QUANTUM LOGIC AND PARTIALLY ORDERED ABELIAN GROUPS

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1 INTRODUCTION

Our purpose in this expository article is to give an account of the connection between quantum logics\(^1\) and partially ordered abelian groups. In what follows, we use the word "logic" in the sense of a partially ordered algebraic structure \(L\) that could be interpreted as a semantic model for a formal symbolic logic \(\mathcal{L}\). This usage is customary in the literature of algebraic logic [Halmos, 1962], and it is consistent with the nomenclature of Birkhoff and von Neumann in their seminal article on the logic of quantum mechanics [Birkhoff and von Neumann, 1936]. Our emphasis is on the mathematical structure of \(L\) rather than on the interpretation of its elements as propositions pertinent to a physical system. An account of \(L\) and \(\mathcal{L}\) from the point of view of logic as an instrument of reasoning, as well as more comprehensive treatment of the general notion of a quantum logic, can be found in [Dalla Chiara et al., 2004].

We begin, in Sections 2–5 below, with a brief sketch touching on some of the basic notions of orthodox quantum mechanics (QM). By orthodox QM, we mean non-relativistic Hilbert-spaced-based QM without superselection rules and for which the observables are represented by projection-valued (PV) measures. Readers already knowledgeable about quantum physics can skim these sections rapidly to familiarize themselves with our notation. The relevant quantum logic (QL) for orthodox QM is the complete atomic orthomodular lattice \(\mathbb{P}(\mathcal{H})\) of projection operators on a Hilbert space \(\mathcal{H}\). The QL \(\mathbb{P}(\mathcal{H})\) is a subset of the partially ordered abelian group \(G(\mathcal{H})\) of Hermitian (i.e., bounded self-adjoint) operators on \(\mathcal{H}\). A comprehensive and authoritative account of the standard quantum logic \(\mathbb{P}(\mathcal{H})\) in the context of orthodox QM can be found in [Beltrametti and Cassinelli, 1981].

In Section 6, we turn our attention to the much more general and flexible positive-operator-valued (POV) measures for which the corresponding QL is the so-called effect algebra \(\mathbb{E}(\mathcal{H})\) of all positive semidefinite Hermitian operators on \(\mathcal{H}\) that are dominated by the identity operator. Evidently \(\mathbb{P}(\mathcal{H}) \subseteq \mathbb{E}(\mathcal{H}) \subseteq G(\mathcal{H})\). As "quantum-logical propositions", effect operators in \(\mathbb{E}(\mathcal{H})\) can manifest "fuzziness" or "unsharpness", while the projection operators in \(\mathbb{P}(\mathcal{H})\) can be regarded as

\(^1\)We follow common usage in which the term "quantum logic" refers not only to a logic associated with a quantum-mechanical system, but indeed to any physical system whatsoever.
“sharp,” even though their “truth values” might be subject to statistical fluctuations. For an exposition of POV measures and the contemporary quantum theory of measurements, see [Busch et al., 1991].

The mathematical structures of \( \mathbb{P} \), \( \mathbb{E} \), and \( \mathbb{G} \) are so rich and the quantum-mechanical interpretations of these structures are so intriguing that one cannot resist the temptation to formulate and study more general triples \( P \subseteq E \subseteq G \) as abstractions or analogues of \( \mathbb{P} \subseteq \mathbb{E} \subseteq \mathbb{G} \). Thus, in Sections 7 through 16 we conduct an abstraction process, motivated and guided by suitable examples, that will ultimately result in the notion of a so-called “CB-triple” \( P \subseteq E \subseteq G \) consisting of a regular orthomodular poset \( P \), an effect algebra \( E \), and a CB-group \( G \), i.e., a partially ordered abelian group enriched by a so-called “compression base.” The notion of a CB-group is both very general and mathematically attractive, and may be considered as an appropriate basis for the general study of quantum logics. In Sections 17 and 18, we introduce and study a class of CB-groups, called archimedean RC-groups, for which a spectral theory has been developed.

Because of the expository nature of this article, we give proofs only when we regard them as being illuminating or when they could be difficult to locate in the existing literature. Background material on Hilbert spaces can be found in [Halmos, 1998], a source for basic facts in regard to \( \sigma \)-fields, measurable functions, and so on, is [Halmos, 1950], and [Halmos, 1963] is an authoritative exposition of the theory of Boolean algebras—all three of these classic references were authored by P.R. Halmos. The monographs of R.V. Kadison and J.R. Ringrose [Kadison and Ringrose, 1983] provide comprehensive expositions of both the basic and advanced theory of operator algebras, and the treatise of G. Emch [Emch, 1972] is an excellent reference for applications of operator algebras to physics. In our treatment of partially ordered abelian groups, we follow the monograph [Goodearl, 1985] of K.R. Goodearl. For general lattice theory, the classic [Birkhoff, 1979] of G. Birkhoff is the standard reference.

2 ORTHODOX QUANTUM MECHANICS

Orthodox quantum mechanics (QM) is founded on the assumption that there is a correspondence \( \mathcal{S} \rightarrow \mathfrak{H} \) assigning a complex separable Hilbert space \( \mathfrak{H} \) to a QM-system \( \mathcal{S} \). Let \( \langle \cdot, \cdot \rangle \) be the inner product\(^2\) on \( \mathfrak{H} \) and denote the additive abelian group of all bounded Hermitian operators on \( \mathfrak{H} \) by \( \mathbb{G}(\mathfrak{H}) \). The tenets of orthodox QM specify that bounded (real) observables for \( \mathcal{S} \) are represented by operators \( A \in \mathbb{G}(\mathfrak{H}) \), (pure) states for \( \mathcal{S} \) are represented by unit vectors \( \psi \in \mathfrak{H} \), and the expectation value of an observable \( A \) when the system \( \mathcal{S} \) is in the state \( \psi \) is given by \( \langle A\psi, \psi \rangle \).

**DEFINITION 1.** If \( \psi \) is a state vector in \( \mathfrak{H} \), i.e., \( \|\psi\| = 1 \), and \( \mathbb{R} \) denotes the system of real numbers, we define the *expectation mapping* \( \omega_\psi: \mathbb{G}(\mathfrak{H}) \rightarrow \mathbb{R} \) by

\[^2\text{Physicists usually use the Dirac bra-ket notation } \langle \cdot | \cdot \rangle \text{ for the inner product.}\]
\[ \omega_{\psi}(A) := \langle A\psi, \psi \rangle \text{ for all } A \in G(\mathfrak{f}).^3 \]

The expectation value \( \omega_{\psi}(A) \) is usually interpreted as the limit as \( n \to \infty \) of the arithmetic mean of \( n \) independent measurements of the bounded observable \( A \) on the QM-system \( \mathcal{S} \), or replicas thereof, in the fixed state \( \psi \).

The expectation mappings \( \omega_{\psi} \) are used to organize the Hermitian group \( G(\mathfrak{f}) \) into a partially ordered abelian group [Goodearl, 1985] as follows: if \( A, B \in G(\mathfrak{f}) \), then by definition \( A \leq B \) iff \( \omega_{\psi}(A) \leq \omega_{\psi}(B) \) for every state vector \( \psi \in \mathfrak{f} \). In fact, under the partial order \( \leq \), \( G(\mathfrak{f}) \) is an order-unit-normed real Banach space with the identity operator \( 1 \in G(\mathfrak{f}) \) as the order unit [Alfsen, 1971, page 69]. Furthermore, the order-unit norm coincides with the uniform operator norm \( \| \cdot \| \) on \( G(\mathfrak{f}) \), and therefore, for \( A \in G(\mathfrak{f}) \),

\[(1) \quad \| A \| = \inf \{ m/n \mid 0 \leq m \in \mathbb{Z}, \ 0 < n \in \mathbb{Z}, \ \text{and} -m1 \leq nA \leq m1 \}, \]

where, as usual, \( \mathbb{Z} \) is the ring of integers. If \( 0 \) is the zero operator in \( G(\mathfrak{f}) \), then (by a slight abuse of language), observables \( A \in G(\mathfrak{f}) \) with \( 0 \leq A \) are called positive. We note that the norm of a positive bounded observable is the maximum expectation value of that observable for all state vectors, i.e.,

\[(2) \quad 0 \leq A \in G(\mathfrak{f}) \Rightarrow \| A \| = \max \{ \omega_{\psi}(A) \mid \psi \in \mathfrak{f}, \| \psi \| = 1 \} . \]

The positive operators \( A \) in \( G(\mathfrak{f}) \) are exactly the operators of the form \( A = B^2 \) for \( B \in G(\mathfrak{f}) \). Indeed, if \( B \in G(\mathfrak{f}) \), then for every state vector \( \psi \),

\[(3) \quad \omega_{\psi}(B^2) = \langle B^2\psi, \psi \rangle = \langle B\psi, B\psi \rangle = \| B\psi \|^2 \geq 0, \]

whence \( 0 \leq B^2 \). Conversely, if \( 0 \leq A \in G(\mathfrak{f}) \), then there is a unique operator \( \sqrt{A} \in G(\mathfrak{f}) \) such that \( 0 \leq \sqrt{A} \) and \( A = (\sqrt{A})^2 \) [Riesz and Nagy, 1955, page 265].

If \( A \in G(\mathfrak{f}) \), then \( 0 \leq A^2 \), and one defines \( |A| := \sqrt{A^2}, A^+ := \frac{1}{2}(|A| + A) \), and \( A^- := \frac{1}{2}(|A| - A) \). Then

\[(4) \quad 0 \leq A^+, A^- \text{ and } A = A^+ - A^- . \]

Thus, every bounded observable is the difference of two bounded positive observables.

Suppose that \( A_1 \leq A_2 \leq \cdots \) is an ascending sequence of bounded observables. Then the sequence is bounded above in norm (i.e., there is a real number \( \beta \) with \( \| A_n \| \leq \beta \) for all \( n \in \mathbb{N} := \{1, 2, 3, \ldots \} \)) iff it is bounded above in \( G(\mathfrak{f}) \) (i.e., there is a bounded observable \( B \in G(\mathfrak{f}) \) such that \( A_n \leq B \) for all \( n \in \mathbb{N} \)). Furthermore, if the sequence is bounded above in either sense, then by Vigier's theorem [Riesz and Nagy, 1955, page 263], there is a uniquely determined bounded observable \( A \in G(\mathfrak{f}) \) such that, for every \( \psi \in \mathfrak{f} \), \( \| A_n\psi - A\psi \| \to 0 \) as \( n \to \infty \), i.e. the sequence \( A_1, A_2, \ldots \) converges to \( A \) in the strong operator topology. But, if \( A_n \to A \) in the strong operator topology, then \( A_n \to A \) in the weak operator

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^3The notation := means "equal by definition."

^4We use "iff" as an abbreviation for "if and only if."
topology, and it follows that \( \omega_\psi(A_n) \) converges monotonically up to \( \omega_\psi(A) \) for each state vector \( \psi \in \mathcal{H} \). The latter condition implies that \( A \) is the supremum (least upper bound) of the sequence \( A_1, A_2, ... \) in \( \mathcal{G}(\mathcal{H}) \); hence the Hermitian group \( \mathcal{G}(\mathcal{H}) \) is (Dedekind) monotone \( \sigma \)-complete, i.e., every bounded ascending sequence in \( \mathcal{G}(\mathcal{H}) \) has a supremum.

Now suppose that \( B_1, B_2, ... \) is a sequence of positive bounded observables, define

\[
\sum_{i=1}^\infty B_i := \lim_{n \to \infty} \sum_{i=1}^n B_i
\]

and, in this sense, the expectation mappings \( \omega_\psi \) are countably additive on positive observables.

According to orthodox QM, the spectrum \( \text{spec}(A) \) of an observable \( A \in \mathcal{G}(\mathcal{H}) \) can be interpreted as the subset of the real number system \( \mathbb{R} \) consisting of all possible numerical values of individual measurements of \( A \) in various states\(^5\). Thus, if \( P \in \mathcal{G}(\mathcal{H}) \) and \( \text{spec}(P) \subseteq \{0, 1\} \), then we can regard \( P \) as a "two-valued proposition" about the system \( \mathcal{S} \) with possible truth values 1 (true) and 0 (false). As is easily seen, if \( P \in \mathcal{G}(\mathcal{H}) \), then \( \text{spec}(P) \subseteq \{0, 1\} \) iff \( P = P^2 \). Thus, we define

\[
\mathbb{P}(\mathcal{H}) := \{ P \in \mathcal{G}(\mathcal{H}) \mid P = P^2 \}
\]

and regard observables \( P \in \mathbb{P}(\mathcal{H}) \) as experimentally testable true/false (or yes/no) propositions about the QM-system \( \mathcal{S} \). The zero operator 0 and the identity operator 1 on \( \mathcal{H} \) belong to \( \mathbb{P}(\mathcal{H}) \), and they can be regarded as propositions that are always false and always true, respectively. The truth values of propositions \( P \in \mathbb{P}(\mathcal{H}) \) other than 0 and 1 may vary from measurement to measurement, even if the state \( \psi \) remains constant. Operators \( P \in \mathbb{P}(\mathcal{H}) \) are called projections (or, by some authors, projectors).

If \( P \in \mathbb{P}(\mathcal{H}) \) and \( \psi \) is a state vector in \( \mathcal{H} \), then by Equation (3), \( \omega_\psi(P) = \omega_\psi(P^2) = \|P\psi\|^2 \). The expectation value \( \omega_\psi(P) \) is the long-run relative frequency with which the result 1 (true) will be obtained for independent measurements of \( P \) in state \( \psi \). In other words, \( \omega_\psi(P) = \|P\psi\|^2 \) is the probability that the proposition \( P \) is true (or more accurately, will be found to be true if tested) when the QM-system \( \mathcal{S} \) is in state \( \psi \). Thus, we shall refer to the restriction of the expectation mapping \( \omega_\psi \) to \( \mathbb{P}(\mathcal{H}) \) as the probability measure or as the probability state determined by the state vector \( \psi \).

Suppose that \( \psi \) and \( \phi \) are state vectors in \( \mathcal{H} \) and denote by \( \mathbb{C} \) the system of complex numbers. Then

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\(^5\)For short, a measurement of \( A \) when the system \( \mathcal{S} \) is in state \( \psi \) is often referred to as a measurement of \( A \) in state \( \psi \).
(7) \( \omega_\psi = \omega_\phi \) on \( \mathbb{P}(5) \) \( \iff \exists \theta \in \mathbb{C} \) with \( |\theta| = 1 \) and \( \psi = \theta \phi \).

For a general quantum logic \( L \), a state is often defined as a "probability measure" \( \pi : L \to [0, 1] \subseteq \mathbb{R} \) (see Section 7 below). By (7), in the prototype quantum logic \( \mathbb{P}(5) \), this entails disregarding phase relations among the state vectors. The identification of physical states with probability states might be innocuous as long as one is considering independent measurements of one and only one observable, but the phase relations could be critical when non-independent or sequential measurements are involved. However, we shall not address this issue in the present article.

The partial order on the Hermitian group \( \mathbb{G}(5) \) induces (by restriction) a partial order on \( \mathbb{P}(5) \) and, for \( P, Q \in \mathbb{P}(5) \), \( P \leq Q \) if the probability \( \omega_\psi(P) \) is less than or equal to the probability \( \omega_\psi(Q) \) for every state vector \( \psi \). It is not difficult to show that

(8) \( P \leq Q \iff P = P Q \iff P = Q P \).

If \( P, Q \in \mathbb{P}(5) \) we say that \( P \) and \( Q \) are orthogonal, in symbols \( P \perp Q \), iff \( P + Q \in \mathbb{P}(5) \). Thus,

(9) \( P \perp Q \iff P + Q \leq 1 \iff P \leq 1 - Q \iff P Q = Q P = 0 \).

It is convenient to have a special symbol for the sum of two orthogonal projections, and we use \( \oplus \) for this purpose. Thus:

(10) If \( P \perp Q \), we define \( P \oplus Q := P + Q \in \mathbb{P}(5) \).

We refer to \( P \oplus Q \) as the orthogonal sum, or for short the orthosum, of the orthogonal projections \( P \) and \( Q \).

Suppose that \( P, Q \in \mathbb{P}(5) \). In regarding the system \( \mathbb{P}(5) \) as a "logic" and \( P, Q \) as "propositions," we regard \( 1 - P \) as a "negation" or "logical denial" of \( P \), and we interpret \( P \leq Q \) to mean that \( P \) "implies" \( Q \). Since

\[
P \perp Q \iff P \leq 1 - Q \iff Q \leq 1 - P,
\]

the condition \( P \perp Q \) means that, in some sense, \( P \) and \( Q \) "refute each other." Thus, if \( P \perp Q \), we may think of \( P \oplus Q \) as a kind of "disjunction" of the mutually refuting propositions \( P \) and \( Q \). We shall refer to \( \mathbb{P}(5) \), equipped with the "negation mapping" \( P \mapsto 1 - P \), the partially defined binary operation \( \oplus \), and the partial order relation \( \leq \), as the sharp quantum logic associated with the Hilbert space \( \mathcal{F} \).

(In Section 6 below, an "unsharp" or "fuzzy" quantum logic will be introduced.)

Let \( P_1, P_2, P_3, \ldots \) be a sequence of pairwise orthogonal propositions in \( \mathbb{P}(5) \). Then, it can be shown that \( \sum_{i=1}^{n} P_i \in \mathbb{P}(5) \) for every positive integer \( n \), whence it is natural to define

\[
P_1 \oplus P_2 \oplus \cdots \oplus P_n := \sum_{i=1}^{n} P_i,
\]

\footnote{Note that the implication relation \( \leq \) is not a logical connective on the quantum logic \( \mathbb{P}(5) \).}
or with alternative notation, \( \bigoplus_{i=1}^{n} P_i := \sum_{i=1}^{n} P_i \). As the partial sums \( \sum_{i=1}^{n} P_i \) are bounded above by 1, it follows that the infinite series \( \sum_{i=1}^{\infty} P_i \) converges (in the strong and weak senses) to an observable in \( \mathcal{G}(\mathcal{H}) \) which is, in fact, the supremum of the partial sums. It can be shown that \( \sum_{i=1}^{\infty} P_i \in \mathcal{P}(\mathcal{H}) \), whence it is natural to define

\[
\bigoplus_{i=1}^{\infty} P_i := \sum_{i=1}^{\infty} P_i \in \mathcal{P}(\mathcal{H})
\]

and to regard \( \bigoplus_{i=1}^{\infty} P_i \) as a kind of “infinite orthogonal disjunction” of the propositions \( P_1, P_2, P_3, \ldots \). By (5), the probability measures determined by the state vectors are countably additive in the sense that

\[
\omega_\psi(\bigoplus_{i=1}^{\infty} P_i) = \sum_{i=1}^{\infty} \omega_\psi(P_i)
\]

for each state vector \( \psi \).

