgamation induces a structure that is of the same type as that of the amalgamated structures. We also study circumstances under which common properties of the state spaces (such as unital, full, and strongly order determining) of the amalgamated structures are inherited by the amalgamation.

KEY WORDS: amalgamation; bounded poset; graph; involution poset; orthoalgebras; orthomodular lattice; orthomodular poset; states.

1. INTRODUCTION

In this paper, we define an amalgamation of bounded involution posets over a strictly directed graph. When applied to classes of orthomodular lattices, orthomodular posets, or orthoalgebras, the amalgamation is an involution poset of the same type as the amalgamated posets. If, in addition, the graph is a tree, we study states on this amalgamation and show that every state on a subalgebra of $\mathbf{L}$ induced by a subtree of the tree T can be extended to a state on $\mathbf{L}$. In particular, we show that every state on the amalgamated (pointed) orthoalgebra L_α can be extended to a state on $\mathbf{L}$. We show that sets of positive or unital states on each L_α induce, respectively, a positive or unital set of states on $\mathbf{L}$. Also we show, with some adaptability conditions, that sets of states on each L_α which are strongly order-determining induce a set of states on $\mathbf{L}$ which is strongly order-determining as well. The paper is concluded by proving that if each family of states on L_α is strongly adaptable and full, then the corresponding set of states on $\mathbf{L}$ is full.

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2. BASIC DEFINITIONS

Throughout this paper the symbol ":=" will mean equal by definition. Let $G := (V, E)$ be a *strictly directed graph*, that is, G is a pair of disjoint sets (V, E) such that the set of *vertices* $V \neq \emptyset$, the set of *edges* $E \subseteq (V \times V)\backslash\Delta$, where $\Delta := \{(u, u) \mid u \in V\}$, and such that $(u, v) \in E$ implies $(v, u) \notin E$. An edge $\alpha := (u, v)$ is said to *link* u to v; and we define $\pi_1(\alpha) := u$, $\pi_2(\alpha) := v$. We define $E^{-1} := \{(v, u)|(u, v) \in E\}$ and $(v, u)^{-1} = (u, v)$. A *path in* G *of length* $n \geq 1$ is a sequence of distinct vertices $v_0, v_1, v_2, \ldots, v_n$, $n \geq 1$, such that $(v_i, v_{i+1}) \in E \cup E^{-1}$. Thus, in discussing a path in G, we ignore the direction of the arrows, ignoring the usual convention. A *cycle* in G is a path $v_0 v_1 v_2 \ldots v_n$ with $n \geq 3$, and $(v_0, v_n) \in E \cup E^{-1}$. A graph G is *connected* if any two distinct vertices are joined by a path. *All graphs considered in this paper are strictly directed connected graphs.* A *tree* is a connected graph with no cycles. For distinct $u, v \in V$, the *distance* $d_v(u, v)$ is the length of the shortest path π joining them, if π exists; otherwise $d_v(u, v) = \infty$. We define $d_v(u, u) := 0$. A *rooted tree* T is a tree with a distinguished vertex, its root $r(T)$, such that $d_v(r(T), \pi_1(\alpha)) \leq d_v(r(T), \pi_2(\alpha))$ for every $\alpha \in E$. *All trees considered in this paper are rooted trees.* If $r(T) = \pi_1(\alpha)$ for exactly one α, then T is called a *trunked tree with trunk* α, denoted by T_α. We view a rooted tree T as a partially ordered set $(T, \leq_v)$ with the root $r(T)$ as the bottom element, where $u \leq_v v$ means that $u = r(T)$ or there is a path from $r(T)$ to the vertex v passing through the vertex u. In particular, $\pi_1(\alpha) \leq_v \pi_2(\alpha)$ for every $\alpha \in E$. For $V_1 \subseteq V$, $E_1 \subseteq E$ with $V_1 = \bigcup_{\alpha \in E_1}\{\pi_1(\alpha), \pi_2(\alpha)\}$, if (V_1, E_1) is connected then $T_1 := (V_1, E_1)$ is a *subtree* of T. A tree may be infinite but every subtree that forms a chain is well ordered in the induced ordering on the chain.

An *orthoalgebra* (OA) is a structure $\mathbf{Q} := (Q, \oplus, \mathbf{0}, \mathbf{1})$, where Q is a set with two special elements $\mathbf{0}, \mathbf{1}$ and $\oplus$ is a partially defined binary operation on Q satisfying the following conditions for all $x, y, z \in Q$:

(i) If $x \oplus y$ is defined, then $y \oplus x$ is defined and $x \oplus y = y \oplus x$. (*Commutativity*)
(ii) If $y \oplus z$ and $x \oplus (y \oplus z)$ are both defined, then $x \oplus y$ and $(x \oplus y) \oplus z$ are both defined, and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$. (*Associativity*)
(iii) For every $x \in Q$ there exists a unique $y \in Q$ such that $x \oplus y$ is defined and $x \oplus y = \mathbf{1}$. (*Orthocomplementation*) [Define $x' := y$.]
(iv) If $x \oplus x$ is defined, then $x = \mathbf{0}$. (*Consistency*)

A *bounded involution poset*, $\mathbf{Q} := (Q, \leq, ', \mathbf{0}, \mathbf{1})$, is a poset $(Q, \leq)$ together with a unary mapping $' : Q \to Q$ with $x'' = x$ and if $x \leq y$ then $y' \leq x'$ such that Q contains a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$. We follow the usual convention in referring to Q in place of $\mathbf{Q}$. It is an *orthoposet* if $x \wedge x' = \mathbf{0}$ for every $x \in Q$. An orthoalgebra $(Q, \oplus, \mathbf{0}, \mathbf{1})$ induces an orthoposet $(Q, \leq, ', \mathbf{0}, \mathbf{1})$ by defining $x \leq z$ to mean $x \oplus y = z$ for some $y \in Q$. In any orthoposet Q,

$x \perp y$ means $x \leq y'$. An OA is an *orthomodular poset* (OMP) in case $x \vee y$ exists whenever $x \perp y$. Note that in an OA, $x \oplus y$ is defined precisely when $x \perp y$; and, in an OMP, when $x \perp y,\ x \oplus y = x \vee y$. An *orthomodular lattice* (OML) is an OMP which is a lattice. A *Boolean algebra* is a distributive OML. For Background on orthomodular structures, see Kalmbach (1983).

Let L be a bounded involution poset, and let $A(L)$ be the set of atoms of L, that is, the elements of L immediately above $\mathbf{0}$. We say that L is *atomistic* (respectively, *atomic*) in case every element in L is the join of some set of atoms (respectively, every nonzero element of L dominates an atom). An OML is atomic iff it is atomistic; there are OMPs which are atomic but not atomistic. Define $y\uparrow := \{x \in P \mid y \leq x\}$, and define $y\downarrow$ dually. For $M \subseteq L$, $M' := \{x' \mid x \in M\}$. An atom a is *isolated* if $a\uparrow \cup a\downarrow = \{\mathbf{0}, a, \mathbf{1}\}$. Let $A^*(L) := \{a \in A(L) \mid a$ is not isolated$\}$. For $(a, b) \in A^*(L) \times A^*(L)$, define *the distance between a and b*, denoted by $d^*(a, b)$, by $d^*(a, b) := \min\{n \mid$ there is a sequence $a_0, a_1, \ldots, a_n \in A^*(L)$ with $a_i \perp a_{i+1},\ a = a_0,$ and $b = a_n\}$ or ∞ if no such sequence exists. A *pointed involution poset* $(L, (\iota, \tau))$ is a bounded involution poset L with a distinguished ordered pair of elements $(\iota, \tau) \in A^*(L) \times A^*(L)$ with $\iota \neq \tau$ and $d^*(\iota, \tau) < \infty$; ι and τ are called the *initial* and *terminal* points of L, respectively. Clearly, any involution poset having an atomic horizontal summand having more than four elements can be made into a pointed involution poset.