**DEFINITION 2.** Let \( \mathcal{B}(\mathbb{R}) \) be the \( \sigma \)-field of Borel subsets of \( \mathbb{R} \). A **projection-valued (PV) measure** is a mapping \( M \mapsto P_M \) from Borel sets \( M \in \mathcal{B}(\mathbb{R}) \) to projections \( P_M \in \mathcal{P}(\mathcal{H}) \) that has the following properties: (i) \( P_\emptyset = 1 \). (ii) If \( M_1, M_2, M_3, \ldots \) is a pairwise disjoint sequence in \( \mathcal{B}(\mathbb{R}) \) and \( M = \bigcup_{i=1}^{\infty} M_i \), then \( P_{M_1}, P_{M_2}, \ldots \) is a pairwise orthogonal sequence in \( \mathcal{P}(\mathcal{H}) \) and \( P_M = \bigoplus_{i=1}^{\infty} P_{M_i} \). The PV-measure \( M \mapsto P_M \) is said to be **bounded** iff there are real numbers \( \alpha < \beta \) such that \( P_{[\alpha, \beta]} = 1 \).

As a consequence of the spectral theorem, a bounded observable \( A \in \mathcal{G}(\mathcal{H}) \) determines and is determined by a bounded PV-measure \( M \mapsto P_M \) in such a way that \( A = \int_{\mathbb{R}} \iota \, dP_M \), where \( \iota \) is the identity function on \( \mathbb{R} \) and the integral converges in the uniform norm topology. Here the proposition \( P_M \) is to be interpreted as asserting that the measured value of \( A \) belongs to the Borel set \( M \). Likewise, unbounded observables also correspond to PV-measures, but without the boundedness condition. If \( M \mapsto P_M \) is a PV-measure and \( \psi \) is a state vector, then \( M \mapsto \omega_\psi(P_M) \) is a countably additive probability measure on the \( \sigma \)-field \( \mathcal{B}(\mathbb{R}) \); thus by combining an observable (bounded or unbounded) and a state we obtain a conventional Kolmogorovian probability measure.

Operations that are only partially defined are always something of an annoyance, and it is reassuring to keep in mind that the partial binary operation \( \oplus \) on \( \mathcal{P}(\mathcal{H}) \) can be extended to a *bona fide* binary operation \( + \) on the whole Hermitian group \( \mathcal{G}(\mathcal{H}) \); furthermore, for each state vector \( \psi \), the probability measure (or probability state) \( P \mapsto \omega_\psi(P) \) on \( \mathcal{P}(\mathcal{H}) \) can be extended to the expectation mapping \( A \mapsto \omega_\psi(A) \) on \( \mathcal{G}(\mathcal{H}) \).

3 PROJECTIONS AND COMPRESSIONS

Closed linear subspaces \( \mathcal{M} \) of \( \mathcal{H} \) play important roles in the study of the QM-system \( S \). It is often necessary or desirable to drop down to a subspace \( \mathcal{M} \subseteq \mathcal{H} \)
to eliminate degrees of freedom that are not of current interest. For instance, in considering the spin components of a spin-1/2 particle \( S \), it is appropriate to drop down to a two-dimensional subspace \( M \) of \( \mathfrak{h} \) where the spin observables "live" [Beltrametti and Cassinelli, 1981, Chapter 4].

Let \( T \) be a bounded linear operator (not necessarily self-adjoint) on \( \mathfrak{h} \) and let \( M \) be a closed linear subspace of \( \mathfrak{h} \). Then \( M \) is said to be invariant under \( T \) iff \( T(M) \subseteq M \), in which case the restriction \( T|_M \) of \( T \) to \( M \) is a bounded linear operator on \( M \). If \( M^\perp := \{ \xi \in \mathfrak{h} \mid \langle \xi, \eta \rangle = 0, \forall \eta \in M \} \) is the orthogonal complement of \( M \), and if both \( M \) and \( M^\perp \) are invariant under \( T \), then \( T \) decomposes into sort of a "direct product" of \( T|_M \) and \( T|_{M^\perp} \), and we say that \( M \) reduces \( T \). If \( T \) is Hermitian, then \( M \) is invariant under \( T \) iff \( M \) reduces \( T \).

If \( P \in \mathbb{P}(\mathfrak{h}) \) and \( M \) is a closed linear subspace of \( \mathfrak{h} \), then \( P \) is said to be the projection onto \( M \) iff the range of \( P \) is \( M \), i.e., \( P(\mathfrak{h}) = M \). There is a one-to-one correspondence \( P \leftrightarrow M \) between projections \( P \in \mathbb{P}(\mathfrak{h}) \) and closed linear subspaces \( M \) of \( \mathfrak{h} \) given by \( P(\mathfrak{h}) = M \). If \( P \) is the projection onto \( M \), then \( 1 - P \) is the projection onto \( M^\perp \). If \( T \) is a bounded linear operator on \( \mathfrak{h} \) and \( P \) is the projection onto \( M \), then \( T \) is invariant under \( T \) iff \( PTP = TP \). The mapping \( T \mapsto PTP \), called the Naimark compression determined by \( P \), maps \( G(\mathfrak{h}) \) into itself, and we shall refer to its restriction to \( G(\mathfrak{h}) \) as a compression. Compressions on \( G(\mathfrak{h}) \) will play an important role in our subsequent development, and we shall use the notation in the following definition for them.

**DEFINITION 3.** If \( P \in \mathbb{P}(\mathfrak{h}) \) we define the compression \( J_P : G(\mathfrak{h}) \rightarrow G(\mathfrak{h}) \) by \( J_P(A) := PAP \) for all \( A \in G(\mathfrak{h}) \).

The compressions \( J_P \) provide a convenient way for dealing algebraically with closed linear subspaces of \( \mathfrak{h} \). For instance, let \( P \in \mathbb{P}(\mathfrak{h}) \) and let \( M = P(\mathfrak{h}) \). If \( A \in G(\mathfrak{h}) \), then the restriction \( J_P(A)|_M \) of \( J_P(A) = PAP \) to \( M \) is a Hermitian operator on the Hilbert space \( M \). Moreover, under restriction to \( M \), the subgroup \( J_P(G(\mathfrak{h})) = \{ PAP \mid A \in G(\mathfrak{h}) \} \) of \( G(\mathfrak{h}) \) is isomorphic as a partially ordered abelian group to the Hermitian group \( G(M) \).

The compression mappings \( J_P \) on \( G(\mathfrak{h}) \) are characterized algebraically as follows (see [Foulis, 2004a]).

**THEOREM 4.** Let \( J : G(\mathfrak{h}) \rightarrow G(\mathfrak{h}) \) and let \( P := J(1) \). Then \( P \in \mathbb{P}(\mathfrak{h}) \) and \( J = J_P \) iff \( J \) is an order-preserving group homomorphism, \( P \leq 1 \), and \( J(E) = E \) for all \( E \in G(\mathfrak{h}) \) with \( 0 \leq E \leq P \).

## 4 SYMMETRIES

In this section, we give a very superficial sketch to indicate how physical symmetries of the QM-system \( S \) interact with the Hermitian group \( G(\mathfrak{h}) \) and the projection lattice \( \mathbb{P}(\mathfrak{h}) \). According to the so-called "Heisenberg picture," a physical symmetry \( \xi \) of \( S \) (e.g., a rotation of \( S \) in physical space) induces a mapping \( A \mapsto \xi A \) on the observables \( A \) for \( S \), \( \xi A \) being the observable into which \( A \) is transformed when \( S \) is transformed by the symmetry \( \xi \). If \( A \) is a bounded observable,
then of course $\xi A$ is again a bounded observable, and plausible arguments can be made to show that the mapping $A \mapsto \xi A$ is a "unital automorphism" as per the next definition.

**DEFINITION 5.** A mapping $\xi : \mathcal{G}(\mathcal{H}) \to \mathcal{G}(\mathcal{H})$ is a unital automorphism iff $\xi$ is an automorphism of the abelian group $\mathcal{G}(\mathcal{H})$ such that $\xi 1 = 1$ and both $\xi$ and $\xi^{-1}$ are order-preserving mappings on $\mathcal{G}(\mathcal{H})$.

In what follows, we shall assume that physical symmetries for $\mathcal{S}$ correspond to unital automorphisms $\xi$ of $\mathcal{G}(\mathcal{H})$.

**THEOREM 6.** Suppose that $\xi$ is a unital automorphism on $\mathcal{G}(\mathcal{H})$. Then there exists a unitary or antiunitary operator $U$ on $\mathcal{H}$ such that $\xi A = U^{-1} A U$ for all $A \in \mathcal{G}(\mathcal{H})$. Conversely, each unitary or antiunitary operator $U$ on $\mathcal{H}$ gives rise to a unital automorphism $\xi$ on $\mathcal{G}(\mathcal{H})$ defined by $\xi A := U^{-1} A U$ for all $A \in \mathcal{G}(\mathcal{H})$.

**Proof.** If the dimension of $\mathcal{H}$ is three or more, then the desired conclusion follows from G. Ludwig's version of Wigner's theorem on symmetry transformations (see [Casinelli et al., 1997]). That the conclusion also holds when the dimension of $\mathcal{H}$ is 2 follows from [Molnár and Páles, 2001]. If $\mathcal{H}$ has dimension 1, then $\mathbb{R}$ is isomorphic under $\lambda \mapsto \lambda 1$ to $\mathcal{G}(\mathcal{H})$ as an ordered group, and the desired conclusion follows from the fact that the only order-preserving group automorphisms of $\mathbb{R}$ are of the form $\lambda \mapsto \kappa \lambda$ where $\kappa \in \mathbb{R}$ is a positive constant. The converse is obvious. ■

**COROLLARY 7.** If $\xi$ is a unital automorphism on $\mathcal{G}(\mathcal{H})$, then $\xi$ maps $\mathcal{P}(\mathcal{H})$ onto itself, and the restriction of $\xi$ to $\mathcal{P}(\mathcal{H})$ is a order automorphism that preserves orthogonality and orthogonal sums.

In the sense of Corollary 7, a physical symmetry induces a "logical symmetry" on the sharp quantum logic $\mathcal{P}(\mathcal{H})$.

As a consequence of Theorem 6, if $\Xi$ is a multiplicatively-written group (e.g., a group of physical symmetries of $\mathcal{S}$), then a unitary/antiunitary representation of $\Xi$ on $\mathcal{H}$ is mathematically equivalent to a unital action of $\Xi$ on $\mathcal{G}(\mathcal{H})$, i.e., a mapping $(\xi, A) \mapsto \xi A$ from $\Xi \times \mathcal{G}(\mathcal{H})$ to $\mathcal{G}(\mathcal{H})$ satisfying the following conditions for all $\xi, \xi' \in \Xi$ and all $A, B \in \mathcal{G}(\mathcal{H})$: (i) $(\xi \xi') A = \xi (\xi' A)$. (ii) If $1$ is the identity in $\Xi$, then $1 A = A$. (iii) $\xi 1 = 1$. (iv) $\xi (A + B) = \xi A + \xi B$. (v) $A \leq B \Rightarrow \xi A \leq \xi B$.

In the theory of unitary/antiunitary group representations, the notion of reduction of a representation by a closed linear subspace is of special significance [Schroock, 1996, Chapter III, Section 1]. If a unitary/antiunitary representation of a group $\Xi$ on $\mathcal{H}$ is expressed as a unital action of $\Xi$ on $\mathcal{G}(\mathcal{H})$, then a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ reduces the representation iff the compression $J_P$ corresponding to the projection $P$ onto $\mathcal{M}$ commutes with the unital action in the sense that $\xi (J_P(A)) = J_P(\xi A)$ for all $\xi \in \Xi$ and all $A \in \mathcal{G}(\mathcal{H})$.

Sometimes it is convenient to adopt an alternative point of view, namely the so-called "Schrödinger picture." According to the Schrödinger picture, a physical symmetry $\xi$ induces a mapping $\psi \mapsto \xi \psi$ on the state vectors $\psi$ to be interpreted as follows: if the system $\mathcal{S}$ is in state $\psi$ before the application of the physical
symmetry $\xi$, it will be in state $\xi \psi$ after the application of $\xi$. The Heisenberg and Schrödinger pictures are physically equivalent, and the relation between them is simply that $\omega_\psi(\xi A) = \omega_{\xi \psi}(A)$ for all $A \in G(H)$ and all state vectors $\psi$. From this relation, it follows that if $U$ is the unitary or antiunitary transformation on $H$ representing $\xi$ as in Theorem 6, then $\xi \psi = U(\psi)$.

Temporal transformations of states (Schrödinger picture) or observables (Heisenberg picture) according to a (reversible) dynamical law are represented by a one-parameter family $(\xi_t)_{t \in \mathbb{R}}$ of symmetries satisfying $\xi_{t+t'} = \xi_t \circ \xi_{t'}$ for all $t, t' \in \mathbb{R}$. For instance, in the Schrödinger picture, starting from an initial state $\psi_0$, the state of the system $S$ evolves in time $t'$ to the state $\psi_{t'} = \xi_{t'} \psi_0$. In an additional $t$ units of time, the system will evolve from the state $\psi_{t'}$ to the state $\xi_t \psi_{t'} = \xi_t (\xi_{t'} \psi_0) = \xi_{t+t'} \psi_0$.

By Theorem 6, a (reversible) dynamical law corresponds to a one-parameter family $(U_t)_{t \in \mathbb{R}}$ of unitary or antiunitary transformations on $H$ such that $U_{t+t'} = U_t \circ U_{t'}$ for all $t, t' \in \mathbb{R}$. However, since the square of a unitary or antiunitary operator is always unitary, and since $U_t = (U_t^*)^2$ for all $t \in \mathbb{R}$, all operators in the family are necessarily unitary. If the one-parameter unitary group $(U_t)_{t \in \mathbb{R}}$ is continuous in the strong operator topology, then by Stone’s theorem [Kadison and Ringrose, 1983, Theorem 5.6.36], there is a (possibly unbounded) self-adjoint operator $H$ on $H$ such that $U_t = e^{-iHt}$ for all $t \in \mathbb{R}$. The operator $H$ is the Hamiltonian for the physical system $S$, and it corresponds to the energy observable for $S$.

5 MIXED STATES AND DENSITY OPERATORS

If the QM-system $S$ is coupled with another QM-system system $S'$ to form a composite system $S + S'$, then the tenets of QM stipulate that the Hilbert space for $S + S'$ is the Hilbert-space tensor product $H \otimes H'$ of the Hilbert spaces $H$ and $H'$ for $S$ and $S'$, respectively [Aerts et al., 1978]. Even if the composite system $S + S'$ is in a pure state represented by a unit vector in $H \otimes H'$, the subsystem $S$ need not be in a pure state. In general, the subsystem $S$ will be in a so-called mixed state in the sense that the expectation values for an observable $A$ measured on $S$ will be a convex linear combination of expectation values $\omega_\phi(A)$, $\omega_\psi(A)$, ... for suitable state vectors $\psi, \phi, ... \in H$. Such a mixed state is represented by a so-called density operator.

Let $T(H)$ be the (real) linear subspace of $G(H)$ consisting of the Hermitian trace-class operators, and let $D(H)$ be the convex subset of $T(H)$ consisting of the positive trace-class operators $D$ with $\text{tr}(D) = 1$. Operators $D \in D(H)$ are called (von Neumann) density operators on $H$, and the (possibly) mixed states for the QM-system $S$ are represented by these density operators$^7$. By definition, the expectation value $\omega_D(A)$ of the bounded observable $A \in G(H)$ when the system $S$ is in the state $D \in D(H)$ is given by the Born formula

$^7$Physicists usually denote density operators by the Greek letter $\rho$. 
(1) \( \omega_D(A) = \text{tr}(DA) \).

As in Equation (5), the expectation mapping \( \omega_D \) is countably additive.

If \( \psi \in \mathcal{H} \) is a state vector, then the projection\(^8\) \( P_\psi \in \mathbb{P}(\mathcal{H}) \) onto the one-dimensional subspace of \( \mathcal{H} \) spanned by \( \psi \) is a density operator on \( \mathcal{H} \), and the density operators of the form \( P_\psi \) for \( \psi \in \mathcal{H} \) with \( \| \psi \| = 1 \) are the extreme points of the convex set \( \mathbb{D}(\mathcal{H}) \). Furthermore, \( \omega_{P_\psi} \) as given by the Born formula (1) coincides with the expectation mapping \( \omega_\psi \) as in Definition 1. Thus (neglecting phase factors), the pure states of the QM-system \( S \) are represented by the extreme points of the convex set \( \mathbb{D}(\mathcal{H}) \).

Mixed states also arise in connection with ensembles of replicas of the QM-system \( S \). If \( \psi_i, i = 1, 2, \ldots, n, \) are state vectors, and if for each \( i \) a fraction \( p_i \) of the systems in the ensemble are in the pure state \( \psi_i \), then the state of the ensemble can be considered to be the mixed state represented by the convex linear sum

\[
(2) \quad D = \sum_{i=1}^{n} p_i P_{\psi_i} \in \mathbb{D}(\mathcal{H}).
\]

If \( A \in \mathcal{G}(\mathcal{H}) \), then \( \omega_D(A) \) can be interpreted as the expectation value for a measurement of the bounded observable \( A \) on a system chosen at random from the ensemble.

If \( P \in \mathbb{P}(\mathcal{H}) \) and \( D \in \mathbb{D}(\mathcal{H}) \) is a density operator, then the expectation value \( \omega_D(P) \) is the long-run relative frequency with which the result 1 (true) will be obtained for independent measurements of \( P \) in the mixed state \( D \), i.e., \( \omega_D(P) = \text{tr}(DP) \) is the probability that the proposition \( P \) is true (or, more accurately, will be found to be true if tested) when the QM-system \( S \) is in the mixed state \( D \). Thus, we shall refer to the restriction of the expectation mapping \( \omega_D \) to \( \mathbb{P}(\mathcal{H}) \) as the probability measure or as the probability state determined by the density operator \( D \).

An alternative interpretation for the mixed state \( D \) in Equation (2) is afforded by the notion that an agent might be in possession of information that the QM-system \( S \) is in one of the pure states \( \psi_i \), and that \( p_i \) is the probability that \( S \) is in state \( \psi_i \) for \( i = 1, 2, \ldots, n \). Then \( \omega_D(P) = \text{tr}(DP) \) is the probability that the agent should assign to the proposition \( P \in \mathbb{P}(\mathcal{H}) \). According to this interpretation, although the pure states \( P_{\psi_i} \) may be ontic, i.e., “real physical states,” the mixed state \( D \) is epistemic, i.e., it is a state of incomplete knowledge about the ontic state of \( S \). For an exposition of the epistemic interpretation of states, see [Fuchs, 2002; Spekkens, 2004].

With the partial order inherited from \( \mathcal{G}(\mathcal{H}) \), \( \mathbb{T}(\mathcal{H}) \) is a base-normed Banach space with the density operators \( \mathbb{D}(\mathcal{H}) \) as the cone base [Alfsen, 1971, page 77], and the base norm coincides with the trace norm. The base-normed Banach space \( \mathbb{T}(\mathcal{H}) \) is in separating order and norm duality with the order-unit Banach space \( \mathcal{G}(\mathcal{H}) \) under the bilinear form \( \langle \cdot, \cdot \rangle \) defined by \( \langle C, A \rangle := \text{tr}(CA) \) for \( C \in \mathbb{T}(\mathcal{H}) \) and \( A \in \mathcal{G}(\mathcal{H}) \); in fact, under this duality, \( \mathcal{G}(\mathcal{H}) \) may be identified with the Banach

\(^8\)Physicists use the Dirac notation \( |\psi\rangle \langle \psi| \) for the projection \( P_\psi \).
dual space of $T(\mathfrak{H})$. This duality is the basis for the formulation of *operational quantum mechanics* by Davies, Lewis, *et al.* [Davies and Lewis, 1970].

6 EFFECT OPERATORS AND POV-MEASURES

In general, if $A \in G(\mathfrak{H})$, the expectation values $\omega_D(A)$ for density operators $D \in D(\mathfrak{H})$ cannot be construed as probabilities because they will not necessarily belong to the unit interval $[0,1] \subseteq \mathbb{R}$. But, if $E \in G(\mathfrak{H})$ with $0 \leq E \leq 1$, then $0 \leq \omega_D(E) \leq 1$ for every density operator $D$, and it becomes feasible to think of $E$ as some kind of "proposition" and of $\omega_D(E)$ as its probability in state $D$. Thus, we define

$$(1) \quad \mathbb{E}(\mathfrak{H}) := \{E \in G(\mathfrak{H}) \mid 0 \leq E \leq 1\},$$

and we think of $\mathbb{E}(\mathfrak{H})$ as a sort of "logic." Following G. Ludwig [Ludwig, 1983], operators $E \in \mathbb{E}(\mathfrak{H})$ are called *effect operators* on $\mathfrak{H}$.

Evidently, $\mathbb{P}(\mathfrak{H}) \subseteq \mathbb{E}(\mathfrak{H})$, and we can extend the notion of orthogonality and the orthogonal sum $\oplus$ to $\mathbb{E}(\mathfrak{H})$ as follows: if $E,F \in \mathbb{E}(\mathfrak{H})$, then by definition, $E \perp F$ iff $E + F \in \mathbb{E}(\mathfrak{H})$, in which case $E \oplus F := E + F$. However, the extended notion of orthogonality is not as well-behaved as orthogonality on $\mathbb{P}(\mathfrak{H})$. Indeed, the sum of a finite pairwise orthogonal sequence of projections is again a projection, but, in general, the analogous condition fails spectacularly for effect operators. For instance, if $E_1 = E_2 = E_3 = \frac{1}{2} 1$, then $E_1, E_2, E_3$ are pairwise orthogonal effect operators, but $E_1 + E_2 + E_3 = \frac{3}{2} 1 \notin \mathbb{E}(\mathfrak{H})$. Thus, for $\mathbb{E}(\mathfrak{H})$, the notion of orthogonality of three or more elements must be handled more delicately.

We say that a finite sequence $E_1, E_2, ..., E_n \in \mathbb{E}(\mathfrak{H})$ is *jointly orthogonal* iff $\sum_{i=1}^{n} E_i \notin \mathbb{E}(\mathfrak{H})$. Evidently, every subsequence of a finite jointly orthogonal sequence is again jointly orthogonal. If the finite sequence $E_1, E_2, ..., E_n$ is jointly orthogonal, we define $E_1 \oplus E_2 \oplus \cdots \oplus E_n := \sum_{i=1}^{n} E_i$, or with alternative notation, $\bigoplus_{i=1}^{n} E_i := \sum_{i=1}^{n} E_i$; and we say that an infinite sequence $E_1, E_2, ... \in \mathbb{E}(\mathfrak{H})$ is *jointly orthogonal* iff every finite subsequence is jointly orthogonal. Clearly, the infinite sequence $E_1, E_2, ...$ is jointly orthogonal iff every finite initial sequence $E_1, E_2, ..., E_n$ is jointly orthogonal.

Suppose $E_1, E_2, ... \in \mathbb{E}(\mathfrak{H})$ is an infinite jointly orthogonal sequence of effect operators. As the partial sums $\sum_{i=1}^{n} E_i \in \mathbb{E}(\mathfrak{H})$ are bounded above by 1, it follows that the infinite series $\sum_{i=1}^{\infty} E_i$ converges (in the strong and weak senses) to an operator in $G(\mathfrak{H})$ which is, in fact, the supremum of the partial sums. Since the partial sums are positive and bounded above by 1, their supremum is again an effect operator, whence it is natural to define $\bigoplus_{i=1}^{\infty} E_i := \sum_{i=1}^{\infty} E_i \in \mathbb{E}(\mathfrak{H})$. The probability measures on $\mathbb{E}(\mathfrak{H})$ determined by the density operators $D \in D(\mathfrak{H})$ are countably additive in the sense that

$$(2) \quad \omega_D\left(\bigoplus_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \omega_D(E_i)$$
for each density operator $D$.

The notion of a PV-measure (Definition 2) extends in a natural way to a measure on the real Borel field $\mathcal{B}(\mathbb{R})$ taking on values in $\mathbb{E}(\mathfrak{f})$.

**DEFINITION 8.** A (normalized) positive-operator-valued (POV) measure is a mapping $M \mapsto E_M$ from Borel sets $M \in \mathcal{B}(\mathbb{R})$ to effect operators $E_M \in \mathbb{E}(\mathfrak{f})$ that has the following properties: (i) $E_{\mathbb{R}} = 1$. (ii) if $M_1, M_2, M_3, \ldots$ is a pairwise disjoint sequence in $\mathcal{B}(\mathbb{R})$ and $M = \bigcup_{i=1}^{\infty} M_i$, then $E_{M_1}, E_{M_2}, \ldots$ is a jointly orthogonal sequence in $\mathbb{E}(\mathfrak{f})$ and $E_M = \bigoplus_{i=1}^{\infty} E_{M_i}$. The POV-measure $M \mapsto E_M$ is said to be bounded iff there are real numbers $\alpha < \beta$ such that $E_{[\alpha, \beta]} = 1$.

As with PV-measures and state vectors, by combining a POV-measure $M \mapsto E_M$ and a density operator $D \in \mathbb{D}(\mathfrak{f})$, we obtain a conventional Kolmogorovian probability measure $M \mapsto \omega_D(E_M)$ on the $\sigma$-field $\mathcal{B}(\mathbb{R})$. A POV-measure $M \mapsto E_M$ is a PV-measure (i.e., $E_M \in \mathbb{P}(\mathfrak{f})$ for all $M \in \mathcal{B}(\mathbb{R})$) iff $E_{M \cap N} = E_ME_N = E_N E_M$ for all $M, N \in \mathcal{B}(\mathbb{R})$.

The POVMs provide a much more general notion of an observable than do the PV-measures. In fact, they

"allow for the possibility of joint measurements of noncommuting observables, especially complementary observables, in full accordance with the uncertainty relations"

[Busch et al., 1991, pp. 12–13]. Thus, unlike projections $P \in \mathbb{P}(\mathfrak{f})$, effect operators $E \in \mathbb{E}(\mathfrak{f})$, regarded as propositions, can manifest "uncertainty" or "unsharpness." Whereas $\mathbb{P}(\mathfrak{f})$ is a quantum logic consisting only of sharp propositions, the larger system $\mathbb{E}(\mathfrak{f})$ also admits unsharp propositions.

If $P, Q \in \mathbb{P}(\mathfrak{f})$, then $J_Q(P) = QPQ$ is again a projection iff $PQ = QP$, but $J_Q(P)$ is always an effect operator on $\mathfrak{f}$, and its restriction $J_Q(P)|_{\mathcal{M} := Q(\mathfrak{f})}$ to $\mathcal{M} := Q(\mathfrak{f})$ is an effect operator on the Hilbert space $\mathcal{M}$. Furthermore, if $M \mapsto P_M$ is a PV-measure, then $M \mapsto J_Q(P_M)|_{\mathcal{M}}$ is a POVM-measure for $\mathcal{M}$, and in this sense the compression $J_Q$ maps PV-measures to POVM-measures. Conversely, by the Naimark dilation theorem if $M \mapsto E_M$ is a POVM-measure, then $\mathfrak{f}$ can be realized as a closed linear subspace of a larger Hilbert space $\hat{\mathfrak{f}}$ in such a way that $M \mapsto E_M$ is the image under a compression of a PV-measure for $\hat{\mathfrak{f}}$ [Schroeck, 1996, Chapter 2].

The following theorems provide connections among effect operators, projections, compressions, and positive Hermitian operators which we shall be using in the sequel.

**THEOREM 9.** If $E \in \mathbb{E}(\mathfrak{f})$ and $P \in \mathbb{P}(\mathfrak{f})$, then: (i) $E \leq P \iff E = EP$. (ii) $P \leq E \iff P = EP$. (iii) $E + P \in \mathbb{E}(\mathfrak{f}) \iff EP = 0$. (iv) $J_P(E) = 0 \iff E \leq 1 - P$.

**Proof.** Part (i) follows from [Greechie et al., 1995, Theorem 6.6 (iii)]. (ii) By (i), $P \leq E \iff 1 - E \leq 1 - P \iff 1 - E = (1 - E)(1 - P) \iff P = EP$. (iii) $E + P \in \mathbb{E}(\mathfrak{f}) \iff E + P \leq 1 \iff E \leq 1 - P$, hence (iii) follows from (i). (iv) If $J_P(E) = PEP = 0$, then, since $0 \leq E$, it follows that $EP = 0$, whence $E \leq 1 - P$.
THEOREM 10. Let $R \in \mathbb{E}(\mathcal{F})$. Then the following conditions are mutually equivalent: (i) $R \in \mathbb{P}(\mathcal{F})$. (ii) If $E \in \mathbb{E}(\mathcal{F})$, then $E \leq R$, $1 - R \Rightarrow E = 0$. (iii) If $E, F \in \mathbb{E}(\mathcal{F})$ and $E, F \leq R$, then $E \perp F \Rightarrow E \oplus F \leq R$.

**Proof.** See [Greechie et al., 1995, Theorem 6.8].

THEOREM 11. Let $D, E, F \in \mathbb{E}(\mathcal{F})$ with $D + E + F \in \mathbb{E}(\mathcal{F})$, and suppose that $P := D + E \in \mathbb{P}(\mathcal{F})$ and $Q := D + F \in \mathbb{P}(\mathcal{F})$. Then $D \in \mathbb{P}(\mathcal{F})$ and $J_P \circ J_Q = J_D$.

**Proof.** Assume the hypotheses. Then $E + Q = D + E + F \in \mathbb{E}(\mathcal{F})$, $D \leq D + F = Q$, and $D \leq D + E = P$, so $EQ = 0$, $DQ = D$, and $DP = D$ by Theorem 9. Define $J : \mathbb{G}(\mathcal{F}) \to \mathbb{G}(\mathcal{F})$ by $J := J_P \circ J_Q$. Then $J$ is an order-preserving group endomorphism on $\mathbb{G}(\mathcal{F})$ and $J(1) = J_P(J_Q(1)) = J_P(Q) = PQP = (E + D)QP = DP = D$. Suppose $C \in \mathbb{E}(\mathcal{F})$ with $C \leq D$. Then $C \leq P, Q$, whence $J(C) = J_P(J_Q(C)) = J_P(C) = C$, and it follows from Theorem 4 that $D \in \mathbb{P}(\mathcal{F})$ and $J = J_D$.

THEOREM 12. Let $P \in \mathbb{P}(\mathcal{F})$. Then the condition $J_P(A) \leq A$ holds for all positive Hermitian operators $A$ iff $P = 0$ or $P = 1$.

**Proof.** Suppose that, for all $A \in \mathbb{G}(\mathcal{F})$, $0 \leq A \Rightarrow J_P(A) = PAP \leq A$ and define $J' : \mathbb{G}(\mathcal{F}) \to \mathbb{G}(\mathcal{F})$ by $J'(A) := A - J_P(A)$ for all $A \in \mathbb{G}(\mathcal{F})$. Then $J'$ is a group endomorphism on $\mathbb{G}(\mathcal{F})$ and, since it maps positive operators to positive operators, it is order preserving. Suppose $0 \leq E \in \mathbb{G}(\mathcal{F})$ with $E \leq J'(1) = 1 - J_P(1) = 1 - P$. Then $0 \leq J_P(E) \leq J_P(1 - P) = 0$, so $J_P(E) = 0$, and it follows that $J'(E) = E$. Therefore, by Theorem 4, $J' = J_{1-P}$, whence $A = J_P(A) + J'(A) = J_P(A) + J_{1-P}(A) = PAP + (1 - P)A(1 - P)$ for all $A \in \mathbb{G}(\mathcal{F})$. Consequently, $PA = PAP$ and $AP = PAP$, whence $PA = AP$ for all $A \in \mathbb{G}(\mathcal{F})$, i.e., the projection $P$ commutes with every Hermitian operator on $\mathcal{F}$. It follows that $P$ commutes with every bounded operator on $\mathcal{F}$, hence that $P = \lambda 1$ for some complex number $\lambda$. As $P = P^2$, we have $\lambda = \lambda^2$, hence $\lambda = 0$ or $\lambda = 1$, i.e., $P = 0$ or $P = 1$. The converse is obvious.

THEOREM 13. Let $A \in \mathbb{G}(\mathcal{F})$. Then $A \geq 0$ iff there is a finite sequence $E_1, E_2, ..., E_n \in \mathbb{E}(\mathcal{F})$ such that $A = E_1 + E_2 + \cdots + E_n$.

**Proof.** Suppose that $A \geq 0$ and choose a positive integer $n \geq \|A\|$. Then $(1/n)A \in \mathbb{E}(\mathcal{F})$ and with $E_i := (1/n)A$ for $i = 1, 2, ..., n$, we have $A = E_1 + E_2 + \cdots + E_n$. The converse is obvious.
7 ABSTRACTION FROM $\mathcal{P}(\mathcal{F})$ AND $\mathbb{E}(\mathcal{F})$—EFFECT ALGEBRAS

The notion of an "effect algebra," as per the following definition, is abstracted from the prototype quantum logics $\mathcal{P}(\mathcal{F})$ and $\mathbb{E}(\mathcal{F})$.

**Definition 14.** An effect algebra is a system $(E, 0, u, \perp, \oplus)$ consisting of a set $E$, special elements $0, u \in E$ called the zero element and the unit element, a binary relation $\perp$ on $E$ called orthogonality, a unary operation $p \mapsto p^\perp$ on $E$ called orthosupplementation, and a partially defined binary operation $\oplus$ on $E$ called orthosummation such that, for all $d, e, f \in E$,

(i) $e \oplus f$ is defined iff $e \perp f$. (orthogonality law)

(ii) If $e \perp f$, then $f \perp e$ and $e \oplus f = f \oplus e$. (commutativity law)

(iii) If $e \perp f$ and $d \perp (e \oplus f)$, then $d \perp e$, $(d \oplus e) \perp f$, and $d \oplus (e \oplus f) = (d \oplus e) \oplus f$. (associativity law)

(iv) For each $e \in E$, $e^\perp$ is the unique element in $E$ such that $e \perp e^\perp$ and $e \oplus e^\perp = u$. (orthosupplementation law)

(v) If $e \perp u$, then $e = 0$. (zero-unit law)

If $(E, 0, u, \perp, \oplus)$ is an effect algebra, the relation $\leq$ is defined on $E$ by $e \leq f$ iff there exists $d \in E$ such that $e \perp d$ and $e \oplus d = f$.

If an effect algebra $(E, 0, u, \perp, \oplus)$ is regarded as a "logic" (i.e., as an algebraic model for a deductive logical calculus), then elements $e, f \in E$ can be thought of as "propositions," $e \leq f$ means that $e$ "implies" $f$, and $0, u \in E$ are "anti-tautological" and "tautological" constants, respectively. The orthogonality condition $e \perp f$ means that, in some sense, the propositions $e$ and $f$ "refute" each other. The orthosupplementation mapping $e \mapsto e^\perp$ is a version of "logical negation," and the "double negation law" $e = (e^\perp)^\perp$ holds. If $e \perp f$, then $e \oplus f$ is a version of "logical disjunction" of the "mutually refuting" propositions $e$ and $f$. Basic properties of effect algebras, e.g., the fact that an effect algebra $E$ is partially ordered by $\leq$ and that, for $e, f \in E$, $e \perp f \iff e \leq f^\perp$, can be found in [Foulis and Bennett, 1994].

In what follows, we regard all quantum logics as being effect algebras (but not necessarily conversely\(^9\)). If no confusion threatens, we usually refer to an effect algebra $(E, 0, u, \perp, \oplus)$ simply as $E$. Of course, both the prototype quantum logics $\mathcal{P}(\mathcal{F})$ and $\mathbb{E}(\mathcal{F})$ are effect algebras, and the partial order on the prototypes coincides with the relation $\leq$ in Definition 14.

Let $E$ be an effect algebra. Elements $e, f \in E$ are said to be orthogonal iff $e \perp f$. By recursion, a finite sequence $e_1, e_2, \ldots, e_n$ in $E$ is said to be jointly orthogonal with orthosum $e_1 \oplus e_2 \oplus \cdots \oplus e_n$ iff $e_1, e_2, \ldots, e_{n-1}$ is jointly orthogonal and $(e_1 \oplus e_2 \oplus \cdots \oplus e_{n-1}) \perp e_n$, in which case $e_1 \oplus e_2 \oplus \cdots \oplus e_n := (e_1 \oplus e_2 \oplus \cdots \oplus e_{n-1}) \oplus e_n$.

\(^9\)There are effect algebras that are quite bizarre and perhaps do not arise as a logic of a physical system.
\[ \cdots \oplus e_{n-1} \oplus e_n. \] The alternative notation \( \bigoplus_{i=1}^{n} e_i := e_1 \oplus e_2 \oplus \cdots \oplus e_n \) is also used for the orthosum. The condition of joint orthogonality and the orthosum are independent of the order in which the elements of the finite sequence are written.