3. AMALGAMATIONS OVER STRICTLY DIRECTED GRAPHS

A relation on a set X is a subset of $X \times X$. For relations R and S on an involution poset Q, define $R' := \{(x', y') \mid (y, x) \in R\}$, and $R^{-1} := \{(x, y) \mid (y, x) \in R\}$. Let $\{(L_\alpha, (\iota_\alpha, \tau_\alpha)) \mid \alpha \in E\}$ be a family of disjoint pointed involution posets indexed by the edges of the directed graph $G = (V, E)$. Let $L_\circ := \bigcup_{\alpha \in E} L_\alpha$. For $x \in L_o$, we write x_α for x when $x_\alpha \in L_\alpha$. Define $\rho : E \longrightarrow A^*(L) \times A^*(L) \subseteq L_\circ \times L_\circ$ by $\rho(\alpha) := (\iota_\alpha, \tau_\alpha)$ for every $\alpha \in E$. Effectively, ρ identifies the initial point of α with $\iota_\alpha \in L_o$ and the terminal point of α with $\tau_\alpha \in L_o$. Since the L_α's are disjoint, ρ is a one-to-one function. Now use ρ to make the corresponding identifications in L_o, that is, if $\pi_1(\alpha) = \pi_1(\beta)$ then ι_α is identified with ι_β, and so on. Technically, we construct an involution poset L as follows: let $R_0 := \{(\mathbf{0}_\alpha, \mathbf{0}_\beta) \mid$ for every $\alpha, \beta \in E\}$, $R_1 := \{(\tau_\alpha, \iota_\beta) \in L_\circ \times L_\circ \mid \pi_2(\alpha) = \pi_1(\beta)\}$, $R_2 := \{(\tau_\alpha, \tau_\beta) \in L_\circ \times L_\circ \mid \pi_2(\alpha) = \pi_2(\beta)\}$, $R_3 := \{(\iota_\alpha, \iota_\beta) \in L_\circ \times L_\circ \mid \pi_1(\alpha) = \pi_1(\beta)\}$, and $R_4 := \{(\iota_\alpha, \tau_\beta) \in L_\circ \times L_\circ \mid \pi_1(\alpha) = \pi_2(\beta)\}$. Define a relation $\equiv$ on $L_\circ \times L_\circ$ by $\equiv := \Delta \cup \bigcup_{i=0}^{4} (R_i \cup R_i' \cup R_i^{-1} \cup (R_i^{-1})')$. A tedious but elementary argument (Al-Agha and Greechie, 2003) shows that $\equiv$ is an equivalence relation on $L_\circ \times L_\circ$. Denote the equivalence class of x by $[x]$. Let $L := L_\circ/\equiv$. Define $' : L \rightarrow L$ by $[x]' = [x^{'\alpha}]$ if $x \in L_\alpha$. It is not hard to check that $'$ is well defined, and that $[x] \equiv [y]$ iff $[x]' \equiv [y]'$. *For convenience we write x' for* $x^{'\alpha}$, where $x \in L_\alpha$. We say that $[x]$ has a *witness in L_α* or $[x]$ has an *α-witness* whenever there exists $x_\alpha \in L_\alpha$ with $x_\alpha \equiv x$.

Define a relation $\leq$ on L as follows: for $[x], [y] \in L$, write $[x] \leq [y]$ if $[x]$ and $[y]$ have α-witnesses x_α, y_α with $x_\alpha \leq_{L_\alpha} y_\alpha$. It is easy to see that $\leq$ is a partial order on L. We call L the *atomic amalgamation* of $L_\alpha,\ \alpha \in E$, over the strictly directed graph G via ρ, and write $(L; L_\alpha, G, \rho)$ to indicate that $L := L_\circ/\!\equiv$, where $L_\circ$ and $\equiv$ are defined as above using $L_\alpha,\ G$ and ρ. Let $\mathbf{L} := (L; L_\alpha, T, \rho)$ be the atomic amalgamation of pointed involution posets $L_\alpha, \alpha \in E$, over a tree T via ρ. Note that if each $(L_\alpha, \leq_\alpha, {}'^{\alpha})$ is an orthoposet, then $(L, \leq, ')$ is an orthoposet. *For convenience we write A_α for $A(L_\alpha)$.*

An element $x \in K$ is a *middle element* of an OML K if there exist $a, b \in K$ with $\mathbf{0} < a < x < b < \mathbf{1}$. If $[x], [y], [z]$ are distinct elements of $L \backslash \{[\mathbf{0}], [\mathbf{1}]\}$ with $[x] < [y]$ and $[y] < [z]$, then y is a middle element of L_α and $[y] = \{y_\alpha\}$ for exactly one $\alpha \in E$.

Recall that $G = (V, E)$ is a strictly directed graph. For $\alpha = (a, b) \in E$, define $\varphi(\alpha) := \{a, b\}$. Observe that $x \in \varphi(\rho(\alpha)) \cap \varphi(\rho(\beta))$ precisely when one of the following four conditions holds: $\tau_\alpha = x = \iota_\beta$, $\tau_\alpha = x = \tau_\beta$, $\iota_\alpha = x = \iota_\beta$, or $\iota_\alpha = x = \tau_\beta$.

Lemma 3.1. *Let $(L; L_\alpha, G, \rho)$ be the atomic amalgamation of a family of pointed orthomodular posets $\{L_\alpha\}_{\alpha \in E}$ over G via ρ. If $x_\alpha, y_\alpha \in A_\alpha,\ x_\beta, y_\beta \in A_\beta$ with $x_\alpha \equiv x_\beta$, $y_\alpha \equiv y_\beta$, and $x_\alpha \neq y_\alpha$, then $\alpha = \beta$, so $x_\alpha = x_\beta$ and $y_\alpha = y_\beta$.*

Proof. Suppose $x_\alpha \equiv x_\beta,\ y_\alpha \equiv y_\beta$, and $x_\alpha \neq y_\alpha$; then $x_\beta \neq y_\beta$ since $x_\alpha \not\equiv y_\alpha$. If $x_\alpha = \iota_\alpha = \tau_\beta = x_\beta$ and $y_\alpha = \tau_\alpha = \iota_\beta = y_\beta$, then $\rho(\alpha) = (\iota_\alpha, \tau_\alpha) = (\tau_\beta, \iota_\beta) = \rho(\beta^{-1})$. Thus, $\alpha = \beta^{-1}$, so that $\beta, \beta^{-1} \in E$ contradicting the fact that G is strictly directed. Thus, we may assume, by possibly interchanging α and β, that $x_\alpha = \iota_\alpha = \iota_\beta = x_\beta$ and $y_\alpha = \tau_\alpha = \tau_\beta = y_\beta$; then, as above, $\alpha = \beta$ and it follows that $x_\alpha = x_\beta$ and $y_\alpha = y_\beta$. □

Theorem 3.2. *If $(L; L_\alpha, G, \rho)$ is the atomic amalgamation of a family of pointed orthoalgebras $\{L_\alpha\}_{\alpha \in E}$ over G via ρ, then L is an orthoalgebra.*

Proof. For $[x], [y] \in L$ with common α-witnesses x_α, y_α, respectively, such that $x_\alpha \perp_{L_\alpha} y_\alpha$, define $[x] \oplus [y] := [x_\alpha \oplus_{L_\alpha} y_\alpha]$. Then $\oplus$ is well defined and L is an OA (Al-Agha and Greechie, 2003). □

In Greechie (1971), it is proved that, under certain conditions, the union of Boolean algebras is an orthomodular poset (respectively, lattice) iff the order of every atomistic loop in this union is at least 4 (respectively 5). An understanding of the proof of this result indicates that some conditions, on the amalgamation of pointed involution posets over a strictly directed graph, are needed in order to prove that the amalgamation of pointed orthomodular posets (respectively lattices) is a poset of the same type. It will be shown that, under the conditions which we shall present, the amalgamation of pointed orthoalgebras, pointed orthomodular posets, or pointed orthomodular lattices, is a structure of the same type.