**DEFINITION 15.** An infinite sequence \( e_1, e_2, \ldots \) in the effect algebra \( E \) is said to be **jointly orthogonal** iff every finite subsequence is jointly orthogonal. If \( e_1, e_2, \ldots \) is jointly orthogonal, and if the partial orthosums \( s_n := \bigoplus_{i=1}^{n} e_i \) have a supremum (least upper bound) \( s \) in the partially ordered set \( E \), we say that the sequence \( e_1, e_2, \ldots \) is **orthosummable** and we define its **orthosum** to be \( \bigoplus_{i=1}^{\infty} e_i := s \). An effect algebra \( E \) in which every jointly orthogonal infinite sequence is orthosummable is called a **\( \sigma \)-effect algebra**.

We note that both of our prototypes \( \mathbb{P}(\mathcal{I}) \) and \( \mathbb{E}(\mathcal{I}) \) are \( \sigma \)-effect algebras.

**EXAMPLE 16.** As usual, a Boolean algebra is a bounded, complemented, and distributive lattice. If \( \wedge_B \) and \( \vee_B \) are the infimum (greatest lower bound) and supremum (least upper bound) operations in the Boolean algebra \( B \) and if 0 and 1 are the smallest and largest elements in \( B \), then \( B \) is organized into an effect algebra \((B, 0, 1, \perp, \oplus)\) by defining \( a \perp b \) iff \( a \wedge_B b = 0 \), in which case \( a \oplus b := a \vee_B b \), and by defining \( a^\perp \) to be the Boolean complement of \( a \) for all \( a, b \in B \). A sequence in \( B \) is jointly orthogonal iff its elements are pairwise orthogonal, i.e., pairwise disjoint. Also, \( B \) is a \( \sigma \)-effect algebra iff it is \( \sigma \)-complete as a Boolean algebra (i.e., iff every countable subset of \( B \) has a supremum and an infimum in \( B \)).

**DEFINITION 17.** A **Boolean effect algebra** is an effect algebra \((B, 0, u, \perp, \oplus)\) such that the bounded partially ordered set, \((B, 0, u, \leq)\) is a complemented and distributive lattice—hence a Boolean algebra—and, for each \( b \in B \), \( b^\perp \) is the Boolean complement of \( b \) in \( B \). A **Boolean \( \sigma \)-effect algebra** is a Boolean effect algebra \( B \) such that, as a Boolean algebra, \( B \) is \( \sigma \)-complete.

If a Boolean algebra \( B \) (respectively, a \( \sigma \)-complete Boolean algebra) is organized into an effect algebra as in Example 16, then the resulting effect algebra is a Boolean effect algebra (respectively, a **Boolean \( \sigma \)-effect algebra**). Thus, Boolean algebras and Boolean effect algebras are essentially equivalent notions. **Caution:** There are simple examples [Greechie et al., 1995, Example 7.4] of effect algebras \((E, 0, u, \perp, \oplus)\) such that \((E, 0, u, \leq)\) is a Boolean algebra, but \( E \) is not a Boolean effect algebra because the orthosupplementation mapping \( e \mapsto e^\perp \) does not coincide with the Boolean complementation.

Partially ordered by set-inclusion, a field of subsets \( B \) of a set \( X \) is a Boolean algebra, and thus can be organized into a Boolean effect algebra as in Example 16. If \( B \) is a \( \sigma \)-field, then, as a Boolean effect algebra, \( B \) is a \( \sigma \)-effect algebra. **Caution:** As a Boolean effect algebra, a field of subsets \( B \) of a set \( X \) can be a \( \sigma \)-effect algebra even though \( B \) is not a \( \sigma \)-field. Redressing this bothersome situation is really what the Loomis-Sikorski theorem is all about [Dvurečenskij, 2000].

Boolean algebras are semantic models for classical propositional calculi, so Boolean effect algebras may be regarded as **classical "quantum" logics** (e.g., the
logic of experimentally testable propositions pertaining to a classical mechanical system).

**DEFINITION 18.** Let \( E_1 \) and \( E_2 \) be effect algebras with units \( u_1 \) and \( u_2 \), respectively. A mapping \( \phi : E_1 \to E_2 \) is called an **effect-algebra morphism** iff \( \phi(u_1) = u_2 \) and \( e \perp f \Rightarrow \phi(e) \perp \phi(f) \) with \( \phi(e \oplus f) = \phi(e) \oplus \phi(f) \) for all \( e, f \in E \). An **effect-algebra isomorphism** is a bijective effect-algebra morphism \( \phi \) such that \( \phi^{-1} \) is also an effect-algebra morphism. If \( E_1 \) and \( E_2 \) are \( \sigma \)-effect algebras, then an effect-algebra morphism \( \phi : E_1 \to E_2 \) is called a **\( \sigma \)-effect-algebra morphism** iff it preserves jointly orthogonal sequences and preserves the orthosums of such sequences. By a **symmetry** of an effect algebra \( E \), we mean an effect-algebra automorphism of \( E \), i.e., an effect-algebra isomorphism \( \xi : E \to E \).

We note that an effect-algebra morphism \( \phi : E_1 \to E_2 \) is order preserving, i.e., if \( e, f \in E \), then \( e \leq f \Rightarrow \phi(e) \leq \phi(f) \). As a consequence, an effect algebra isomorphism preserves all existing suprema and infima. In particular, every symmetry of a \( \sigma \)-effect algebra is a \( \sigma \)-effect-algebra morphism.

**DEFINITION 19.** If \( E \) is an effect algebra, regarded as a quantum logic, then a (finitely additive) \( E \)-valued observable is defined to be an effect-algebra morphism \( \alpha : B \to E \) from a Boolean effect algebra \( B \) to \( E \). If both \( B \) and \( E \) are \( \sigma \)-effect algebras and \( \alpha : B \to E \) is a \( \sigma \)-effect-algebra morphism, then the \( E \)-valued observable \( \alpha \) is said to be **\( \sigma \)-additive**.

If \( \alpha : B \to E \) is an observable and elements of \( B \) are interpreted as classical propositions, then for each \( b \in B \), the proposition \( p = \alpha(b) \) is a possibly "nonclassical" counterpart of \( b \) in the quantum logic \( E \). For the prototypes \( \mathbb{P}(\mathcal{H}) \) and \( \mathbb{E}(\mathcal{H}) \), the PV-measures and POV-measures, respectively, are \( \sigma \)-additive observables defined on the real Borel field \( B(\mathbb{R}) \).

**EXAMPLE 20.** The real unit interval \([0, 1] \subseteq \mathbb{R} \) is organized into an effect algebra \(([0, 1], 0, 1, \perp, \oplus) \) as follows: If \( p, q \in [0, 1] \), define \( p \perp q \) iff \( p + q \leq 1 \), in which case \( p \oplus q := p + q \), and define \( p^\perp := 1 - p \). Then \([0, 1] \) is a \( \sigma \)-effect algebra, where a sequence \( p_1, p_2, \ldots \) in \([0, 1] \) is jointly orthogonal iff the infinite series \( \sum_{i=1}^{\infty} p_i \) converges and \( \sum_{i=1}^{\infty} p_i \leq 1 \), in which case \( \bigoplus_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} p_i \).

If \( \mathcal{H} \) is a one-dimensional Hilbert space, then the unit interval \([0, 1] \subseteq \mathbb{R} \) is isomorphic as an effect algebra to \( \mathbb{E}(\mathcal{H}) \) under the mapping \( \lambda \mapsto \lambda \mathbf{1} \), and \( \mathbb{P}(\mathcal{H}) \) is the two-element Boolean algebra \( \{0, 1\} \). Thus, \([0, 1] \subseteq [0, 1] \subseteq \mathbb{R} \) is actually a special case of our prototypic triple \( \mathbb{P}(\mathcal{H}) \subseteq \mathbb{E}(\mathcal{H}) \subseteq \mathbb{G}(\mathcal{H}) \).

**DEFINITION 21.** If \( E \) is an effect algebra, regarded as a quantum logic, and \([0, 1] \subseteq \mathbb{R} \) is organized into an effect algebra as in Example 20, then a (finitely additive) **probability state**, or **probability measure**, for (or on) \( E \) is defined to be an effect-algebra morphism \( \pi : E \to [0, 1] \). If no confusion threatens, a probability state \( \pi \) on \( E \) is often called a **state** for short. If \( E \) is a \( \sigma \)-effect algebra and \( \pi : E \to [0, 1] \) is a \( \sigma \)-effect-algebra morphism, then the probability state \( \pi \) is said to be **\( \sigma \)-additive**.
If $p \in E$ and $\pi$ is a probability state for $E$, then $\pi(p)$ may be regarded as the probability assigned by the probability model $\pi$ to the (possibly) "nonclassical" proposition $p$. For the prototypes $\mathbb{P}(\mathfrak{H})$ and $\mathbb{E}(\mathfrak{H})$, the probability measures $\omega_D$ determined by the density operators $D \in \mathcal{D}(\mathfrak{H})$ are $\sigma$-additive probability states. Conversely, by a celebrated theorem of A. Gleason [Dvurečenskij, 1993], if $\mathfrak{H}$ has dimension 3 or more, then every $\sigma$-additive probability state on the prototypes has the form $\omega_D$ for a uniquely determined density operator $D$.

The interplay among observables, symmetries, probability states, Boolean algebras, quantum logics, and the real unit interval $[0,1] \subseteq \mathbb{R}$ is indicated by the diagram in Figure 1.

In Figure 1, the observable $\alpha$ is a morphism from the Boolean algebra, regarded as a Boolean effect algebra, into the quantum logic, the symmetry $\xi$ is an automorphism of the quantum logic, and the probability state $\pi$ is a morphism from the quantum logic to the unit interval $[0,1]$. The composition $\pi \circ \alpha$ is a (perhaps only finitely additive) probability measure on the Boolean algebra. The symmetry $\xi$ acts on the quantum logic, and induces an action on observables via $\alpha \mapsto \xi \circ \alpha$ (Heisenberg picture) and an action on states via $\pi \mapsto \pi \circ \xi$ (Schrödinger picture). If the Boolean algebra is a $\sigma$-field of sets, the quantum logic is a $\sigma$-effect algebra, and both $\alpha$ and $\pi$ are $\sigma$-morphisms, then $\pi \circ \alpha$ is a Kolmogorovian probability measure.

8 CLASSIFICATION OF EFFECT ALGEBRAS

In Section 7, we defined the notion of a $\sigma$-effect algebra and observed that both the sharp and unsharp prototypes, $\mathbb{P}(\mathfrak{H})$ and $\mathbb{E}(\mathfrak{H})$, are $\sigma$-effect algebras. In this section we set forth a number of additional conditions that an effect algebra $E$ might or
might not satisfy, and indicate whether or not the prototypes $\mathbb{P}(\mathfrak{F})$ and $\mathbb{E}(\mathfrak{F})$ satisfy these conditions. Here we only sketch some of the basic ideas. For a deeper and more systematic account, see [Dalla Chiara et al., 2004] and [Dvurečenskij and Pulmannová, 2000].

**Standing Assumption:** Henceforth we assume that $(E, 0, u, \bot, \oplus)$ is an effect algebra.

We write the infimum (greatest lower bound) of elements $e$ and $f$ in $E$, if it exists, as $e \land_E f$. Likewise, the supremum (least upper bound) of $e$ and $f$ in $E$, if it exists, is written as $e \lor_E f$. If we write an equation $e \land_E f = d$ (respectively, $e \lor_E f = d$) with $d, e, f \in E$, we mean that $e \land_E f$ (respectively, $e \lor_E f$) exists and equals $d$.

**DEFINITION 22.** The effect algebra $E$ is said to be *lattice ordered* if, as a partially ordered set, it is a lattice, i.e., iff $e \land_E f$ and $e \lor_E f$ exist for all $e, f \in E$. We say that $E$ is *distributive* (respectively, *modular*) iff it is lattice ordered and, as a lattice, it is distributive (respectively, modular). If every subset of $E$ has an infimum and a supremum, then $E$ is *complete* as a lattice.

Clearly, every complete lattice-ordered effect algebra is a $\sigma$-effect algebra. A Boolean effect algebra (Example 16) is lattice ordered and distributive, and the real unit interval $[0, 1]$ (Example 20) is lattice ordered, distributive, and complete. The sharp quantum logic $\mathbb{P}(\mathfrak{F})$ is lattice ordered and complete, it is modular iff $\mathfrak{F}$ is finite dimensional, and it is distributive iff $\mathfrak{F}$ has dimension 1 or 0. On the other hand, the unsharp quantum logic $\mathbb{E}(\mathfrak{F})$ is never lattice ordered unless $\mathfrak{F}$ is of dimension 1 or 0. The structure of finite modular effect algebras has been worked out by S. Sykes in [Sykes, 1997].

**DEFINITION 23.** (i) Elements $e, f \in E$ are said to be *complements* (or *complementary*) iff $e \land_E f = 0$ and $e \lor_E f = u$. (ii) $E$ is *complemented* iff every element $e \in E$ has at least one complement $f \in E$. (iii) A mapping $\prime : E \rightarrow E$ is a *complementation* iff $e'$ is a complement of $e$ for all $e \in E$. (iv) The orthosupplementation mapping $e \mapsto e' \bot$ is called an *orthocomplementation* iff it is a complementation mapping.

If $E$ is a Boolean effect algebra, then $E$ is not only complemented, but each element $e \in E$ has a unique complement, namely $e'$. If an element in a distributive effect algebra has a complement, then it is unique; however a distributive effect algebra need not be complemented—for instance, in the real unit interval $[0, 1]$, only 0 and 1 have complements. The mapping $P \mapsto P' = 1 - P$ is an orthocomplementation on the sharp quantum logic $\mathbb{P}(\mathfrak{F})$; however, if the dimension of $\mathfrak{F}$ is 2 or more, then every $P \in \mathbb{P}(\mathfrak{F})$, other than $P = 0$ and $P = 1$, has infinitely many complements. On the other hand, the unsharp quantum logic $\mathbb{E}(\mathfrak{F})$ fails to be complemented; however, if $P \in \mathbb{P}(\mathfrak{F})$, then $P$ has a complement in $\mathbb{E}(\mathfrak{F})$; indeed, the orthocomplement $P' = 1 - P$ of $P$ in $\mathbb{P}(\mathfrak{F})$ is also a complement of $P$ in $\mathbb{E}(\mathfrak{F})$.

**DEFINITION 24.** (i) Elements $e, f \in E$ are said to be *disjoint* iff $e \land_E f = 0$. (ii) An element $s \in E$ is said to be *sharp* iff $s \land_E s' = 0$, i.e., iff $s$ and $s'$ are
disjoint. (ii) The effect algebra $E$ is called an orthoalgebra (OA) iff every element in $E$ is sharp. (iii) An element $p \in E$ is said to be principal iff for all $e, f \in E$ with $e \perp f$, the condition $e, f \leq p$ implies that $e \oplus f \leq p$. (iv) The effect algebra $E$ is called an orthomodular poset (OMP) iff every element in $E$ is principal. (v) A lattice-ordered OMP is called an orthomodular lattice (OML) [Beran, 1985; Kalmbach, 1983].

Let $e, f \in E$ with $e \perp f$. Then $e \oplus f$ is an upper bound, but not necessarily the supremum of $e$ and $f$ in $E$. However, if $E$ is an orthoalgebra, then $e \oplus f$ is a minimal upper bound for $e$ and $f$ in $E$; and if $E$ is an OMP, then $e \oplus f = e \vee_E f$. The effect algebra $E$ is an orthoalgebra iff the orthosupplementation $e \mapsto e^\perp$ on $E$ is an orthocomplementation.

We can now be more precise about the meaning of "sharp" versus "unsharp." The effect algebra $E$, regarded as a quantum logic, is sharp iff every element in $E$ is sharp, i.e., iff $E$ is an orthoalgebra; otherwise it is unsharp. It is in this sense that $P(\mathcal{S})$ is sharp and $E(\mathcal{S})$ is unsharp. In fact, $P(\mathcal{S})$ is a complete orthomodular lattice. If $R \in E(\mathcal{S})$, then by Theorem 10, $R$ is sharp $\iff$ $R$ is principal $\iff$ $R \in P(\mathcal{S})$.

Every principal element in $E$ is sharp; hence every OMP is an orthoalgebra and, by definition, every OML is an OMP. Every Boolean effect algebra is an OML. Thus if $\text{Boole}$, $\text{OML}$, $\text{OMP}$, $\text{OA}$, and $\text{EA}$ represent the class of all Boolean algebras, orthomodular lattices, orthomodular posets, orthoalgebras, and effect algebras, respectively, we have

$$\text{Boole} \subseteq \text{OML} \subseteq \text{OMP} \subseteq \text{OA} \subseteq \text{EA}.$$ For the basic theory of OML's and OMP's, see [Beran, 1985; Kalmbach, 1983]; for OA's, see [Foulis et al., 1992].

If $E$ is an orthoalgebra, then for all $e, f \in E$, $e \perp f$ $\Rightarrow$ $e \wedge_E f = 0$, i.e., orthogonal elements are disjoint; however the converse condition may fail.

**DEFINITION 25.** (i) $E$ is pseudo-Boolean iff, for all $e, f \in E$, $e \wedge_E f = 0$ $\Rightarrow$ $e \perp f$, i.e., iff disjoint elements in $E$ are always orthogonal. (ii) An $\text{MV-algebra}$ is a pseudo-Boolean lattice-ordered effect algebra.

Every MV-algebra is distributive, every Boolean effect algebra is an MV-algebra, and every pseudo-Boolean OMP is Boolean, but there are pseudo-Boolean orthoalgebras that are not Boolean. With a different but equivalent set of axioms, MV-algebras were introduced by C.C. Chang [1957; 1958] to serve as semantic models for Łukasiewicz multi-valued logics. In [1965], D. Mundici used MV-algebras to classify AF $C^*$-algebras, and there is now a substantial literature on MV-algebras. The proof that MV-algebras, as originally defined by Chang, are mathematically the same as pseudo-Boolean lattice-ordered effect algebras follows from the work of Chovanec and Kópka in [1997]. We note that both of the "quantum logics" at the bottom of the diagram in Figure 1 (Section 7), i.e., the Boolean algebra and the real unit interval, are MV-algebras, hence are actually "classical logics." The observables and states link these classical logics to the quantum logic.
DEFINITION 26. The effect algebra $E$ has the Riesz decomposition property iff, whenever $a, b, c \in E$, $b \perp c$, and $a \leq b + c$, there exist $b_1, c_1 \in E$ such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 \oplus c_1$. An effect algebra with the Riesz decomposition property is called an RD-algebra.