The distinct edges $\alpha, \beta, \gamma \in E$ are said to *form a triangle*, denoted by $\triangle(\alpha, \beta, \gamma)$, if there exist $a, b, c \in V$ with $\varphi(\alpha) = \{a, b\}$, $\varphi(\beta) = \{b, c\}$, and $\varphi(\gamma) = \{c, a\}$; and the distinct edges $\alpha, \beta, \gamma, \delta \in E$ are said to *form a square*, denoted by $\square(\alpha, \beta, \gamma, \delta)$, if there exist $a, b, c, d \in V$ with $\varphi(\alpha) = \{a, b\}$, $\varphi(\beta) = \{b, c\}$, $\varphi(\gamma) = \{c, d\}$ and $\varphi(\delta) = \{d, a\}$. Note that, $|\{a, b, c\}| = 3$ when $\triangle(\alpha, \beta, \gamma)$ and $|\{\alpha, \beta, \gamma, \delta\}| = 4$ when $\square(\alpha, \beta, \gamma, \delta)$ since G is strictly directed with no loops. In what follows we write $d^*_\theta(x, y)$ for the distance in L_θ between the non-isolated atoms x, y of L_θ.

3.1. Distancing Conditions

D_1. If $\triangle(\alpha, \beta, \gamma)$, then $d^*_\mu(\iota_\mu, \tau_\mu) \geq 2$ for some $\mu \in \{\alpha, \beta, \gamma\}$,
D_2. If $\triangle(\alpha, \beta, \gamma)$, then $d^*_\nu(\iota_\mu, \tau_\mu) \geq 2$ and $d^*_\nu(\iota_\nu, \tau_\nu) \geq 2$ for distinct $\mu, \nu \in \{\alpha, \beta, \gamma\}$,
D_3. If $\triangle(\alpha, \beta, \gamma)$, then $d^*_\mu(\iota_\mu, \tau_\mu) \geq 3$ for some $\mu \in \{\alpha, \beta, \gamma\}$, and
D_4. If $\square(\alpha, \beta, \gamma, \delta)$, then $d^*_\mu(\iota_\mu, \tau_\mu) \geq 2$ for some $\mu \in \{\alpha, \beta, \gamma, \delta\}$.

If $\{L_\alpha\}_{\alpha \in E}$ satisfies some conditions D_i then we say that the amalgamation $(L; L_\alpha, G, \rho)$, or simply L, satisfies D_i. Note that D_2 or D_3 implies D_1.

Lemma 3.3. *Let $(L; L_\alpha, G, \rho)$ be the atomic amalgamation of a family of pointed orthomodular posets $\{L_\alpha\}_{\alpha \in E}$ over G via ρ satisfying the distancing condition D_1, D_2, or D_3. If $[x], [y], [z] \in L$ with $[x] \perp [y]$ and $[x], [y] \leq [z]$, then there exists $\alpha \in E$ such that $x_\alpha \equiv x$, $y_\alpha \equiv y$, and $z_\alpha \equiv z$ with $x_\alpha \vee_{L_\alpha} y_\alpha \leq_{L_\alpha} z_\alpha$. Moreover, if $[x], [y] \neq [\mathbf{0}]$, then α is unique.*

Proof. Suppose that $[x], [y], [z] \in L$ such that $[x] \perp [y]$ and $[x], [y] \leq [z]$. Then there exist $\alpha, \beta, \gamma \in E$ such that $x_\alpha \equiv x \equiv x_\beta$, $y_\alpha \equiv y \equiv y_\gamma$, and $z_\beta \equiv z \equiv z_\gamma$ with $x_\alpha \leq_{L_\alpha} y'_\alpha$, $x_\beta \leq_{L_\beta} z_\beta$ and $y_\gamma \leq_{L_\gamma} z_\gamma$. We may assume that $x, y, z \neq \mathbf{0}, \mathbf{1}$. If α, β, γ are distinct, then it follows that x, y, z are distinct and $y_\alpha \equiv y_\gamma \perp_{L_\gamma} z'_\gamma \equiv z'_\beta \perp_{L_\beta} x_\beta \equiv x_\alpha \perp_{L_\alpha} y_\alpha$, so we have $x_\alpha, x_\beta, y_\alpha, y_\gamma, z'_\beta, z'_\gamma$ are all atoms and α, β, γ form a triangle with $d^*_\mu(\iota_\mu, \tau_\mu) = 1$ for every $\mu \in \{\alpha, \beta, \gamma\}$ contradicting each D_i, $i = 1, 2, 3$. If $\alpha = \beta$, then $y_\alpha \equiv y_\gamma$ and $z_\alpha \equiv z_\gamma$ with $y_\alpha \neq z_\alpha$ so that $\alpha = \gamma$ by Lemma 3.1. Thus, $\alpha = \beta = \gamma$. It follows that $x_\alpha \equiv x$, $y_\alpha \equiv y$, and $z_\alpha \equiv z$ and $x_\alpha, y_\alpha \leq z_\alpha$. Since $[x] \perp [y]$, $x_\alpha \vee_{L_\alpha} y_\alpha$ exists and $x_\alpha \vee_{L_\alpha} y_\alpha \leq z_\alpha$. The cases $\alpha = \gamma$ and $\beta = \gamma$ are similar and hence are omitted.

Now assume $[x], [y] \neq \mathbf{0}$ and suppose there exists $\beta \in E$ and $x_\beta, y_\beta \in L_\beta$ with $x_\beta \equiv x_\alpha \equiv x$ and $y_\beta \equiv y_\alpha \equiv y$. Since $[x] \perp [y]$, we have $x \neq y$ and $\alpha = \beta$ by Lemma 3.1. $\square$

For any poset $P = (P, \leq)$, two elements $x, y \in P$ are said to be *incomparable*, denoted by $x \| y$, if neither $x \leq y$ nor $y \leq x$ holds; let $\mathrm{inc}(P) := \{(x, y) \in P \times P : x \| y\}$.

For $N \subset L$, define $U(N) := \{m \in L \mid n \leq m$ for every $n \in N\}$.

Theorem 3.4. *Let $(L; L_\alpha, G, \rho)$ be the atomic amalgamation of a family of pointed atomic orthomodular posets $\{L_\alpha\}_{\alpha \in E}$ over G via ρ satisfying the distancing condition D_1, D_2, or D_3. Then L is an orthomodular poset.*

Proof. By Theorem 3.2, it suffices to show that for every $[x], [y] \in L$ with $[x] \perp [y]$, $[x] \vee [y]$ exists in L. We may assume that $[x], [y] \neq \mathbf{0}$. If $[x], [y] \in L$ with $[x] \perp [y]$, then there exists $\alpha \in E$ and α-witnesses x_α, y_α with $x_\alpha \perp_{L_\alpha} y_\alpha$. Since each L_α is an orthomodular poset, $x_\alpha \vee_{L_\alpha} y_\alpha$ exists. By Lemma 3.1, such an α is unique. We will show that $[x] \vee [y] = [x_\alpha \vee_{L_\alpha} y_\alpha]$. Since G is strictly directed graph and L satisfies D_1, D_2, or D_3, it follows that if $x_\alpha \vee_{L_\alpha} y_\alpha = 1_\alpha$, then $U_L([x], [y]) = \{[1]\}$. Thus, we may assume $x_\alpha \vee_{L_\alpha} y_\alpha < 1_\alpha$. Since $x_\alpha, y_\alpha \leq_{L_\alpha} x_\alpha \vee_{L_\alpha} y_\alpha$, we have $[x], [y] \leq [x_\alpha \vee_{L_\alpha} y_\alpha]$. Suppose there exists $[z] \in L$ such that $[x], [y] \leq [z]$. We will show that $[x_\alpha \vee_{L_\alpha} y_\alpha] \leq [z]$. If not, then either $[z] < [x_\alpha \vee_{L_\alpha} y_\alpha]$ or $[z] \| [x_\alpha \vee_{L_\alpha} y_\alpha]$. Suppose $[z] < [x_\alpha \vee_{L_\alpha} y_\alpha]$; then $[x], [y] \leq [z] < [x_\alpha \vee_{L_\alpha} y_\alpha] < [\mathbf{1}]$; by Lemma 3.1, $z \equiv z_\alpha$ and $x_\alpha, y_\alpha \leq z_\alpha < x_\alpha \vee_{L_\alpha} y_\alpha$ which is a contradiction. Thus, we may assume that $[z] \| [x_\alpha \vee_{L_\alpha} y_\alpha]$. Then $[z]$ has no witness in L_α and $[z] \supseteq \{z_\beta, z_\gamma\}$ with $\alpha \neq \beta, \gamma$ and $\beta \neq \gamma$ such that $x_\beta \equiv x_\alpha \equiv x$, $y_\gamma \equiv y_\alpha \equiv y$, and $z_\beta \equiv z_\gamma \equiv z$ with $x_\beta \perp_{L_\beta} z'_\beta$, $x_\alpha \perp_{L_\beta} y_\alpha$, and $y_\gamma \perp_{L_\gamma} z'_\gamma$. Hence, α, β, γ form a triangle with $d^*_\mu(\iota_\mu, \tau_\mu) = 1$ for every $\mu \in \{\alpha, \beta, \gamma\}$, contradicting each of D_1, D_2, and D_3. Therefore, we have $[x_\alpha \vee_{L_\alpha} y_\alpha] \leq [z]$. □

Corollary 3.5. *Let $T = (V, E)$ be a rooted tree and let $\{L_\alpha\}_{\alpha \in E}$ be a family of pointed atomic orthomodular posets. If $(L; L_\alpha, T, \rho)$ is the atomic amalgamation of L_α over T via ρ, then L is an orthomodular poset.*

The above corollary follows from the preceeding theorem since all the distancing conditions are satisfied in every tree. The following lemma is an immediate consequence of the definition of $\leq$ on L.

Lemma 3.6. *Let $(L; L_\alpha, G, \rho)$ be the atomic amalgamation of a family of pointed atomic orthomodular lattices $\{L_\alpha\}_{\alpha \in E}$ over G via ρ. If $[x], [y] \in \mathrm{inc}(L)$ such that x and y have no common α-witness, then $U(\{[x], [y]\}) \subset A(L)' \cup \{\mathbf{1}\}$.*

The proof of the following lemma is an immediate consequence of Lemma 3.1.

Lemma 3.7. *Let* $(L; L_\alpha, G, \rho)$ *be the atomic amalgamation of a family of pointed atomic orthomodular lattices* $\{L_\alpha\}_{\alpha\in E}$ *over* G *via* ρ. *Let* $[x], [y], [z] \in L$ *with* $[x] \neq [y]$ *and* $[\mathbf{0}] < [x], [y] \leq [z] < [\mathbf{1}]$. *If both* $[x], [y]$ *have an* α*-witness, then* $[z]$ *has an* α*-witness.*

Lemma 3.8. *Let* $(L; L_\alpha, G, \rho)$ *be the atomic amalgamation of a family of pointed atomic orthomodular lattices* $\{L_\alpha\}_{\alpha\in E}$ *over* G *via* ρ *satisfying the distancing conditions* D_2, D_3, *or both* D_1 *and* D_4. *If* $[x], [y] \in L$ *such that* $[x] \vee [y]$ *does not exist in* L, *then there exist distinct* $[w], [z] \in A(L)'$ *with* $[w], [z] \in U(\{[x], [y]\})$.

Proof. Suppose that $[x] \vee [y]$ does not exist in L. Then $U(\{[x], [y]\})\backslash\{[\mathbf{1}]\} \neq \emptyset$ and for every $[z] \in L$ with $[x], [y] < [z] < [\mathbf{1}]$ there exists $[w] \in L$ with $[x], [y] < [w]$ but $[z] \not\leq [w]$. Thus, for such elements, $[w] < [z]$ or $[w] \| [z]$. If $[w] < [z]$, then there exists two coatoms greater than $[w]$ and hence greater than $[x]$ and $[y]$. Thus, we may assume that $[w] \| [z]$. If $[z], [w] \in A'_L$ we are done; and if not there exist two coatoms above whichever is not a coatom. □

Theorem 3.9. *Let* $(L; L_\alpha, G, \rho)$ *be the atomic amalgamation of a family of pointed atomic orthomodular lattices* $\{L_\alpha\}_{\alpha\in E}$ *over* G *via* ρ. *If* L *satisfies the distancing conditions* D_2, D_3, *or both* D_1 *and* D_4, *then* L *is an orthomodular lattice.*

Proof. Let $[x], [y] \in L$ and suppose that $[x] \vee [y]$ does not exist in L. Then $\{[x], [y]\} \cap \{[\mathbf{0}], [\mathbf{1}]\} = \emptyset$, and $[x], [y] \notin A(L)'$. We have the following mutually exhaustive cases.

Case 1. $[x], [y] \in A(L)$. If $[x], [y]$ have no common α-witness, then there exist distinct α and β such that $[x]$ has an α-witness and $[y]$ has a β-witness. By Lemma 3.8, there exist distinct $[z], [w] \in A'(L)$ such that $[x], [y] \leq [z], [w]$; that is, there exist $L_\alpha, L_\beta, L_\gamma, L_\delta$ and there exist $x_\alpha, z'_\alpha \in L_\alpha$, $y_\beta, z'_\beta \in L_\beta$, $y_\gamma, w'_\gamma \in L_\gamma$, and $x_\delta, w'_\delta \in L_\delta$ with $x_\alpha \equiv x_\delta \equiv x$, $z_\alpha \equiv z_\beta \equiv z$, $y_\beta \equiv y_\gamma \equiv y$, and $w_\gamma \equiv w_\delta \equiv w$ such that $x_\alpha \perp_{L_\alpha} z'_\alpha$, $y_\beta \perp_{L_\beta} z'_\beta$, $y_\gamma \perp_{L_\gamma} w'_\gamma$, and $x_\delta \perp_{L_\delta} w'_\delta$. We claim that $\beta \neq \gamma$, $\gamma \neq \delta$, and $\alpha \neq \delta$, else say $\beta = \gamma$, the edges α, β, γ form a triangle with $d^*_\beta(\iota_\beta, \tau_\beta) = 2$ contradicting D_1, D_2, and D_3. Proving $\gamma \neq \delta$ and $\alpha \neq \delta$ follows by symmetry. Thus, $\alpha, \beta, \gamma, \delta$ are distinct with $d^*_\theta(\iota_\theta, \tau_\theta) = 1$ for every $\theta \in \{\alpha, \beta, \gamma, \delta\}$ contradicting D_4. If $[x]$ and $[y]$ have a common α-witness then we may deduce, by a similar argument, the existence of a triangle or a square, contradicting D_1 (and therefore D_2 and D_3) or D_4, respectively.