An MV-algebra is the same thing as a lattice-ordered RD-algebra [Bennett and Foulis, 1995, Theorem 3.11]. A Boolean effect algebra is the same thing as an RD-algebra that is also an orthoalgebra. If $E$ satisfies the chain conditions (i.e., there are no properly ascending or descending infinite chains in $E$), then $E$ is an RD-algebra iff $E$ is an MV-algebra.

DEFINITION 27. A subset $S \subseteq E$ of the effect algebra $E$ is called a sub-effect algebra of $E$ iff (i) $0, u \in S$, (ii) $s \in S \Rightarrow s^\perp \in S$, and (iii) $s \oplus t \in S$ whenever $s, t \in S$ with $s \perp t$. A sub-effect algebra $S$ of $E$ is said to be normal iff, whenever $d, e, f \in E$, $d \oplus e \oplus f$ is defined, $d \oplus e \in S$, and $d \oplus f \in S$, it follows that $d \in S$.

For instance, by Theorem 11, $\mathbb{P}(5)$ is a normal sub-effect algebra of $\mathbb{E}(5)$. If $S$ is a sub-effect algebra of $E$, then $S$ is an effect algebra in its own right with 0 and $u$ as zero and unit, the restriction of $\perp$ to $S$ as the orthogonality relation, $s \mapsto s^\perp$ as the orthosupplementation, and the restriction to $S$ of $\oplus$ as the orthosummation. If $E$ is a Boolean effect algebra and $S$ is a normal sub-effect algebra of $E$, then $S$ is again a Boolean effect algebra; however, in general, a sub-effect algebra of a Boolean effect algebra need not be Boolean.

An orthoalgebra is covered by sub-effect algebras each of which is Boolean; hence, as a quantum logic, every orthoalgebra is "locally classical." Similarly, by a theorem of Riečanová [Riečanová, 2000], a lattice-ordered effect algebra is covered by sub-effect algebras each of which is an MV-algebra. Thus, as a quantum logic, a lattice-ordered effect algebra is "locally classical," although the classical subalgebras might involve propositions with multiple truth values.

DEFINITION 28. The center of $E$ is the subset $C$ of $E$ consisting of all elements $c \in E$ such that both $c$ and $c^\perp$ are principal and, for each element $e \in E$, there are orthogonal elements $e_1, e_2 \in E$ such that $e = e_1 \oplus e_2$ with $e_1 \leq c$, and $e_2 \leq c^\perp$. If 0 and $u$ are the only elements in $C$, then $E$ is irreducible.

The center $C$ of $E$ is a normal sub-effect algebra of $E$ and, as such, it is a Boolean effect algebra [Foulis, to appear(a)]. If $E$ is an orthoalgebra, then the center $C$ of $E$ is precisely the set of elements in $E$ that have unique complements in $E$. If $E$ is an MV-algebra, then the center $C$ of $E$ is precisely the set of elements in $E$ that have complements in $E$. Elements in the center of the effect algebra $E$ correspond to decompositions of $E$ as a direct product in the category of all effect algebras [Greechie et al., 1995], hence $E$ is irreducible iff it cannot be factored nontrivially as a direct product of effect algebras. Both of the prototypes $\mathbb{P}(5)$ and $\mathbb{E}(5)$ are irreducible.

DEFINITION 29. A set $A \subseteq E$ is coexistent iff it is contained in the range of an observable, i.e., iff there is a Boolean effect algebra $B$ and an effect-algebra

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10Since $b_1 \leq b$, $c_1 \leq c$, and $b \perp c$, it is necessary that $b_1 \perp c_1$. 
morphism \( \alpha : B \to E \) such that \( A \subseteq \alpha(B) \) [Lahti et al., 1998].

Clearly, a subset of a coexistent set is coexistent, and every Boolean sub-effect algebra \( B \) of \( E \) is coexistent. However, the Boolean sub-effect algebras of \( E \) do not necessarily cover \( E \). For instance, in the prototype \( \mathbb{E}(5) \), the singleton set \( \{ \frac{1}{2}1 \} \) is coexistent, but \( \frac{1}{2}1 \) does not belong to any Boolean sub-effect algebra of \( \mathbb{E}(5) \). However, if \( E \) is an orthoalgebra, then a subset \( A \) of \( E \) is coexistent if there is a Boolean sub-effect algebra \( B \) of \( E \) with \( A \subseteq B \), and of course this applies to the prototype \( \mathbb{P}(5) \).

**Lemma 30.** If \( a, b \in E \), then \( a \) and \( b \) are coexistent (i.e., \( \{ a, b \} \) is a coexistent set\(^{11} \)) iff there are elements \( d, e, f \in E \) such that \( d \oplus e \oplus f \) exists in \( E \), \( a = d \oplus e \), and \( b = d \oplus f \).

**Proof.** If elements \( d, e, f \in E \) exist satisfying the given conditions, let \( c := (d \oplus e \oplus f)^{+} \) and let \( B \) be the Boolean algebra of all subsets of \( \{1, 2, 3, 4\} \). Then there is a uniquely determined effect-algebra morphism \( \alpha : B \to E \) such that \( \alpha(\{1\}) = d \), \( \alpha(\{2\}) = e \), \( \alpha(\{3\}) = f \), and \( \alpha(\{4\}) = c \), whence \( \alpha(\{1, 2\}) = d \oplus e = a \) and \( \alpha(\{1, 3\}) = d \oplus f = b \), so \( \{a, b\} \) is contained in the range of the observable \( \alpha \).

Conversely, suppose that \( B \) is a Boolean algebra, \( \alpha : B \to E \) is an effect-algebra morphism, and there are elements \( p, q \in B \) such that \( \alpha(p) = a \) and \( \alpha(q) = b \). Then, with \( d := \alpha(p \wedge_B q) \), \( e := \alpha(p \wedge_B q^{+}) \), and \( f := \alpha(p^{+} \wedge_B q) \), we have \( d \oplus e \oplus f = \alpha(p \vee_B q) \in E \), \( d \oplus e = a \), and \( d \oplus f = b \).

As an immediate consequence of Lemma 30, if \( S \) is a normal sub-effect algebra of \( E \) and \( s, t \in S \), then \( s \) and \( t \) are coexistent in \( S \) iff \( s \) and \( t \) are coexistent in \( E \). Also, if \( a \in E \), then the singleton set \( \{a\} \) is coexistent since \( a = d \oplus e = d \oplus f \) with \( d := a \), \( e := 0 \), and \( f := 0 \).

A Boolean effect algebra is the same thing as an OMP in which every pair of elements is coexistent, and an MV-algebra is the same thing as a lattice-ordered effect algebra in which every pair of elements is coexistent\(^{12} \). There are orthoalgebras in which every pair of elements is coexistent but which are not Boolean effect algebras.

The unsharp quantum logic \( \mathbb{E}(5) \) is convex in the sense that it is closed under the formation of convex combinations \( \lambda A + (1 - \lambda)B \) for \( \lambda \in [0, 1] \subseteq \mathbb{R} \) and \( A, B \in \mathbb{E}(5) \), but the sharp quantum logic \( \mathbb{P}(5) \) is not. Convex effect algebras have been studied in [Beltrametti et al., 2000; Bugajski et al., 2000; Gudder et al., 1999; Lahti et al., 1998] and they admit a very satisfactory structure theory.

There are always plenty of \( E \)-valued observables, but there are effect algebras, even finite OML's, that do not carry any probability states at all. Such "stateless" effect algebras may not be of much interest as quantum logics since they do not admit (non-contextual) probability models. In general, if an effect algebra \( E \) is

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\(^{11}\)Note that we do not necessarily assume that \( a \neq b \).

\(^{12}\)In the literature, coexistent pairs \( a, b \in E \) are sometimes called "compatible pairs", or "Mackey compatible pairs," but later we shall be using the word "compatible" in a different sense, so we shall avoid this terminology.
to be regarded as a bona fide quantum logic, it should admit a good supply of probability states.

**DEFINITION 31.** Let $\Delta$ be a set of probability states on $E$. (i) We say that $\Delta$ is full, or order-determining, iff, for all $e, f \in E$, the condition $\pi(e) \leq \pi(f)$ for all $\pi \in \Delta$ implies that $e \leq f$. (ii) We say that $\Delta$ is a unital set of probability states iff, whenever $0 \neq e \in E$, there exists $\pi \in \Delta$ such that $\pi(e) = 1$.

For both of the prototypes $\mathbb{P}(\mathfrak{F})$ and $\mathbb{E}(\mathfrak{F})$ the set $\Psi$ of all countably additive probability states $\omega_\psi = \omega_{P_\psi}$ determined by state vectors $\psi$ in $\mathfrak{F}$ is full. For the sharp quantum logic $\mathbb{P}(\mathfrak{F})$, $\Psi$ is unital; but for the unsharp quantum logic $\mathbb{E}(\mathfrak{F})$, it is not. More generally, if an effect algebra $E$ carries a unital set of probability states, it is an orthoalgebra.

Every Boolean effect algebra $B$ carries a full and unital set of probability states, but even if $B$ is complete as a lattice, it might not admit any $\sigma$-additive probability states at all. For example, the reduced Borel algebra, i.e., the Boolean algebra of all Borel subsets of $[0, 1] \subseteq \mathbb{R}$ modulo the ideal of meager Borel subsets of $[0, 1]$, is a complete Boolean effect algebra, but it has no countably additive probability states. (See [Horn and Tarski, 1948].) On the other hand, a $\sigma$-field of subsets of a set $X$, regarded as a Boolean $\sigma$-effect algebra as in Example 16, admits an order-determining set of $\sigma$-additive probability states, namely the $\{0, 1\}$-valued “Dirac states” determined by the points $x \in X$.

**9 PARTIALLY ORDERED ABELIAN GROUPS**

With the prototype $G(\mathfrak{F})$ in mind, we recall that a partially ordered abelian group is an abelian group $G$ equipped with a partial order relation $\leq$ that is translation invariant in the sense that, for all $g, h, k \in G$, $g \leq h \Rightarrow g + k \leq h + k$. If $G$ is a partially ordered abelian group, then $G^+ := \{g \in G \mid 0 \leq g\}$, called the positive cone in $G$, has the properties (i) $0 \in G^+$, (ii) $g, h \in G^+ \Rightarrow g + h \in G^+$, and (iii) $g, -g \in G^+ \Rightarrow g = 0$. Conversely, if $G$ is an (additively written) abelian group and $G^+ \subseteq G$ has properties (i), (ii), and (iii), then there is one and only one way to organize $G$ into a partially ordered abelian group with $G^+$ as its positive cone, namely by defining $g \leq h \Leftrightarrow h - g \in G^+$ for all $g, h \in G$. In this section, we review some basic facts about partially ordered abelian groups. Details can be found in [Goodearl, 1985, Chapter 1].

**Standing Assumption:** In this section, we assume that $G$ is a partially ordered abelian group with positive cone $G^+$.

**DEFINITION 32.**

(i) If $G^+$ generates $G$ as a group, then $G$ is said to be directed [Goodearl, 1985, page 4].

(ii) An element $u \in G^+$ is called an order unit\textsuperscript{13} iff, for every $g \in G$, there is a

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\textsuperscript{13}Some authors use the terminology “strong order unit.”
positive integer \( n \) such that \( g \leq nu \) [Goodearl, 1985, page 4].

(iii) We say that \( G \) is archimedean [Goodearl, 1985, page 20] iff, whenever \( g, h \in G \) and \( ng \leq h \) for all positive integers \( n \), it follows that \( -g \in G^+ \).

(iv) As in [Goodearl, 1985, page 19], \( G \) is called unperforated iff, for all \( g \in G \) and every positive integer \( n, 0 \leq ng \Rightarrow 0 \leq g \).

(v) The abelian group \( G \) is torsion free iff, for all \( g \in G \) and every positive integer \( n, ng = 0 \Rightarrow g = 0 \).

It is easy to see that a partially ordered abelian group \( G \) is directed iff \( G = \{ a - b \mid a, b \in G^+ \} \); for instance, by (4), the Hermitian group \( G(\mathcal{F}) \) is directed. Also, \( 1 \) is an order unit in \( G(\mathcal{F}) \) and \( G(\mathcal{F}) \) is archimedean, unperforated, and torsion free.

If a partially ordered abelian group \( G \) has an order unit, then it is directed [Goodearl, 1985, page 4]. If \( G \) is unperforated, then it is obviously torsion free. If \( G \) is directed and archimedean, then it is unperforated [Goodearl, 1985, Proposition 1.24], hence torsion free.

Let \( A \subseteq G \). We understand that \( A \) is partially ordered by the induced partial order, i.e., the restriction to \( A \) of the partial order \( \leq \) on \( G \). If \( a, b \in A \), then the infimum (greatest lower bound), if it exists, of \( a \) and \( b \) as calculated in \( A \) will be written as \( a \wedge_A b \), and a similar convention applies to the supremum (least upper bound) \( a \vee_A b \) if it exists. Similarly, if \( (a_i)_{i \in I} \) is a family of elements in \( A \), then an existing infimum or supremum of the family as calculated in \( A \) will be written as \( \bigwedge_A \{ a_i \mid i \in I \} \) or \( \bigvee_A \{ a_i \mid i \in I \} \), respectively. As the mapping \( g \mapsto -g \) on \( G \) is order reversing and of period two, there is a duality on \( G \) whereby properties of suprema are converted to properties of infima and vice versa.

DEFINITION 33. The subset \( A \subseteq G \) is said to be lattice ordered iff \( A \) is a lattice under the induced partial order, i.e., \( a \wedge_A b \) and \( a \vee_A b \) exist for all \( a, b \in A \). If \( G \) itself is lattice ordered, it is called an (abelian) lattice-ordered group or, for short, an (abelian) \( \ell \)-group [Darnel, 1995].

If \( G \) is an \( \ell \)-group, then it is directed [Goodearl, 1985, Proposition 1.5] and unperforated [Goodearl, 1985, Proposition 1.22], and therefore it is torsion free.

DEFINITION 34. A subset \( A \subseteq G \) has the Riesz interpolation property iff, for all \( a, b, x, y \in A \) with \( a, b \leq x, y \) (i.e., \( a \leq x, a \leq y, b \leq x, \) and \( b \leq y \)) there exists \( t \in A \) such that \( a, b \leq t \leq x, y \). If \( G \) itself has the Riesz interpolation property, it is called an (abelian) interpolation group [Goodearl, 1985, Chapter 2].

Let \( A \subseteq G \). If \( A \) is lattice ordered, then it obviously has the Riesz interpolation property (in Definition 34, just take \( t \) to be any element in \( A \) such that \( a \vee_A b \leq t \leq a \wedge_A y \)). In particular, every abelian \( \ell \)-group is an interpolation group. If \( A \) satisfies the chain conditions (i.e., there are no properly ascending or descending infinite chains in \( A \)), then \( A \) is lattice ordered iff it has the Riesz interpolation property.
Unless the Hilbert space $\mathcal{H}$ has dimension 0 or 1, the Hermitian group $\mathbb{G}(\mathcal{H})$ is not an interpolation group, hence it is not an \ell-group; in fact, $\mathbb{G}(\mathcal{H})$ is an anti-lattice, i.e., two operators in $\mathbb{G}(\mathcal{H})$ have a supremum (or an infimum) in $\mathbb{G}(\mathcal{H})$ iff one of the operators is less than or equal to the other$^{14}$ [Kadison, 1951]. However, even though $\mathbb{G}(\mathcal{H})$ is not lattice ordered (unless the dimension of $\mathcal{H}$ is less than 2), the norm closure of the set of all real polynomials in a fixed Hermitian operator is a lattice-ordered subgroup of $\mathbb{G}(\mathcal{H})$, so $\mathbb{G}(\mathcal{H})$ is covered by subgroups that are lattice ordered.

**DEFINITION 35.** The set $A \subseteq G$ is said to be **monotone $\sigma$-complete**$^{15}$ iff every ascending (respectively, descending) sequence $a_1 \leq a_2 \leq \cdots$ (respectively, $a_1 \geq a_2 \geq \cdots$) in $A$ that is bounded above (respectively, below) in $A$ has a supremum $\bigvee_A \{a_i \mid i=1,2,\ldots\}$ (respectively an infimum $\bigwedge_A \{a_i \mid i=1,2,\ldots\}$) in $A$.

If $G$ is an \ell-group, then $G$ is monotone $\sigma$-complete iff it is Dedekind $\sigma$-complete, i.e., if every countable subset with an upper (respectively lower) bound in $G$ has a supremum (respectively, an infimum) in $G$ [Goodearl, 1985, Lemma 16.7]. The Hermitian group $\mathbb{G}(\mathcal{H})$ is monotone $\sigma$-complete, as is the unsharp quantum logic $\mathbb{E}(\mathcal{H}) \subseteq \mathbb{G}(\mathcal{H})$. Of course, the sharp quantum logic $\mathbb{P}(\mathcal{H})$ is a complete lattice; hence, a fortiori, it is monotone $\sigma$-complete.

If $G$ and $H$ are partially ordered abelian groups, and $\theta: G \rightarrow H$ is a group homomorphism, then it is clear that the condition $\theta(G^+) \subseteq H^+$ holds iff $\theta$ is order preserving, i.e., for all $a, b \in G$, $a \leq b \Rightarrow \theta(a) \leq \theta(b)$. Simple examples show that even if $\theta: G \rightarrow H$ is an order-preserving group isomorphism, $\theta^{-1}: H \rightarrow G$ need not be order preserving. If $\theta: G \rightarrow H$ is a group isomorphism and both $\theta$ and $\theta^{-1}$ are order preserving, then $\theta$ is called an **isomorphism of partially ordered abelian groups**.

10 **UNIVERSAL GROUPS**

The sharp and unsharp prototypes $\mathbb{P}(\mathcal{H})$ and $\mathbb{E}(\mathcal{H})$ are subsets of the Hermitian group $\mathbb{G}(\mathcal{H})$, and their structures as effect algebras are determined by the structure of $\mathbb{G}(\mathcal{H})$. With this in mind, we proceed to study relations between effect algebras and abelian groups—especially partially ordered abelian groups. We maintain our convention that $E$ is an effect algebra with unit $u$.

**DEFINITION 36.** If $K$ is an abelian group, then a mapping $\phi: E \rightarrow K$ is called a $K$-valued measure on $E$ iff $\phi(e \oplus f) = \phi(e) + \phi(f)$ whenever $e, f \in E$ with $e \perp f$.

For example, a probability state $\pi: E \rightarrow [0,1] \subseteq \mathbb{R}$ is an $\mathbb{R}$-valued measure on $E$.