Case 2. $[x] \in A(L)$ and $[y] \notin A(L)$ or $[y] \in A(L)$ and $[x] \notin A(L)$. By symmetry, we may assume that $[x] \in A(L)$ and $[y] \notin A(L)$. It follows that $[y]$ is a singleton. Suppose that $[x]$ and $[y]$ have no common α-witness. Since $[x] \vee [y]$ does not exist, Lemma 3.8 implies that there exist distinct $[z], [w] \in A'(L)$ such that $[x], [y] \leq [z], [w]$; necessarily $[z] \| [w]$. Thus, there exist $L_\alpha, L_\beta, L_\gamma$ and $x_\alpha, z_\alpha \in L_\alpha,\ z_\beta, w_\beta \in L_\beta,\ x_\gamma, w_\gamma \in L_\gamma$ with $x_\alpha < z_\alpha,\ x_\gamma < w_\gamma$, and $y_\beta < z_\beta, w_\beta$. Thus, α, β, γ form a triangle with $d^*_\theta(\iota_\theta, \tau_\theta) = 1$ for every $\theta \in \{\alpha, \beta, \gamma\}$ contradicting D_1, D_2, and D_3. Thus, we may assume that $[x]$ and $[y]$ have common α-witnesses, and $[y] = \{y_\alpha\}$ and $x \equiv x_\alpha$. Since $[x] \vee [y]$ does not exist, $[x_\alpha \vee_{L_\alpha} y_\alpha] \neq [x] \vee [y]$. Then there exists $[z] \in L$ with $[x], [y] \leq [z]$ but $[x_\alpha \vee_{L_\alpha} y_\alpha] \not\leq [z]$. Since L_α is an OML, $[z]$ has no α-witness. Thus, $[y]$ and $[z]$ have no common witness, contradicting $[y] \leq [z]$.

Case 3. $[x], [y] \notin A(L)$. It follows that both $[x]$ and $[y]$ are singletons. Suppose $[x]$ and $[y]$ have no common α-witness. Then there exist $\alpha \neq \beta,\ x_\alpha \in L_\alpha, y_\beta \in L_\beta$ such that $[x] = \{x_\alpha\}$ and $[y] = \{y_\beta\}$. Since $[x] \vee [y]$ does not exist, Lemma 3.8 implies that there exist distinct $[z], [w] \in A'(L)$ such that $[x], [y] \leq [z], [w]$. Since $[x], [y]$ are singletons, $[z]$ and $[w]$ have witnesses in L_α and in L_β, respectively; that is, there exist $z_\alpha, w_\alpha \in L_\alpha,\ z_\beta, w_\beta \in L_\beta,\ z_\alpha \equiv z_\beta \equiv z$ and $w_\alpha \equiv w_\beta \equiv w$ with $x_\alpha < z_\alpha, w_\alpha$ and $y_\beta < z_\beta, w_\beta$. Hence, $\iota_\alpha = \tau_\beta$ and $\tau_\alpha = \iota_\beta$ contradicting the fact that G is a strictly directed graph. Thus, we may assume that $[x]$ and $[y]$ have a common α-witness in which case a proof similar to that of Case 1 provides a contradiction.

Since we obtain a contradiction in each case, it follows that $[x] \vee [y]$ exists in L for all $[x], [y] \in L$ so that, in the light of Theorem 3.2, L is a lattice. □

Corollary 3.10. *Let $T = (V, E)$ be a rooted tree and let $\{L_\alpha\}_{\alpha \in E}$ be a family of pointed atomic orthomodular lattices. If $(L; L_\alpha, T, \rho)$ is the atomic amalgamation of L_α over T via ρ. Then L is an orthomodular lattice.*

Theorem 3.11. *Let $(L; L_\alpha, G, \rho)$ be the atomic amalgamation of a family of pointed orthomodular posets $\{L_\alpha\}_{\alpha \in E}$ over G via ρ. For $\alpha \in E$, let $[L_\alpha] := \{[x] \mid x \in L_\alpha\}$. Then, for each $\alpha \in E$, $L_\alpha \simeq [L_\alpha]$ and $[L_\alpha]$ is a subalgebra of L.*

Proof. Fix $\alpha \in E$ and define a mapping $f\colon L_\alpha \to [L_\alpha]$ via $f(x) := [x]$. Let $x, y \in L_\alpha$. We claim that $x \leq_{L_\alpha} y$ iff $[x] \leq [y]$. We may assume that $\mathbf{0} < [x] < [y] < \mathbf{1}$. Clearly, if $x \leq_{L_\alpha} y$, then $[x] \leq [y]$. Now suppose that $[x] \leq [y]$. Then $[x]$ and $[y]$ have common β-witnesses, say x_β and y_β such that $x_\beta \leq_{L_\beta} y_\beta$. Since x and y are common α-witnesses to $[x]$ and $[y]$, respectively, and at most one atom (respectively, coatom) of L_α is equivalent to an atom (respectively, coatom) of L_β because G is a strictly directed graph, we have

$\alpha = \beta$ and hence $x \leq_{L_\alpha} y$. Therefore, $x \leq_{L_\alpha} y$ iff $f(x) \leq f(y)$ so that f is an order embedding. Clearly $f(x') = (f(x))'$ and f is onto, so that $L_\alpha \simeq [L_\alpha]$. It follows easily that $[L_\alpha]$ is a subalgebra of L. □

4. STATES ON AMALGAMATIONS

In what follows we assume that $G = T = (V, E)$ is a tree and that $(L = L_T; L_\alpha, T, \rho)$ is the atomic amalgamation of a family of pointed orthomodular posets $\{L_\alpha\}_{\alpha \in E}$ over T via ρ. Hence, every edge β of G has a unique distance $d(\alpha, \beta)$ from a fixed edge α. *Henceforth, for convenience, we write x for $[x]$ and L_α for $[L_\alpha]$.* Recall that L_α is a pointed orthoalgebra for every $\alpha \in E$ with the two distinguished points ι_α and τ_α such that $\iota_\alpha \neq \tau_\alpha$. Note that L_α may contain only two atoms and $\cup_{\alpha \in E}\{\iota_\alpha, \tau_\alpha\} \subseteq A^*(L)$. Also, *now assume that $\iota_\alpha \not\perp \tau_\alpha$.*

A *state on* an orthoalgebra K is a mapping $s : K \to [0, 1]$ such that $s(0) = 0$, $s(1) = 1$ and for $x, y \in K$, $s(x \vee y) = s(x) + s(y)$ whenever $x \perp y$. Let $\mathcal{S}_K$ be a set of states on **K**. Then $\mathcal{S} \subseteq \mathcal{S}_K$ is *full* if, for every $x, y \in K$, $s(x) \leq s(y)$ for every $s \in \mathcal{S}$ implies $x \leq y$. (Note that $\mathcal{S}$ is full iff $x \not\perp y$ implies there exists $s \in \mathcal{S}$ such that $s(x) + s(y) > 1$.) $\mathcal{S}$ is *strongly order determining* (SOD) if, for all $x, y \in K$, if $s(x) = 1$ implies $s(y) = 1$ for every $s \in \mathcal{S}$ then $x \leq y$. (Note that $\mathcal{S}$ is SOD iff $x \not\leq y$ implies there exists $s \in \mathcal{S}$ such that $s(x) = 1$ and $s(y) > 0$.) $\mathcal{S}$ is *unital* if for every $x \neq 0$ there exists $s \in \mathcal{S}$ such that $s(x) = 1$. $\mathcal{S}$ is *positive* if, for every $x \neq 0$, there exists $s \in \mathcal{S}$ such that $s(x) > 0$. And $\mathcal{S}$ is *dispersion free* if $s(x) \in \{0, 1\}$ for every $s \in \mathcal{S}$ and for every $x \in K$. Let $\mathcal{S}_K^{\mathrm{DF}}$ be the set of all dispersion free states on K. For $x \in K$ and $\mathcal{S} \subseteq \mathcal{S}_K$, define the *$\mathcal{S}$-spectrum of x*, denoted by $\mathrm{spec}_{\mathcal{S}}(\mathrm{x})$, by $\mathrm{spec}_{\mathcal{S}}(x) := \{s(x) \mid s \in \mathcal{S}\}$. We use $\mathrm{spec}(x)$ when $\mathcal{S}$ is understood.