**DEFINITION 37.** A pair $(\mathcal{G}, \gamma)$ is called a **universal group** for the effect algebra $E$ iff (i) $\mathcal{G}$ is an abelian group, (ii) $\gamma: E \rightarrow \mathcal{G}$ is a $\mathcal{G}$-valued measure, (iii) $\gamma(E)$

$^{14}$More generally, in [Kadison, 1951] it is shown that the self-adjoint part of a von Neumann algebra $A$ is an anti-lattice iff $A$ is a factor.

$^{15}$Some authors refer to this condition as “Dedekind monotone $\sigma$-complete” to emphasize the boundedness requirement, but we prefer to follow the terminology in [Goodearl, 1985, page 269].
generates $\mathcal{G}$ as a group, and (iv) whenever $\phi: E \to K$ is a $K$-valued measure, there exists a group homomorphism $\bar{\phi}: \mathcal{G} \to K$ such that $\bar{\phi} \circ \gamma = \phi$.

Every effect algebra $E$ has a universal group $(\mathcal{G}, \gamma)$ which is uniquely determined by $E$ up to an isomorphism [Foulis and Bennett, 1994, Theorem 9.2 and ff.]. In view of the uniqueness, we often refer to $(\mathcal{G}, \gamma)$ as the universal group for $E$. Suppose that $(\mathcal{G}, \gamma)$ is the universal group for $E$ and let $K$ be an abelian group. If $\bar{\phi}: \mathcal{G} \to K$ is a group homomorphism, then $\phi: E \to K$ defined by $\phi := \bar{\phi} \circ \gamma$ is a $K$-valued measure and, since $\gamma(E)$ generates $\mathcal{G}$, the group homomorphism $\bar{\phi}$ is uniquely determined by $\phi$. Consequently, there is a bijective correspondence $\phi \leftrightarrow \bar{\phi}$ between $K$-valued measures $\phi$ on $E$ and group homomorphisms $\bar{\phi}: \mathcal{G} \to K$ given by $\phi = \bar{\phi} \circ \gamma$. Thus, the universal group is a device for converting group-valued measures into group homomorphisms.

**LEMMA 38.** Let $G$ be a partially ordered abelian group and let $\phi: E \to G$ be a $G$-valued measure on $E$. Then the following conditions are mutually equivalent:

(i) If $e, f \in E$, then $\phi(e) \leq \phi(f) \Rightarrow e \leq f$.

(ii) If $d, e \in E$, then $\phi(d) + \phi(e) \leq \phi(u) \Rightarrow d \perp e$.

(iii) If $e_1, \ldots, e_n \in E$ with $\sum_{i=1}^{n} \phi(e_i) \leq \phi(u)$, it follows that $e_1, \ldots, e_n$ is a jointly orthogonal sequence in $E$.

**Proof.** Assume (i) and suppose $d, e \in E$ with $\phi(d) + \phi(e) \leq \phi(u)$, i.e., $\phi(d) \leq \phi(u) - \phi(e)$. As $e \ominus e^\perp = u$ and $\phi$ is a $G$-valued measure, we have $\phi(e) + \phi(e^\perp) = \phi(u)$, whence $\phi(u) - \phi(e) = \phi(e^\perp)$, so $\phi(d) \leq \phi(e^\perp)$, and it follows from (i) that $d \perp e^\perp$, i.e., $d \perp e$. Conversely, assume (ii) and suppose $e, f \in E$ with $\phi(e) \leq \phi(f)$. But $\phi(f) = \phi(u) - \phi(f^\perp)$, so $\phi(e) + \phi(f^\perp) \leq \phi(u)$, and it follows from (ii) that $e \perp f^\perp$, i.e., $e \leq f$. Therefore, (i) $\Leftrightarrow$ (ii).

That (ii) $\Rightarrow$ (iii) follows from a straightforward induction on $n$, and that (iii) $\Rightarrow$ (ii) is obvious. ■

If $G$ is a partially ordered abelian group and $v \in G^+$, we define the $v$-interval $G^+[0, v] := \{g \in G \mid 0 \leq g \leq v\}$. The proof of the following lemma is straightforward.

**LEMMA 39.** Let $G$ be a partially ordered abelian group and let $v \in G^+$. Then the $v$-interval in $G$ can be organized into an effect algebra $(G^+[0, v], 0, v, \perp, ^\perp, \ominus)$ as follows: for $g, h \in G^+[0, v]$, $g^\perp := v - g$, and $g \perp h$ iff $g + h \leq v$, in which case $g \ominus h := g + h$.

For instance, $E(\mathcal{J}) = G(\mathcal{J})^+[0, 1]$. Thus, as per the following definition, $E(\mathcal{J})$ is an "interval effect algebra."}

**DEFINITION 40.** If there is a partially ordered abelian group $G$ and an element $v \in G^+$ such that $E$ is isomorphic as an effect algebra to $G^+[0, v]$, we say that $E$ is an interval effect algebra (IEA).
The proof of the following theorem is implicit in [Foulis and Bennett, 1994], but for clarity we make it explicit here.

**Theorem 41.** Let \((G, \gamma)\) be the universal group for \(E\) and define \(G^+\) to be the set of all elements of the form \(\sum_{i=1}^{n} \gamma(e_i)\) where \(e_1, e_2, \ldots, e_n\) is a finite sequence of (not necessarily distinct) elements in \(E\). Then the following conditions are mutually equivalent:

(i) \(E\) is an IEA.

(ii) If \(e_1, \ldots, e_m \in E\), then \(\sum_{i=1}^{m} \gamma(e_i) = 0 \Rightarrow e_i = 0\) for \(i = 1, 2, \ldots, m\); and if \(e, f \in E\), then \(\gamma(f) - \gamma(e) \in G^+ \Rightarrow e \leq f\).

(iii) \(G\) is a directed partially ordered abelian group with \(G^+\) as its positive cone and \(\gamma: E \rightarrow G^+[0, \gamma(u)]\) is an effect-algebra isomorphism.

**Proof.** (i) \(\Rightarrow\) (ii). Suppose (i) holds. Then, by Definition 40, there is a partially ordered abelian group \(G\), an element \(v \in G^+\), and an effect-algebra isomorphism \(\phi: E \rightarrow G^+[0, v] \subseteq G\). Evidently, \(\phi: E \rightarrow G\) is a \(G\)-valued measure, whence there is a group homomorphism \(\phi: G \rightarrow G\) such that \(\phi = \phi \circ \gamma\).

Let \(e_1, \ldots, e_m \in E\) with \(\sum_{i=1}^{m} \gamma(e_i) = 0\). Applying the group homomorphism \(\phi\) to the latter equation and using the fact that \(\phi = \phi \circ \gamma\), we find that \(\sum_{i=1}^{m} \phi(e_i) = 0\) in the effect algebra \(G^+[0, v]\), and this implies that \(\phi(e_1) = \cdots = \phi(e_m) = 0\). But \(\phi\) is an effect-algebra isomorphism, so \(e_1 = \cdots = e_m = 0\).

Now suppose \(e, f \in E\) with \(\gamma(f) - \gamma(e) \in G^+\). Then there exist \(f_1, \ldots, f_n \in E\) such that \(\gamma(f) - \gamma(e) = \sum_{i=1}^{n} \gamma(f_i)\), and applying \(\phi\) to the latter equation, we obtain \(\phi(f) - \phi(e) = \sum_{i=1}^{n} \phi(f_i) \in G^+\). Consequently, \(\phi(e) \leq \phi(f)\), and since \(\phi\) is an effect-algebra isomorphism, it follows that \(e \leq f\).

(ii) \(\Rightarrow\) (iii). Suppose that (ii) holds. Obviously \(0 = \gamma(0) \in G^+\) and \(G^+\) is closed under addition. Suppose \(w, -w \in G^+\). Then there exist \(e_1, \ldots, e_n, e_{n+1}, \ldots, e_m \in E\) with \(w = \sum_{i=1}^{n} \gamma(e_i)\) and \(-w = \sum_{i=n+1}^{m} \gamma(e_i)\), whence \(0 = w + (-w) = \sum_{i=1}^{m} \gamma(e_i)\), and it follows from (ii) that \(e_1 = \cdots = e_m = 0\). Consequently \(\gamma(e_i) = 0\) for \(i = 1, 2, \ldots, n\), so \(w = 0\). Therefore, \(G\) is a partially ordered abelian group with positive cone \(G^+\). As \(\gamma(E)\) generates \(G\), it follows that \(G\) is directed.

Let \(w \in G^+[0, \gamma(u)]\). Then \(w, \gamma(u) - w \in G^+\), so there exist \(e_1, \ldots, e_n, \ldots, e_m \in E\) with \(w = \sum_{i=1}^{n} \gamma(e_i)\) and \(\gamma(u) - w = \sum_{i=n+1}^{m} \gamma(e_i)\), whence \(\sum_{i=1}^{m} \gamma(e_i) \leq \sum_{i=1}^{n} \gamma(e_i) = w + (\gamma(u) - w) = \gamma(u)\). Consequently, by Lemma 38, \(e_1, \ldots, e_n\) is a jointly orthogonal sequence in \(E\). Therefore, since \(\gamma\) is a \(G\)-valued measure, we have \(w = \gamma(\bigoplus_{i=1}^{n} e_i)\), and it follows that \(\gamma: E \rightarrow G^+[0, \gamma(u)]\) is surjective.

If \(e, f \in E\) and \(\gamma(e) = \gamma(f)\), then \(e \leq f\) and \(f \leq e\), so \(e = f\), proving that \(\gamma\) is injective. Therefore \(\gamma: E \rightarrow G^+[0, \gamma(u)]\) is a bijection. Now suppose \(e, f \in E\) and \(\gamma(e) \perp \gamma(f)\) in \(G^+[0, \gamma(u)]\). Then \(\gamma(e) + \gamma(f) \leq \gamma(u)\), whence \(e \perp f\) by Lemma 38, so \(\gamma(e) + \gamma(f) = \gamma(e \oplus f)\), and it follows that \(\gamma^{-1}(\gamma(e) + \gamma(f)) = e \oplus f\). Therefore, \(\gamma: E \rightarrow G^+[0, \gamma(u)]\) is an effect-algebra isomorphism.

That (iii) \(\Rightarrow\) (i) is obvious.
THEOREM 42. If $E$ has a full (i.e., order-determining) set of probability states, then $E$ is an IEA.

Proof. Let $(G, \gamma)$ be the universal group for $E$, and suppose that $\Delta$ is a full set of probability states for $E$. Thus, for each $\pi \in \Delta$, there is a group homomorphism $\bar{\pi}: G \to \mathbb{R}$ such that $\pi = \bar{\pi} \circ \gamma$. It will be sufficient to establish condition (ii) in Theorem 41. Thus, let $e_1, ..., e_m, e, f \in E$ with $\sum_{i=1}^{m} \gamma(e_i) = 0$ and $\gamma(f) - \gamma(e) \in G^+$. Then $\sum_{i=1}^{m} \pi(e_i) = \bar{\pi}(\sum_{i=1}^{m} \gamma(e_i)) = 0$, and since each summand $\pi(e_i)$ is a nonnegative real number, it follows that $\pi(e_i) = 0$ for $i = 1, 2, ..., m$. Therefore, since $\Delta$ is full, $e_i = 0$ for $i = 1, 2, ..., m$. Also, there exist $f_1, ..., f_n \in E$ with $\gamma(f) - \gamma(e) = \sum_{i=1}^{n} \gamma(f_i)$, and upon applying the group homomorphism $\bar{\pi}$ to the latter equation, we obtain $\pi(f) - \pi(e) = \sum_{i=1}^{n} \pi(f_i) \geq 0$. Therefore, $\pi(e) \leq \pi(f)$ for all $\pi \in \Delta$, so $e \leq f$.

Suppose that $E$ is an IEA with universal group $(G, \gamma)$. Then by Theorem 41, $\gamma: E \to G^+ [0, \gamma(u)]$ is an effect-algebra isomorphism, hence by identifying each $e \in E$ with $\gamma(e) \in G^+ [0, \gamma(u)]$, we can assume that $\gamma: E \leftrightarrow G$ is the inclusion mapping$^16$, $u \in G$, and $E = G^+ [0, u]$, i.e., $E$ is realized as the $u$-interval in its own universal group. With this identification in force, we say for short that $G$ is the universal group for $E$. If $G$ is the universal group for the IEA $E$, and if $K$ is any abelian group, then there is a bijective correspondence $\phi \leftrightarrow \phi$ between $K$-valued measures $\phi$ on $E$ and group homomorphisms $\phi: G \to K$, where $\phi$ is the restriction of $\phi$ to $E$.

It turns out that $G(\mathcal{F})$ is the universal group for $E(\mathcal{F})$.$^17$ Therefore, every probability state $\pi: E(\mathcal{F}) \to [0, 1] \subseteq \mathbb{R}$ can be extended (uniquely) to a group homomorphism $\bar{\pi}: G(\mathcal{F}) \to \mathbb{R}$, which can be interpreted as an "expectation-value mapping" on bounded observables.$^18$

By [Bennett and Foulis, 1997, Corollary 2.5], every sub-effect algebra of an IEA is again an IEA. Therefore, since $E(\mathcal{F})$ is an IEA, it follows that $P(\mathcal{F})$ is an IEA, so it has its own universal group. As of the writing of this article, a perspicuous characterization of the universal group of $P(\mathcal{F})$ is not known.$^19$

11 ABSTRACTION FROM $G(\mathcal{F})$—UNITAL GROUPS

In this section we formulate and study pairs $E \subseteq G$ that generalize the pair $E(\mathcal{F}) \subseteq G(\mathcal{F})$ consisting of the unsharp quantum logic $E(\mathcal{F})$ and the partially ordered abelian group $G(\mathcal{F})$ of bounded observables on $\mathcal{F}$. Later in Section 16, we complete the abstraction process by extracting from $E$ a subset $P$ that generalizes the sharp quantum logic $P(\mathcal{F})$ of projection operators on $\mathcal{F}$.

$^16$The inclusion mapping $\gamma: E \leftrightarrow G$ is the mapping defined by $\gamma(e) := e$ for all $e \in E$.

$^17$This follows from [Bennett and Foulis, 1997, Corollary 4.7].

$^18$Note that $\bar{\pi}$ might not be $\sigma$-additive.

$^19$In [Harding et al., to appear], it is shown that the universal group of $P(\mathcal{F})$ is a divisible group.
DEFINITION 43. Let \( G \) be a partially ordered abelian group with positive cone \( G^+ = \{ g \in G \mid 0 \leq g \} \) and let \( u \in G^+ \). If every element in \( G^+ \) is the sum of a finite sequence of (not necessarily distinct) elements of the \( u \)-interval \( G^+[0, u] \), i.e., if \( G^+[0, u] \) generates \( G^+ \) as a semigroup, we say that \( u \) is a generative element in \( G^+ \) [Bennett and Foulis, 1997].

By Theorem 13, 1 is a generative element in the Hermitian group \( G(\mathbb{S}) \). If \( E \) is an IEA with unit \( u \) and \( (G, \gamma) \) is the universal group of \( E \), then by Theorem 41, \( \gamma(u) \) is a generative element in \( G \).

DEFINITION 44. A unital group is a directed partially ordered abelian group \( G \) equipped with a distinguished generative element \( u \) called the unit. The \( u \)-interval \( E := G^+[0, u] \subseteq G \), organized into an interval effect algebra as in Lemma 39, is called the unit interval in \( G \), and elements \( e \in E \) are called effects in \( G \). If a unital group \( G \) is the universal group for its own unit interval \( E \), then \( G \) is called a unigroup [Foulis et al., 1998].

If \( G \) is a unital group with unit \( u \) and unit interval \( E \), then it is clear that \( u \) is an order unit in \( G \). Also, \( E \) generates \( G^+ \) as a semigroup and \( G^+ \) generates \( G \) as a group, hence \( E \) generates \( G \) as a group. The unit interval \( E \) is monotone \( \sigma \)-complete iff \( E \) is a \( \sigma \)-effect algebra. Also, if \( G \) is monotone \( \sigma \)-complete, then so is \( E \), but in general, not conversely\(^{20}\).

With \( 1 \) as the unit and \( \mathbb{E}(\mathbb{S}) \) as the unit interval, the Hermitian group \( G(\mathbb{S}) \) is a unigroup. More generally, if \( E \) is an IEA with unit \( u \) and \( G \) is the universal group for \( E \), then \( G \) is a unigroup with unit \( u \) and unit interval \( E \). Here is an example to show that not every unital group is a unigroup.

EXAMPLE 45. Let \( G = \mathbb{Z} \), the additive group of integers, but organized into a partially ordered abelian group with the nonstandard positive cone \( G^+ := \{0, 2, 3, 4, ...\} \) obtained by removing 1 from the standard positive cone \( \mathbb{Z}^+ \) in \( \mathbb{Z} \). Then with unit \( u := 5 \), \( G \) is a (non-archimedean) unital group, and the unit interval \( E = \{0, 2, 3, 5\} \) is a Boolean effect algebra. The only group homomorphisms \( G \rightarrow \mathbb{R} \) are of the form \( n \mapsto cn \) where \( c \) is a real constant. But, for any \( p \in [0, 1] \subseteq \mathbb{R} \), \( \pi_p: E \rightarrow \mathbb{R} \) defined by \( \pi_p(0) := 0 \), \( \pi_p(2) := p \), \( \pi_p(3) := 1 - p \), and \( \pi_p(5) := 1 \) is a probability state on \( E \), and it can be extended to a group homomorphism \( G \rightarrow \mathbb{R} \) (if and) only if \( p = 1/5 \). Therefore, \( G \) is not a unigroup.

Standing Assumption: Henceforth, we assume that \( G \) is a unital group with unit \( u \neq 0 \) and with unit interval \( E \).

From the point of view of quantum logic, it may well be that the only reasonable generalizations of \( G(\mathbb{S}) \) and its unit interval \( \mathbb{E}(\mathbb{S}) \) are archimedean unigroups and their unit intervals. But, as we do not wish to rule out any potentially interesting quantum logics, we shall not necessarily assume that \( G \) is archimedean nor that it is a unigroup. Even if \( G \) is not a unigroup, it is a homomorphic image of a unigroup as per the following theorem.

\(^{20}\) However, see Theorem 63.
THEOREM 46. Let $(G, \gamma)$ be the universal group for the unit interval $E$ in $G$. Then $G$ is a unigroup with unit $\gamma(u)$, and if $E$ is the unit interval in $G$, then $\gamma: E \to E$ is an effect-algebra isomorphism. Furthermore, there exists an order-preserving group homomorphism $\tilde{i}: G \to G$ of $G$ onto $G$ such that $\tilde{i}(G^+) = G^+$ and $\tilde{i}(\gamma(e)) = e$ for all $e \in E$.