For edges $\alpha, \beta \in E$, write $\alpha \sim \beta$ when $\tau_\alpha \equiv \iota_\beta$, $\tau_\beta \equiv \iota_\alpha$, or $\iota_\alpha \equiv \iota_\beta$. Since G is a strictly directed tree, for every $\alpha, \beta \in E$ there exists a unique path $\alpha = \alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_n = \beta$. Define $\pi(\alpha, \beta) := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, with $\alpha = \alpha_1$ and $\beta = \alpha_n$, and let $L_{\alpha\beta} := \cup_{\gamma \in \pi(\alpha,\beta)} L_\gamma$. Note that $L_{\alpha\alpha} = L_\alpha$. Let $\mathcal{S}_{L_{\alpha\beta}}$ be the set of all states on $L_{\alpha\beta}$, let $\mathcal{S}_{\alpha\beta} \subseteq \mathcal{S}_{L_{\alpha\beta}}$, and let $\mathcal{S}_\alpha \subseteq \mathcal{S}_{L_\alpha}$ with $\mathcal{S}_\alpha = \mathcal{S}_{\alpha\alpha}$ when $\alpha = \beta$. A family $\Sigma := \{\mathcal{S}_\alpha \mid \alpha \in E\}$, where $\mathcal{S}_\alpha \subseteq \mathcal{S}_{L_\alpha}$, is an *adaptable family* if (i) $\mathrm{spec}_{\mathcal{S}_\alpha}(\iota_\alpha) = \mathrm{spec}_{\mathcal{S}_\beta}(\tau_\beta)$ whenever $\iota_\alpha \equiv \tau_\beta$, (ii) $\mathrm{spec}_{\mathcal{S}_\alpha}(\tau_\alpha) = \mathrm{spec}_{\mathcal{S}_\beta}(\iota_\beta)$, whenever $\tau_\alpha \equiv \iota_\beta$, or (iii) $\mathrm{spec}_{\mathcal{S}_\alpha}(\iota_\alpha) = \mathrm{spec}_{\mathcal{S}_\beta}(\iota_\beta)$, whenever $\iota_\alpha \equiv \iota_\beta$.

Given an adaptable family Σ, and $s_\alpha \in \mathcal{S}_\alpha$ define $s : L \to [0, 1]$ by $s(x) = s_\alpha(x_\alpha)$, whenever $s_\alpha(x_\alpha) = s_\beta(x_\beta)$ in case $x_\alpha \equiv x_\beta$. Let $\mathcal{S}^{\Sigma(L)}$:= all such states. Let $\mathcal{S}^{\Sigma(L_T)} := \{s \in \mathcal{S}_{L_T} \mid s = (s_\alpha)_{\alpha \in E}$ and each $s_\alpha \in \mathcal{S}_\alpha\}$. We use $\mathcal{S}^{\Sigma(L),\mathrm{DF}}$ instead of $\mathcal{S}^{\Sigma(L)}$ whenever every $\mathcal{S}_\alpha$ is dispersion free.

Define

$$\mathrm{spec}_{\mathcal{S}_\alpha}(t; x, r) := \{s_\alpha(t) \mid s_\alpha \in \mathcal{S}_\alpha \text{ and } s_\alpha(x) > r\},$$

where $t \in \{\iota_\alpha, \tau_\alpha\}$, $x \in L_\alpha$, and $r \in [0, 1]$.

A family $\Sigma_s := \{\mathcal{S}_\alpha | \alpha \in E\}$ is said to be a *strongly adaptable family* if it is adaptable and $(\frac{1}{2}, 1] \cap \text{spec}_{\mathcal{S}_\alpha}(\sigma_\alpha; x, r) \cap \text{spec}_{\mathcal{S}_\beta}(\sigma_\beta; y, r) \neq \emptyset$ where $x \in L_\alpha \backslash \{0\}, y \in L_\beta \backslash \{0\}, r \in [0, 1)$ and one of $\sigma_\alpha = \tau_\alpha \equiv \iota_\beta = \sigma_\beta$, $\sigma_\alpha = \iota_\alpha \equiv \tau_\beta = \sigma_\beta$, or $\sigma_\alpha = \iota_\alpha \equiv \iota_\beta = \sigma_\beta$. Note that strongly adaptable implies adaptable.

Lemma 4.1. *f $\Sigma := \{\mathcal{S}_\alpha\}_{\alpha \in E}$ is an adaptable family and $\alpha, \beta \in E$, then every state $s_\alpha \in \mathcal{S}_\alpha$ extends to a state $s \in \mathcal{S}^{\Sigma(L_{\alpha\beta})}$.*

Proof. Let S_n be the statement that if $d(\alpha, \beta) = n$, then every state on L_α extends to a state on $L_{\alpha\beta}$. Note that S_0 is trivially true. Suppose S_k is true and let $d(\alpha, \beta) = k + 1$. Since T is a tree, there exists a unique $\gamma \in E$ such that $d(\alpha, \gamma) = k$ and $d(\gamma, \beta) = 1$. By the induction hypothesis, s_α extends to a state, say, $s_0 \in \mathcal{S}_{\alpha\gamma}$. Then $\tau_\gamma \equiv \iota_\beta, \iota_\gamma \equiv \tau_\beta$, or $\iota_\alpha \equiv \iota_\beta$. Suppose $\tau_\gamma \equiv \iota_\beta$. Since Σ is adaptable family, we can choose $s_\beta \in \mathcal{S}_\beta$ such that $s_0(\tau_\gamma) = s_\beta(\iota_\beta)$. Then $s := s_0 \cup s_\beta$ is a state on $L_{\alpha\beta}$ extending s_0. The possibilities $\iota_\gamma \equiv \tau_\beta$ and $\iota_\alpha \equiv \iota_\beta$ follow similarly. Hence, S_{k+1} is true, completing the proof. □

The proof of the following corollary is straightforward and hence is omitted.

Corollary 4.2. *If each $\mathcal{S}_\alpha \in \Sigma$ is positive or unital, then $\mathcal{S}^{\Sigma(L)}$ is positive or unital, respectively.*

For a subtree $T_1 = (V_1, E_1)$ of a tree $T = (V, E)$ (that may not have the same root), we define the neighborhood $N(T_1)$ of T_1 by $N(T_1) := (V_1^N, E_1^N)$ where $E_1^N := E_1 \cup \{\alpha \mid \alpha \sim \beta \text{ for some } \beta \in E_1\}$, and $V_1^N := \pi_1(E_1^N) \cup \pi_2(E_1^N)$. For $k \geq 1$, we define $N^k(T_1)$ as follows:

$$N^1(T_1) := N(T_1),$$

and, for $k > 1$,

$$N^k(T_1) := N(N^{k-1}(T_1)).$$

Also, for T_1 a subtree of T and for $k \geq 1$, we define

$$N^k(L_{T_1}) = \bigcup_{\alpha \in N^k(T_1)} L_\alpha.$$

Note that $N^k(L_{T_1})$ is a suborthomodular poset of L for each k. In fact, every subtree $T_1 := (V_1, E_1)$ of T induces a sub-orthomodular lattice L_{T_1} of L.