Proof. The effect algebra $E$ is an IEA, hence by Theorem 41, $G$ is a unigroup with unit $\gamma(u)$, and $\gamma$ is an effect-algebra isomorphism of $E$ onto the unit interval $E$ in $G$.

The inclusion mapping $\iota: E \hookrightarrow G$ is a $G$-valued measure, hence it induces a group homomorphism $\iota: G \to G$ such that $\iota = \tilde{i} \circ \gamma$. Thus, $e \in E \Rightarrow \iota(\gamma(e)) = \iota(e) = e$.

If $p \in G^+$, then $p = \sum_{i=1}^{n} \gamma(e_i)$ with $e_i \in E$, so $\iota(p) = \sum_{i=1}^{n} e_i \in G^+$. Therefore $\iota(G^+) \subseteq G^+$. Conversely, if $g \in G^+$, there exist $e_i \in E$ with $g = \sum_{i=1}^{n} e_i$. But then $p := \sum_{i=1}^{n} \gamma(e_i) \in G^+$ with $\iota(p) = g$, and it follows that $\iota(G^+) = G^+$. Consequently, $\tilde{i}: G \to G$ is an order-preserving group homomorphism.

If $g \in G$, we can write $g = a - b$ with $a, b \in G^+$. As $G^+ = \iota(G^+)$ there exist $x, y \in (G^+)$ with $a = \tilde{i}(x)$ and $b = \tilde{i}(y)$. Hence $g = \tilde{i}(x - y)$, so $\tilde{i}: G \to G$ maps $G$ onto $G$.

EXAMPLE 47. In Example 45, the universal group for $E = \{0, 2, 3, 5\}$ is $(Z \times Z, \gamma)$ with coordinatewise addition and partial order on $Z \times Z$, where $\gamma(2) = (1, 0), \gamma(3) = (0, 1)$, and $\gamma(5) = (1, 1)$. Accordingly, the surjective group homomorphism

$$\tilde{i}: Z \times Z \to G$$

in Theorem 46 is given by $\gamma(m, n) = 2m + 3n$ for all $(m, n) \in Z \times Z$.

DEFINITION 48. If $H$ and $K$ are unital groups with units $v$ and $w$, respectively, then a unital morphism $\phi: H \to K$ is an order-preserving group homomorphism that is normalized in the sense that $\phi(v) = w$. A unital isomorphism is a bijective unital morphism with an inverse that is also a unital morphism, and a unital symmetry of $H$ is a unital automorphism of $H$, i.e., a unital isomorphism $\xi: H \to H$.

THEOREM 49. Suppose that $H$ is a unigroup and $L$ is the unit interval in $H$. Then there is a bijective correspondence $\alpha \leftrightarrow \tilde{\alpha}$ between effect-algebra morphisms $\alpha: L \to E \subseteq G$ and unital morphisms $\tilde{\alpha}: H \to G$ such that $\tilde{\alpha}$ is an extension of $\alpha$.

Proof. Let $v$ be the unit in $H$. If $\alpha: L \to E \subseteq G$ is an effect-algebra morphism, then $\alpha(v) = u, \alpha(L) \subseteq E \subseteq G^+$, and $\alpha: L \to G$ is a $G$-valued measure. Since $H$ is a unigroup, $\alpha$ can be extended (uniquely) to a group homomorphism $\tilde{\alpha}: H \to G$, with $\tilde{\alpha}(v) = u$ and $\tilde{\alpha}(L) \subseteq G^+$. As $L$ generates $H^+$ as a semigroup, it follows that $\tilde{\alpha}(H^+) \subseteq G^+$, hence $\tilde{\alpha}: H \to G$ is an order-preserving group homomorphism. Therefore, each effect-algebra morphism $\alpha: L \to E$ extends uniquely to a unital morphism $\tilde{\alpha}: H \to G$. Conversely, it is clear that the restriction to $L$ of a unital morphism from $H$ to $G$ is an effect-algebra morphism from $L$ to $E$. ■
With the usual total order and the standard positive cone \( \mathbb{R}^+ = \{ \lambda^2 \mid \lambda \in \mathbb{R} \} \),
the additive group \( \mathbb{R} \) of real numbers is an archimedean unigroup with unit 1, and
the unit interval \( \mathbb{R}^+ [0, 1] \) is just the standard unit interval \( [0, 1] \subseteq \mathbb{R} \). In the sequel,
we always regard \( \mathbb{R} \) as a unigroup in this way.

**DEFINITION 50.** A state on \( G \) is a unital morphism \( \omega : G \to \mathbb{R} \) [Goodearl, 1985,
page 60]. The state space \( \Omega(G) \) is the set of all states \( \omega \) on \( G \), and is regarded as
a subset of the locally convex real vector space \( \mathbb{R}^G \) of all real-valued functions on
\( G \) with pointwise operations and the topology of pointwise convergence. A set of
states \( \Delta \subseteq \Omega(G) \) is cone determining (or order determining) iff \( G^+ = \{ g \in G \mid 0 \leq \omega(g) \text{ for all } \omega \in \Delta \} \).

In the study of quantum logics and unital groups, the word "state" is used in
a number of different (but related) senses, however the context usually makes it
clear what is intended. We note that the restriction \( \pi := \omega|_E \) to \( E \) of a state
\( \omega \in \Omega(G) \) is a probability state on \( E \), and if \( G \) is a unigroup \( \pi = \omega|_E \) provides a
bijection \( \pi \leftrightarrow \omega \) between probability states on \( E \) and states on \( G \).

**DEFINITION 51.** If \( H \) and \( K \) are monotone \( \sigma \)-complete unital groups, then a
unital morphism \( \phi : H \to K \) is called a unital \( \sigma \)-morphism iff it preserves the
suprema of all bounded monotone ascending sequences\(^{21}\). If the unital group \( G \) is
monotone \( \sigma \)-complete, then we say that a state \( \omega \in \Omega(G) \) is a \( \sigma \)-state iff it is an
unital \( \sigma \)-morphism \( \omega : G \to \mathbb{R} \).

For the Hermitian group \( G(\mathfrak{H}) \), the expectation mappings \( \omega_D \) corresponding to
density operators \( D \) are \( \sigma \)-states, and if the dimension of \( \mathfrak{H} \) is finite and greater than
2, then by Gleason’s theorem [Dvurečenskij, 1993], \( \Omega(G(\mathfrak{H})) = \{ \omega_D \mid D \in \mathcal{D}(\mathfrak{H}) \} \).
If \( \mathfrak{H} \) is infinite dimensional, there are always states on \( G(\mathfrak{H}) \) that are only finitely
additive, and hence are not of the form \( \omega_D \). In any case, the set \( \Psi \) of expectation
mappings \( \omega_\psi = \omega_{\phi_\psi} \) corresponding to state vectors \( \psi \in \mathfrak{H} \) is cone determining for
the unigroup \( G(\mathfrak{H}) \).

For an arbitrary unital group \( G \), the state space \( \Omega(G) \) is a nonempty convex
(and even \( \sigma \)-convex) subset of \( \mathbb{R}^G \) [Goodearl, 1985, Corollary 4.4, Proposition
6.5] and is a compact Hausdorff space in the topology of pointwise convergence
[Goodearl, 1985, Proposition 6.2]. The set of all extreme points of the convex set
\( \Omega(G) \), called the extreme boundary of \( \Omega(G) \), is denoted by \( \partial_e \Omega(G) \). By the Krein-
Mil'man theorem, \( \Omega(G) \) is the closure of the convex hull of \( \partial_e \Omega(G) \). Evidently,
a subset \( \Delta \subseteq \Omega(G) \) is cone determining iff it determines the partial order on \( G \)
in the sense that, for all \( g, h \in G \), \( g \leq h \iff \omega(g) \leq \omega(h) \) for all \( \omega \in \Omega(G) \). See
[Goodearl, 1985, Theorem 4.14] for a proof of the following important result.

**THEOREM 52.** The unital group \( G \) is archimedean iff \( \Omega(G) \) is cone determining.

The following definition is suggested by Equation (1).

**DEFINITION 53.** Define the order-unit pseudonorm \( \| \cdot | : G \to \mathbb{R}^+ \) by \( \| g \| := \inf \{ m/n \mid 0 \leq m \in \mathbb{Z}, 0 < n \in \mathbb{Z}, \text{ and } -mu \leq ng \leq mu \} \) for all \( g \in G \). If
\(^{21}\)If \( \phi \) is a unital \( \sigma \)-morphism, then by duality it preserves the infima of all bounded monotone
descending sequences.
$\|g\| = 0 \Rightarrow g = 0$ for all $g \in G$, then $\| \cdot \|$ is called the order-unit norm on $G$. If $V$ is a real normed linear space and $f: G \rightarrow V$ is a group homomorphism, then $f$ is bounded if there exists $\beta \in \mathbb{R}^+$ such that $\|f(g)\| \leq \beta\|g\|$ for every $g \in G$, and if $f$ is bounded, then $\|f\|$ is defined to be the infimum of all such nonnegative real numbers $\beta$.

**THEOREM 54.** Let $g, h \in G$ and $m \in \mathbb{Z}$. Then: (i) $\|mg\| = |m| \cdot \|g\|$. (ii) $\|g + h\| \leq \|g\| + \|h\|$. (iii) $-h \leq g \leq h \Rightarrow \|g\| \leq \|h\|$. (iv) $\|u\| = 1$. (v) $\|g\| = \max\{\|\omega(g)\| \mid \omega \in \Omega\}$ (cf. Equation (2)). (vi) If $G$ is archimedean, then $\| \cdot \|$ is a norm on $G$. (vii) If $G$ is archimedean and $0 < n, m \in \mathbb{Z}$, then $\|g\| \leq m/n \Rightarrow -mu \leq ng \leq mu$. (viii) If $\omega: G \rightarrow \mathbb{R}$ is a group homomorphism, then $\omega \in \Omega(G)$ iff $\omega$ is bounded and $\|\omega\| = \omega(u) = 1$.

**Proof.** See [Goodearl, 1985, pp. 120–121 and 123].

If $G$ is archimedean, then the order-unit norm $\| \cdot \|$ can be used to organize $G$ into a metric space, the distance between $g$ and $h$ in $G$ being given by $\|g - h\|$. With the resulting norm topology, $G$ is an abelian topological group.

The following example is of special interest in connection with the so-called algebraic approach to quantum theory [Emch, 1972].

**EXAMPLE 55.** Let $\mathcal{A}$ be a unital $C^*$ algebra and let $G(\mathcal{A})$ be the real Banach algebra of all self-adjoint elements in $\mathcal{A}$. Then, as an additive abelian group and with the unity element 1 of $\mathcal{A}$ as the unit, $G(\mathcal{A})$ is an archimedean unigroup with positive cone $G(\mathcal{A})^+ := \{a^2 \mid a \in G(\mathcal{A})\} = \{bb^* \mid b \in \mathcal{A}\}$ and with unit interval $E(\mathcal{A}) := G(\mathcal{A})^+[0,1]$. Furthermore, the order-unit norm on $G(\mathcal{A})$ coincides with the restriction to $G(\mathcal{A})$ of the $C^*$-algebra norm on $\mathcal{A}$. The set $P(\mathcal{A})$ of all projections (self-adjoint idempotents) in $\mathcal{A}$, which is a normal sub-effect algebra of $E(\mathcal{A})$, is precisely the set of all sharp elements in $E(\mathcal{A})$. Thus, the triple $P(\mathcal{A}) \subseteq E(\mathcal{A}) \subseteq G(\mathcal{A})$ generalizes the triple $\mathbb{P}(\mathcal{H}) \subseteq \mathbb{E}(\mathcal{H}) \subseteq G(\mathcal{H})$.

The real Banach algebra $G(\mathcal{A})$ in Example 55 is actually a Jordan algebra [Topping, 1965], and it is also an order-unit normed Banach space with 1 as the order unit [Alfsen, 1971, page 69]. Moreover, the order-unit norm on $G(\mathcal{A})$ coincides with the restriction to $G(\mathcal{A})$ of the norm on the $C^*$-algebra $\mathcal{A}$.

A condition even stronger than being a $\sigma$-state, and that is especially important in the study of states $\omega$ on the self-adjoint part $G(\mathcal{A})$ of a $C^*$-algebra $\mathcal{A}$, is that $\omega$ is normal in the sense that it preserves the suprema of ascending nets in $G(\mathcal{A})$ [Kadison and Ringrose, 1983, Chapter 7].

### 12 SEMISIMPLICIAL UNITAL GROUPS

Hilbert-space-based quantum mechanics still harbors many mysteries, both philosophically and mathematically, and it can be useful to look at simple "toy" models that manifest some of the salient features of the prototypes $\mathbb{P}(\mathcal{H})$, $\mathbb{E}(\mathcal{H})$, and $G(\mathcal{H})$, but that are easier to contemplate and study. The Kochen-Specker spin-1 model
[Kochen and Specker, 1967], Gudder's pattern-recognition model [Gudder, 1998, Section 2], the $\epsilon$-model of Aerts [Aerts, 1999], Spekkens's toy model [Spekkens, 2004], Foulis's firefly box [Foulis, 1999], and the D-model of Gudder and Foulis [Foulis and Gudder, 2001] are some examples. Toy models often involve finite quantum logics, hence we devote this section to a very brief sketch of some of the theory of unital groups $G$ with finite unit intervals $E$. For more details, see [Foulis, 2003a; Foulis, 2003b; Foulis and Greechie, 2004].

Ordered in the usual way, the system $\mathbb{Z}$ of integers is an archimedean totally ordered additive abelian group with the standard positive cone $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and as $\mathbb{Z}$ is totally ordered, it is an $\ell$-group. Clearly, the order units in $\mathbb{Z}$ are the integers $u \in \mathbb{Z}$ with $1 \leq u$. If $1 \leq u \in \mathbb{Z}$, then, as easily seen, the totally ordered abelian $\ell$-group $\mathbb{Z}$ is a unigroup with unit $u$, and the resulting unit interval is the $(u + 1)$-element chain $C_u := \{0, 1, 2, \ldots, u\}$ with $0 < 1 < 2 < \cdots < u$.

If $n$ is a positive integer, then $\mathbb{Z}^n$ denotes the $n$-fold cartesian product $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ organized into an abelian group with coordinatewise addition. With the coordinatewise partial order, $\mathbb{Z}^n$ is an abelian $\ell$-group, called the standard simplicial group of rank $n$.

Let $\mathbb{Z}^n$ be the standard simplicial group of rank $n$. Then the positive cone in $\mathbb{Z}^n$ is the $n$-fold cartesian product $(\mathbb{Z}^+)^n$ consisting of all vectors $(z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n$ such that $0 \leq z_i$ for $i = 1, 2, \ldots, n$. The infimum and supremum of two vectors in $\mathbb{Z}^n$ is given by the coordinatewise minimum and maximum of the vectors, respectively, and the order units in $\mathbb{Z}^n$ are the vectors $(u_1, u_2, \ldots, u_n)$ such that $1 \leq u_i$ for $i = 1, 2, \ldots, n$. If $(u_1, u_2, \ldots, u_n)$ is an order unit in $\mathbb{Z}^n$, then, as easily seen, $\mathbb{Z}^n$ is a unigroup with $(u_1, u_2, \ldots, u_n)$ as unit, and its unit interval $\mathcal{C}$ is an MV-algebra. Clearly, the unit interval is a cartesian product of chains

$$\mathcal{C} = C_{u_1} \times C_{u_2} \times \cdots \times C_{u_n}$$

with coordinatewise relations and operations.

**DEFINITION 56.** Following [Goodearl, 1985, page 47], we define a simplicial group to be a partially ordered abelian group $H$ for which there exists an isomorphism $\phi: H \rightarrow \mathbb{Z}^n$ of partially ordered abelian groups from $H$ onto the standard simplicial group $\mathbb{Z}^n$ of rank $n$. A simplicial unital group is a unital group that is also a simplicial group.

Since a simplicial group is isomorphic as a partially ordered abelian group to an $\ell$-group, it follows that every simplicial group is an $\ell$-group. By [Goodearl, 1985, Corollary 3.14], a partially ordered abelian group $H \neq \{0\}$ is simplicial iff $H$ is an interpolation group, there is an order unit in $H$, and there are no properly descending infinite chains in $H^+$. If $G$ is a simplicial unital group with unit $u$, then $G$ is an $\ell$-unigroup (i.e., an $\ell$-group that is also a unigroup) and the unit interval $E$ in $G$ is a finite MV-algebra. Conversely, it can be shown that every finite MV-algebra $E$ is isomorphic as an effect algebra to the unit interval $E$ in a

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22In [Goodearl, 1985], the zero group $\{0\}$ is also regarded as a simplicial group.
simplicial unigroup $G$, and it follows that $E$ is isomorphic to a finite product of finite chains.

**DEFINITION 57.** The partially ordered abelian group $H$ is called a semisimplicial group iff there is a positive integer $n$ and a bijective order-preserving group homomorphism $\phi$ from $H$ onto the standard simplicial group $\mathbb{Z}^n$. A semisimplicial unital group (respectively, a semisimplicial unigroup) is a unital group (respectively, a unigroup) that is also a semisimplicial group.

Although the group homomorphism $\phi : G \to \mathbb{Z}^n$ in Definition 57 is a bijection (hence a group isomorphism) that preserves order, it is not required that $\phi$ is an isomorphism of partially ordered abelian groups—specifically, there is no requirement that $\phi^{-1} : \mathbb{Z}^n \to G$ also preserves order. In other words, although the image $\phi(G^+)$ of the positive cone $G^+$ under $\phi$ is contained in the standard positive cone $(\mathbb{Z}^+)^n$ in $\mathbb{Z}^n$, $\phi(G^+)$ can be a proper subcone of $(\mathbb{Z}^+)^n$. The unital group $G$ in Example 45 is semisimplicial; but it is not simplicial, it is not archimedean, it is not an interpolation group, and it is not a unigroup. By [Foulis and Greechie, 2004, Corollary 5.3] a semisimplicial group is simplicial iff it is an interpolation group.

Here is the fundamental theorem for unital groups with finite unit intervals.

**THEOREM 58.** The unital group $G$ is semisimplicial iff the unit interval $E$ in $G$ is finite and $G$ is torsion free.

**Proof.** If $E$ is finite and $G$ is torsion free, then $G$ is semisimplicial by [Foulis, 2003a, Lemma 3.3]. Conversely, suppose $G$ is semisimplicial. Then $G$ is isomorphic as a group to the torsion-free abelian group $\mathbb{Z}^n$, so $G$ is torsion free. Also, by [Foulis and Greechie, 2004, Lemma 1 (ii), Section 5], $E$ is finite.