Lemma 4.3. *If $\Sigma := \{\mathcal{S}_\alpha\}_{\alpha \in E}$ is an adaptable family and T_1 is a subtree of T. Then every state in $\mathcal{S}^{\Sigma(L_{T_1})}$ extends to a state in $\mathcal{S}^{\Sigma(L_T)}$.*

Proof (By induction). Let $N^n(L_{T_1}) := (V_n, E_n)$ and let P_n be the statement that for every state $s_1 \in \mathcal{S}^{\Sigma(L_{T_1})}$ there is a state $s^n \in \mathcal{S}^{\Sigma(N^n(L_{T_1}))}$ with $s^n|_{L_\alpha} = s_\alpha \in \mathcal{S}_\alpha$ for every $\alpha \in E_n$. To prove P_1, let $s_1 \in \mathcal{S}^{\Sigma(L_{T_1})}$. If $L_\alpha \in N^1(L_{T_1})\backslash L_{T_1}$, then there exists $a \in A_\alpha \cap A(L_{T_1})$ and $t_\alpha \in \mathcal{S}_{L_\alpha}$ such that $s_1(a) = t_\alpha(a)$. Now the state $s^1 := s_1 \cup \{t_\alpha \,|\, \alpha \in E_1^N \backslash E_1\}$ is a state on $N^1(L_{T_1})$ extending s_1, so P_1 is true. Next, suppose that P_{n-1} is true; we prove that P_n is true. If $s_1 \in \mathcal{S}^{\Sigma(L_{T_1})}$ then, by the induction hypothesis, we can extend s_1 to a state $s^{n-1} \in \mathcal{S}^{\Sigma(N^{n-1}(L_{T_1}))}$. If $L_\beta \in N^{n-1}(L_{T_1})$, then there exists $b \in A_\beta \cap A(N^{n-1}(L_{T_1}))$ and $s_\beta \in \mathcal{S}_{L_\beta}$ such that $s^{n-1}(b) = s_\beta(b)$. Then $s^n := s^{n-1} \cup \{s_\beta \mid \beta \in E_n^N \backslash E_{n-1}^N\}$ is a state on $N^n(L_{T_1})$ extending s_1. This proves that P_n is true and completes the proof. □

Note that, for any $\alpha, \beta \in E$, $L_{\alpha\beta}$ is a sublattice of L which is the atomic amalgamation of a subtree of T and, by the above lemma, we have the following corollary.

Corollary 4.4. *If $\{\mathcal{S}_\alpha\}_{\alpha \in E}$ is an adaptable family, then any state on $L_{\alpha\beta}$ extends to a state on L. In particular, any state in some $\mathcal{S}_\alpha$ extends to a state in $\mathcal{S}^{\Sigma(L)}$.*

Lemma 4.5. *Let $\mathcal{S}_\alpha$ be strongly order determining for every $\alpha \in E$. If $r \in [0, 1)$, $y \neq \iota_\beta$ and $y \not\perp \iota_\beta$ then there exists $s_\beta \in \mathcal{S}_\beta$ such that $s_\beta(y) > 0$ and $s_\beta(\iota_\beta) = r$.*

Proof. Since $\tau_\beta \neq \iota_\beta$, $\tau_\beta \not\perp (\iota_\beta)'$. Thus, there exists $s_0 \in \mathcal{S}_\beta$ with $s_0((\iota_\beta)') = 1$ and $s_0(\tau_\beta) > 0$. Hence, $s_0(\iota_\beta) = 0$. Also, since $y \not\perp \iota_\beta$, there exists $s_1 \in \mathcal{S}_\beta$ with $s_1(\iota_\beta) = 1$ and $s_1(y) > 0$. For $r > 0$, define $s_\beta := rs_1 + (1 - r)s_0$. Then $s_\beta(y) \geq rs_1(y) > 0$, and $s_\beta(\iota_\beta) = rs_1(\iota_\beta) + (1 - r)s_0(\iota_\beta) = r$. Now suppose that $r = 0$. Since $0 \neq y \neq \iota_\beta$, $y \not\perp (\iota_\beta)'$. Thus, there exists $s_\beta \in \mathcal{S}_\beta$ with $s_\beta((\iota_\beta)') = 1$ and $s_\beta(y) > 0$. So $s_\beta(\iota_\beta) = 0$ and $s_\beta(y) > 0$.

Lemma 4.6. *Let $\mathcal{S}_\beta$ be strongly order determining for every $\beta \in E$. Let $y \in L_\beta$ and let $r \in [0, 1)$ be a real number.*

(1) *If $y \perp \iota_\beta$ and $y \not\perp \tau_\beta$ then there exists $s_\beta \in \mathcal{S}_\beta$ such that $s_\beta(\iota_\beta) = r$ and $s_\beta(y) > 0$;*
(2) *If $y \perp \tau_\beta$ and $y \not\perp \iota_\beta$, then there exists $s_\beta \in \mathcal{S}_\beta$ such that $s_\beta(\tau_\beta) = r$ and $s_\beta(y) > 0$;*
(3) *If $y \perp \iota_\beta, \tau_\beta$, then there exist $s_\beta, t_\beta \in \mathcal{S}_\beta$ such that $s_\beta(\iota_\beta) = r$ and $s_\beta(y) > 0$; and $t_\beta(\tau_\beta) = r$ and $t_\beta(y) > 0$.* □

Proof.

(1) Since $\iota_\beta \not\perp \tau_\beta$, there exists $s_{\beta_1} \in \mathcal{S}_\beta$ such that $s_{\beta_1}(\iota_\beta) = 1$ and $s_{\beta_1}(\tau_\beta) > 0$. Hence, $s_{\beta_1}(y) = 0$ because $y \perp \iota_\beta$. Also $y \not\perp \tau_\beta$ implies that there ex-

ists $s_{\beta_2} \in \mathcal{S}_\beta$ such that $s_{\beta_2}(y) = 1$ and $s_{\beta_2}(\tau_\beta) > 0$. Note that $s_{\beta_2}(\iota_\beta) = 0$ because $y \perp \iota_\beta$. Now define $s_\beta \in \mathcal{S}_\beta$ by $s_\beta := rs_{\beta_1} + (1-r)s_{\beta_2}$. Then $s_\beta(\iota_\beta) = rs_{\beta_1}(\iota_\beta) + (1-r)s_{\beta_2}(\iota_\beta) = r$ and $s_\beta(y) = rs_{\beta_1}(y) + (1-r)s_{\beta_2}(y) = 1 - r > 0$.

(2) Follows from (1) by symmetry of hypotheses.

(3) Since $y \perp \iota_\beta, \tau_\beta$, there exist $s_{\beta_1}, s_{\beta_2}, s_{\beta_3} \in \mathcal{S}_\beta$ such that $s_{\beta_1}(\iota_\beta) = 1$, $s_{\beta_1}(\tau_\beta) > 0$, $s_{\beta_2}(\tau_\beta) = 1$, $s_{\beta_2}(\iota_\beta) > 0$, and $s_{\beta_3}(y) = 1$. Since $y \perp \iota_\beta, \tau_\beta$, we get $s_{\beta_1}(y) = 0 = s_{\beta_2}(y)$ and $s_{\beta_3}(\iota_\beta) = 0 = s_{\beta_3}(\tau_\beta)$. Define $s_\beta, t_\beta \in \mathcal{S}_\beta$ as follows: $s_\beta := rs_{\beta_1} + (1-r)s_{\beta_3}$ and $t_\beta := rs_{\beta_2} + (1-r)s_{\beta_3}$. Then $s_\beta(\iota_\beta) = rs_{\beta_1}(\iota_\beta) + (1-r)s_{\beta_3}(\iota_\beta) = r, s_\beta(y) = rs_{\beta_1}(y) + (1-r)s_{\beta_3}(y) = 1 - r > 0, t_\beta(\tau_\beta) = rs_{\beta_2}(\tau_\beta) + (1-r)s_{\beta_3}(\tau_\beta) = r$, and $t_\beta(y) = rs_{\beta_2}(y) + (1-r)s_{\beta_3}(y) = 1 - r > 0$. □

Theorem 4.7. *If $\{\mathcal{S}_\alpha\}_{\alpha \in E}$ is an adaptable family such that each $\mathcal{S}_\alpha$ is strongly order determining, then $\mathcal{S}_L^\Sigma$ is strongly order determining.*

Proof (By induction). Fix $\alpha \in E$ and, for $n \geq 0$, let S_n be the statement: if $x \in L_\alpha, y \in L_\beta$ with $x, y \neq 0, 1$ and $x \not\perp y$ and $d(\alpha, \beta) = n$, then there exists $s \in \mathcal{S}_{\alpha\beta}^\Sigma$ such that $s(x) = 1$ and $s(y) > 0$. If $n = 0$, then $x, y \in L_\alpha$ and the result follows because $\mathcal{S}_\alpha$ is SOD. Because we essentially need it later in the proof, we make the argument for $n = 1$. Note that, in this case, $L_{\alpha\beta} = L_\alpha \cup L_\beta$. We may assume that $\tau_\alpha \equiv \iota_\beta$, $x \neq \tau_\alpha$ and $y \neq \iota_\beta$. (The cases $\iota_\alpha \equiv \tau_\beta$ and $\iota_\alpha \equiv \iota_\beta$ follow similarly.) We have the following cases:

Case I: $x \perp \tau_\alpha$ and $y \perp \iota_\beta$.
Case II: $x \perp \tau_\alpha$ and $y \not\perp \iota_\beta$.
Case III: $x \not\perp \tau_\alpha$ and $y \perp \iota_\beta$.
Case IV: $x \not\perp \tau_\alpha$, and $y \not\perp \iota_\beta$.

In each case we produce a state $s_0 = s_\alpha \cup s_\beta \in \mathcal{S}^{\Sigma(L_{\alpha\beta})}$ by finding appropriate $s_\alpha \in \mathcal{S}_\alpha$ and $s_\beta \in \mathcal{S}_\beta$. In Case I, any pair s_α, s_β with $s_\alpha(x) = 1 = s_\beta(y)$ works since, in this case for such s_α and s_β, $s_\alpha(\tau_\alpha) = 0 = s_\beta(\iota_\beta)$. In Case II, Since $y \not\perp \iota'_\beta$ there exists s_β such that $s_\beta(y) = 1$ and $s_\beta(\iota_\beta) := r > 0$; by Lemma 4.6, parts (1) and (3), there exists s_α such that $s_\alpha(\tau_\alpha) = r$ and $s_\alpha(x) > 0$. Case III follows by symmetry of hypotheses. In Case IV, since $\mathcal{S}_\alpha$ is SOD and $x \not\perp \tau_\alpha$, there exists $s_\alpha \in \mathcal{S}_\alpha$ such that $s_\alpha(x) = 1$ and $r := s_\alpha(\tau_\alpha) > 0$. We need to show that there exists $s_\beta \in \mathcal{S}_\beta$ such that $s_\beta(\iota_\beta) = s_\alpha(\tau_\alpha)$ and $s_\beta(y) > 0$. To show this, notice that $y \not\perp \iota_\beta, \iota'_\beta$ implies that there exist $\sigma_1, \sigma_2 \in \mathcal{S}_\beta$ with $\sigma_1(\iota_\beta) = 1, \sigma_1(y) > 0, \sigma_2(\iota'_\beta) = 1$ (hence $\sigma_2(\iota_\beta) = 0$), and $\sigma_2(y) > 0$. Let $s_\beta := r\sigma_1 + (1-r)\sigma_2$. Then $s_\beta(\iota_\beta) = r\sigma_1(\iota_\beta) + (1-r)\sigma_2(\iota_\beta) = r = s_\alpha(\tau_\alpha) > 0$ and $s_\beta(y) = r\sigma_1(y) + (1-r)\sigma_2(y) > 0$. In conclusion, S_1 is true.

Next suppose that S_{k-1} is true; we prove that S_k is true. Assume $d(\alpha, \beta) = k$. Then there exists a unique path $\alpha = \alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_k = \beta$ because T is a tree. We may assume that $\tau_{\alpha_{i-1}} \equiv \iota_{\alpha_i}$, for every $i \in \{1, 2, \ldots, k\}$ (the cases $\iota_{\alpha_{i-1}} \equiv \iota_{\alpha_i}$ and $\iota_{\alpha_{i-1}} \equiv \iota_{\alpha_i}$ follow similarly). Let $\gamma =: \alpha_{k-1}$. Then $d(\alpha, \gamma) = k - 1$ and $d(\gamma, \beta) = 1$. It follows that $x \not\perp (\tau_\gamma)'$ because $d^*(x, \tau_\gamma) > d^*(\iota_\gamma, \tau_\gamma)$. By the induction hypothesis, there exists $\bar{s} \in \mathcal{S}_{\alpha\gamma}$ such that $\bar{s}(x) = 1$ and $\bar{s}(\tau_\gamma)' > 0$. Hence, $\bar{s}(\tau_\gamma) < 1$. Since $\tau_\gamma \equiv \iota_\beta$, there exists $s_\beta \in \mathcal{S}_\beta$ such that $s_\beta(\iota_\beta) = \bar{s}(\tau_\gamma) < 1$ and $s_\beta(y) > 0$ by Lemma 4.5 in case $y \not\perp \iota_\beta$ or Lemma 4.6 in case $y \perp \iota_\beta$. Now $s_0 := \bar{s} \cup s_\beta$ is the desired state. Therefore, S_n is true for all n. By Corollary 4.4, we extend each s_0 to a state $s = (s_\alpha)_{\alpha\in E} \in \mathcal{S}_L^\Sigma$ such that $s(x) = 1$ implies $s(y) > 0$; and the proof is complete. □

It follows from the above theorem that if $\{\mathcal{S}_\alpha\}_{\alpha\in E}$ is an adaptable family such that each $\mathcal{S}_\alpha \subseteq \mathcal{S}_{L_\alpha}^{\Sigma,\mathrm{DF}}$ is full, then $\mathcal{S}^{\Sigma(L),\mathrm{DF}}$ is full.

Theorem 4.8. *If $\Sigma_s := \{\mathcal{S}_\alpha\}_{\alpha\in E}$ is a strongly adaptable family and each $\mathcal{S}_\alpha$ is full, then $\{s \in \mathcal{S}^{\Sigma(L)} : s|_{L_\alpha} \in \mathcal{S}_\alpha\}$ is full.*

Proof (By induction). For $n \geq 0, \alpha \in E$, let S_n be the statement if $x \not\perp y$ with $x \in L_\alpha$, $y \in L_\beta$, and $d(\alpha, \beta) = n$, then there exists $s_0 \in \mathcal{S}_{\alpha\beta}^\Sigma$ such that $s_0(x) + s_0(y) > 1$. The fact that S_0 is true follows immediately from the hypotheses. Suppose that S_k is true. That is, if $d(\alpha, \gamma) = k$, then for every $x \not\perp y$ there exists $\bar{s} \in \mathcal{S}_{\alpha\gamma}^\Sigma$ such that $\bar{s}|_{L_\delta} = s_\delta$ for every $\delta \in \pi(\alpha, \gamma)$ with $\bar{s}(x) + \bar{s}(y) > 1$. Now we prove that S_{k+1} is true. If $d(\alpha, \beta) = k + 1$, there exists a unique $\gamma \in E$ with $d(\alpha, \gamma) = k$ and $d(\gamma, \beta) = 1$ because T is a tree. We may assume that $\tau_\gamma \equiv \iota_\beta$ (the cases $\iota_\gamma \equiv \tau_\beta$ and $\iota_\gamma \equiv \iota_\beta$ follow similarly). Since Σ_s is strongly adaptable, there exist $t \in \mathcal{S}_{\alpha\gamma}$ and $s_\beta \in \mathcal{S}_\beta$ with $t(x), s_\beta(y) > \frac{1}{2}$ and $t(\tau_\gamma) = r' = s_\beta(\iota_\beta)$. Let $s_0 \in \mathcal{S}_{\alpha\beta}^\Sigma = t$ and $s_0|_{L_\beta} = s_\beta$. Then $s_0(x) + s_0(y) > 1$. Thus, S_n is true for all n. By Corollary 4.4, we extend each s_0 to a state $s = (s_\alpha)_{\alpha\in E} \in \mathcal{S}_L^\Sigma$ such that $s(x) + s(y) > 1$, completing the proof. □

An amalgamation over strictly directed graphs was introduced. States on this amalgamation were studied and it was shown that, under certain conditions, some common properties of these states on the amalgamated posets are carried over to the amalgamation over a tree. It would be desirable to find weaker conditions so that these properties carry over to the amalgamation over a tree. We have left open the question as to which other properties are inherited by the amalgamation over strictly directed graphs. In a future paper, we will address the order dimension of such amalgamations.

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