To study a semisimplicial unigroup group $G$, it is convenient to use the group isomorphism $\phi : G \to \mathbb{Z}^n$ in Definition 57 to identify each element $g \in G$ with the corresponding vector $\phi(g) = (g_1, g_2, \ldots, g_n) \in \mathbb{Z}^n$. This identification provides a definite computational advantage in that elements of $G$ are vectors $g = (g_1, g_2, \ldots, g_n)$ with integer coordinates, elements of $G^+$ have nonnegative integer coordinates (but not necessarily conversely!), and addition is performed coordinatewise. Furthermore, by [Foulis, 2003b, Theorem 5.1], we have the following criterion for $G$ to be archimedean.

**THEOREM 59.** Let $n$ be a positive integer, let $G = \mathbb{Z}^n$ as an additive abelian group, and suppose that $G^+ \subseteq (\mathbb{Z}^+)^n$. Then $G$ is archimedean iff there is an $n \times m$ matrix $[b_{ij}]$ over $\mathbb{Z}$ such that $G^+ = \{ g \in (\mathbb{Z}^+)^n \mid 0 \leq \sum_{i=1}^n b_{ij} g_i \text{ for } j = 1, 2, \ldots, m \}$.

To paraphrase Theorem 59, a semisimplicial unital group is archimedean iff its positive cone is determined by a finite set of homogeneous linear inequalities over $\mathbb{Z}$. A necessary and sufficient condition for a semisimplicial unital group to be a unigroup can be found in [Foulis, 2003b, Corollary 2.8].
We denote by $2 := \{0, 1\}$ the two-element Boolean algebra (or Boolean effect algebra). As is well-known, a finite Boolean algebra (or Boolean effect algebra) $B$ is isomorphic to an $n$-fold cartesian product $2^n$, with coordinatewise relations and operations. The unigroup for $2^n$ is the standard simplicial group $Z^n$ with the unit $(1,1,\ldots,1)$.

Figures that have come to be called Greechie diagrams are useful for specifying finite orthoalgebras. Let $E$ be a finite orthoalgebra with unit $u$ and let $A$ be the set of atoms (minimal nonzero elements) in $E$. Then the structure of $E$ is determined by the maximal coexistent subsets of $A$. Furthermore, if $a_1, a_2, \ldots, a_k$ are distinct elements of $A$, then $\{a_1, a_2, \ldots, a_k\}$ is a maximal coexistent subset of $A$ iff $a_1, a_2, \ldots, a_k$ are jointly orthogonal and $a_1 \oplus a_2 \oplus \cdots \oplus a_k = u$. In the Greechie diagram for $E$ the atoms of $E$ correspond to nodes lying on line segments (or, in more complicated situations, by smooth curves), and atoms lying on each line segment are understood to be maximal coexistent sets. For instance, Figure 2 shows the Greechie diagram for the 8-element Boolean algebra $2^3$ with three atoms labeled $p, q, r$.

![Figure 2](image)

Figure 3 is the Greechie diagram for an OML consisting of the 12 elements \{0, a, b, c, d, e, a^\perp, b^\perp, c^\perp, d^\perp, e^\perp, u\}, and having the structure of two copies of $2^3$ "glued" together on the zero, the unit, the atom $c$ and the orthocomplement $c^\perp$ of $c$. This OML is isomorphic to the logic of testable propositions for the "firefly box" in [Foulis, 1999], it is an IEA, and it is the unit interval in the archimedean semisimplicial unigroup given in the following example.

EXAMPLE 60. Let $G := Z^4$ as an additive abelian group, and let $G^+$ be the set of all vectors $(x, y, z, w)$ with $0 \leq x, y, z, w$ and $x \leq y + z$. Then $G$ is an archimedean semisimplicial unigroup with unit $u = (1, 1, 1, 1)$ and positive cone $G^+$. The unit interval $E$ in $G$ is the 12-element OML with the Greechie diagram in Figure 3, where $a = (1,1,0,0), b = (0,0,1,0), c = (0,0,0,1), d = (0,1,0,0),$ and $e = (1,0,1,0)$.

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\[23\text{Similar diagrams for more general finite effect algebras, showing the "multiplicities" of atoms, have been proposed, but haven't been widely used.}\]
13 INTERPOLATION UNIGROUPS AND THEIR UNIT INTERVALS

The following theorem was proved by S. Pulmannová in [Pulmannová, 1999]. We reproduce the proof here because it illustrates some important ideas and techniques.

**Theorem 61.** Let $H$ be an interpolation group and let $v$ be an order unit in $H$. Then, with $v$ as the unit, $H$ is a unigroup.

**Proof.** As $H$ has an order unit, it is directed [Goodearl, 1985, page 4]. Let $h \in H^+$. As $v$ is an order unit in $H$, there is a positive integer $n$ such that $h \leq nv$, hence by [Goodearl, 1985, Proposition 2.2 (b)], there exist $e_i \in H^+$, with $e_i \leq v$ for $i = 1, 2, ..., n$, such that $h = \sum_{i=1}^{n} e_i$. Therefore, $H$ is a unital group with unit $v$.

To prove that $H$ is a unigroup, suppose $K$ is an abelian group and $\phi: H^+ [0, v] \rightarrow K$ is a $K$-valued measure. We are going to extend $\phi$ to $\phi^+: H^+ \rightarrow K$ by defining $\phi^+(h) := \sum_{i=1}^{n} \phi(e_i)$ for $h \in H^+$, where $h = \sum_{i=1}^{n} e_i$ with $e_i \in H^+[0, v]$ for $i = 1, 2, ..., n$, but we have to show that this definition is independent of the choice of the $e_i$. If also $h = \sum_{j=1}^{m} f_j$ with $f_j \in H^+[0, v]$ for $j = 1, 2, ..., m$, then by [Goodearl, 1985, Proposition 2.2 (c)], there exist $z_{ij} \in H^+$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ such that $e_i = \sum_{j=1}^{m} z_{ij}$ for $i = 1, 2, ..., n$. Hence, $\sum_{i=1}^{n} \phi(e_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(z_{ij}) = \sum_{j=1}^{m} \sum_{i=1}^{n} \phi(z_{ij}) = \sum_{j=1}^{m} \phi(f_j)$, so $\phi^+$ is well defined. Evidently, $\phi^+$ preserves addition on $H^+$.

We now extend $\phi^+$ to $\bar{\phi}: H \rightarrow K$ by defining $\bar{\phi}(h) := \phi^+(a) - \phi^+(b)$ for $h = a - b$ with $a, b \in H^+$. Since $\phi^+$ preserves addition, it follows that $\bar{\phi}$ is well defined and that $\bar{\phi}: H \rightarrow K$ is a group homomorphism. Therefore, $H$ is a unigroup.

By Theorem 61, an interpolation group $G$ with a specified order unit $u$ is not only a unital group, but a unigroup, and we shall refer to such a $G$ as an *interpolation unigroup*. We recall that an RD-algebra is an effect algebra with the Riesz decomposition property (Definition 26) and that every MV-algebra is an RD-algebra. The following fundamental theorem is due to K. Ravindran.

**Theorem 62.** If $G$ is an interpolation unigroup, then the unit interval $E$ in $G$ is an RD-algebra. Conversely, every RD-algebra is an IEA and is realized as the unit interval in an interpolation unigroup.

**Proof.** As a consequence of [Goodearl, 1985, Proposition 2.1 (a) and (b)], if $G$ is an interpolation unigroup, then $E$ is an RD-algebra. A proof of the converse can be found in [Ravindran, 1996].

As the unigroup for an IEA is uniquely determined up to a unital isomorphism, Theorem 62 provides a categorical equivalence $E \leftrightarrow G$ between RD-algebras $E$. 

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and interpolation unigroups $G$. There are interpolation unigroups—even totally ordered unigroups—that are not archimedean, e.g., [Foulis, 2004b, Example 4.1].

THEOREM 63. Let $G$ be an interpolation unigroup with unit interval $E$. Then:

(i) If $G$ is monotone $\sigma$-complete, then $G$ is archimedean.

(ii) $G$ is monotone $\sigma$-complete iff $E$ is a $\sigma$-effect algebra.

Proof. See [Goodearl, 1985, Proposition 16.9 and Theorem 16.10].

As an MV-algebra is necessarily an RD-algebra, it follows from Theorem 62 that every MV-algebra $E$ is an IEA and can be realized as the unit interval in an interpolation unigroup $G$. For the categorical equivalence $E \leftrightarrow G$ between RD-algebras and interpolation unigroups, the question of just when $E$ is an MV-algebra is settled by the following celebrated theorem of D. Mundici [Mundici, 1965].

THEOREM 64. If $G$ is an $\ell$-group and $u$ is an order unit in $G$, then $G$ is an $\ell$-unigroup (i.e., a lattice-ordered unigroup ) with unit $u$ and the unit interval $E$ in $G$ is an MV-algebra. Conversely, every MV-algebra is an IEA and is realized as the unit interval in an $\ell$-unigroup.

As a Boolean effect algebra is necessarily an MV-algebra, it follows from Mundici's theorem that every Boolean effect algebra $E$ can be realized as the unit interval in an $\ell$-unigroup. For the categorical equivalence $E \leftrightarrow G$ between RD-algebras and interpolation unigroups, N. Ritter [Ritter, 2000] has settled the question of just when $E$ is a Boolean effect algebra. Ritter's theorem involves the notion of a minimal order unit in $G$, i.e., an order unit for which there is no strictly smaller order unit. We shall reproduce the proof of Ritter's theorem (Theorem 68 below) because it illustrates some important ideas and techniques.

Before giving the statement and proof of Ritter's theorem, we shall need some preliminary definitions and a lemma involving the notion of a field $B$ of subsets of a nonempty set $X$. Recall that, partially ordered by set inclusion $\subseteq$, a field $B$ of subsets of $X$ is a Boolean algebra, and hence can be organized into a Boolean effect algebra $(B, \emptyset, X, \perp, 1, \oplus)$ as in Example 16.

DEFINITION 65. If $B$ is a field of subsets of the nonempty set $X$, we define $G(X, B, Z)$ to be the partially ordered abelian group of pointedwise addition and pointwise partial order, of all bounded functions $g : X \to Z$ such that $g^{-1}(n) \in B$ for all $n \in Z$. The constant function assigning the integer 1 to every element $x \in X$ is denoted by $1 \in G(X, B, Z)$.

Since a function $g \in G(X, B, Z)$ is bounded and integer-valued, it can take on only finitely many values. Thus, each function $g \in G(X, B, Z)$ corresponds to a

\footnote{Actually, $G(X, B, Z)$ is a commutative ring with pointwise multiplication. See Example 82 below.}
partition of $X$ into finitely many sets of the form $g^{-1}(n) \in B$ as $n$ runs through the integers in the range $g(X)$ of $g$.

**Lemma 66.** Let $B$ be a field of subsets of a nonempty set $X$. Then, with the constant function 1 as unit, $G(X,B)$ is an archimedean $\ell$-unigroup and 1 is a minimal order unit in $G(X,B,Z)$. The unit interval $E(X,B,Z)$ in $G(X,B,Z)$ is the set of characteristic set functions $\chi_M$ of sets $M \in B$, hence the Boolean effect algebra $B$ is isomorphic to $E(X,B,Z)$ under the mapping $M \mapsto \chi_M$.

**Proof.** That $G(X,B,Z)$ is a partially ordered abelian group is clear, as is the fact that $G(X,B,Z)$ is archimedean. Since the functions in $G(X,B,Z)$ are bounded, 1 is an order unit in $G(X,B,Z)$. Obviously, $G(X,B,Z)$ is lattice ordered, the greatest lower bound and least upper bound of two functions $g,h \in G(X,B,Z)$ being given by the pointwise minimum and maximum, respectively, of $g$ and $h$.

Therefore, by Theorem 61, $G(X,B,Z)$ is an $\ell$-unigroup with unit 1. For all $x \in X$, an effect $e \in E(X,B,Z)$ satisfies $0 \leq e(x) \leq 1$ with $e(x) \in Z$; hence $e(x) = 0$ or $e(x) = 1$, and we have $M := e^{-1}(1) \in B$ and $e = \chi_M$. Conversely, it is clear that $\chi_M \in E(X,B,Z)$ for all $M \in B$. Evidently, $M \mapsto \chi_M$ is an effect-algebra isomorphism of $B$ onto $E(X,B,Z)$.

Finally, suppose $e$ is an order unit in $E(X,B,Z)$, $e \leq 1$, but $e \neq 1$. Then $e = \chi_M$ for some $M \in B$ with $M \neq X$, so $N := X \setminus M \neq \emptyset$ and $\chi_N \in E(X,B)$. As $e = \chi_M$ is an order unit, there is a positive integer $n$ such that $\chi_N \leq n\chi_M$. Choosing $x \in N$, we obtain the contradiction $1 = \chi_N(x) \leq n\chi_M(x) = n \cdot 0 = 0$.  

If $B$ is a field of subsets of a nonempty set $X$, then by Lemma 66, $G(X,B,Z)$ is a **Boolean unigroup** as per the following definition.

**Definition 67.** A **Boolean unigroup** is an interpolation unigroup in which the unit is a minimal order unit.

Ritter's theorem states that, in the correspondence $E \leftrightarrow G$ between RD-algebras $E$ and interpolation unigroups $G$, the Boolean effect algebras correspond to Boolean unigroups. More precisely, here is Ritter's theorem.

**Theorem 68.** Every Boolean effect algebra is an IEA and is realized as the unit interval in an archimedean $\ell$-unigroup in which the unit is a minimal order unit. Conversely, if $G$ is a Boolean unigroup, then $G$ is an archimedean $\ell$-unigroup and the unit interval $E$ in $G$ is a Boolean effect algebra.

**Proof.** Let $B$ be a Boolean effect algebra. By Stone's representation theorem [Stone, 1936], $B$ is isomorphic as a Boolean algebra, hence as a Boolean effect algebra, to a field of subsets $B$ of a nonempty set $X$. If we identify $B$ with the characteristic set functions $\chi_M$ of sets $M \in B$, then by Lemma 66 we obtain a realization of $B$ as the unit interval $E(X,B)$ in the archimedean $\ell$-unigroup $G(X,B)$ for which 1 is a minimal order unit.

Conversely, suppose that $G$ is an interpolation unigroup with unit interval $E$ and that the unit $u$ in $G$ is a minimal order unit. Let $e \in E$ and let $t \in E$ with $t \leq e, u - e$. Then $u - e, e \leq u - t \in E$, and it follows that $u = (u - e) + e \leq 2(u - t)$. 
If \( g \in G \), there is a positive integer \( n \) such that \( g \leq nu \), whence \( g \leq (2n)(u - t) \), so \( u - t \) is an order unit in \( G \). But \( 0 \leq u - t \leq u \), so \( u - t = u \) by the minimality of \( u \), and we conclude that \( t = 0 \). Now suppose \( g \in G \) and \( g \leq e, u - e \). As \( G \) is an interpolation group and \( 0 \leq e, u - e \), there exists \( t \in G \) with \( 0, g \leq t \leq e, u - e \). By the argument given above, \( t = 0 \), so \( g \leq 0 \), and it follows that \( e \land G(u - e) = 0 \). Thus, every element of \( E \) is a so-called characteristic element of \( G \) [Goodearl, 1985, page 127], so by [Goodearl, 1985, Theorem 8.7], \( E \) is a Boolean effect algebra. Hence, by the first part of the theorem, the universal group \( G \) for \( E \) is an archimedean \( \ell \)-unigroup.

In Figure 1, Section 7, denote the Boolean algebra by \( B \), let the IEA \( E \) be the quantum logic, and suppose that \( E \) is the unit interval in the unital group \( G \). By Theorem 68, the Boolean effect algebra \( B \) is realized as the unit interval in a Boolean unigroup \( H \). The observable \( \alpha: B \to E \) is a \( G \)-valued measure on \( B \), hence it can be extended uniquely to a unital morphism \( \bar{\alpha} \). We replace the unit interval \([0, 1]\) in Figure 1 by its unigroup, the real numbers \( \mathbb{R} \), and we replace the probability state \( \pi \) by a state \( \omega \in \Omega(G) \). (If \( G \) is a unigroup, we can take \( \omega \) to be the unique extension \( \pi \) of \( \pi \).) The symmetry \( \xi \) in Figure 1 is replaced by a unital symmetry \( \eta \) on \( G \). (If \( G \) is a unigroup, we can take \( \eta \) to be the unique extension \( \xi \) of \( \xi \).) Thus, Figure 4 below is the counterpart at the level of unital groups for Figure 1 at the level of quantum logics.

![Figure 4](image-url)

If the Boolean algebra \( B \) is \( \sigma \)-complete (e.g., if \( B \) is a \( \sigma \)-field of sets), then by Theorem 63, the Boolean unigroup \( H \) is monotone \( \sigma \)-complete. If also, the unital group \( G \) is monotone \( \sigma \)-complete, \( \bar{\alpha} \) is a \( \sigma \)-morphism, and \( \omega \) is a \( \sigma \)-state, then \( \omega \circ \bar{\alpha} \) is a \( \sigma \)-state, and its restriction to the Boolean \( \sigma \)-algebra \( B \) is a countably additive probability measure.
14 UNITAL GROUPS OF REAL-VALUED FUNCTIONS

If \(X\) is a nonempty set, we understand that the function space \(\mathbb{R}^X\) consisting of all functions \(f: X \to \mathbb{R}\) is organized into a locally convex, archimedean, partially ordered, directed, and lattice-ordered real topological vector space with pointwise operations, pointwise partial order, and the topology of pointwise convergence. Thus, any additive subgroup \(F\) of \(\mathbb{R}^X\) is an archimedean, unperforated, and torsion free partially ordered abelian group under the induced partial order. Conversely, by [Goodearl, 1985, Theorem 4.14], every archimedean unital group is isomorphic as a partially ordered abelian group to a subgroup \(F\) of \(\mathbb{R}^X\) for suitable choice of the set \(X\).

We denote the constant function that maps every \(x \in X\) to the real number 1 simply by \(1 \in \mathbb{R}^X\). Evidently, \(X\) is a finite set iff 1 is an order unit in \(\mathbb{R}^X\). By dropping down to various subgroups \(F\) of \(\mathbb{R}^X\) consisting only of bounded functions and such that 1 \(\in F\), we obtain numerous examples of archimedean unperforated unital groups with 1 as the unit. Standard techniques for dropping down to suitable subgroups \(F \subseteq \mathbb{R}^X\) include the following: (1) equipping \(X\) with a topology and imposing continuity conditions on the functions in \(F\); (2) equipping \(X\) with a convex (or linear) structure and choosing \(F\) to be a suitable set of affine (or linear) functionals on \(X\); (3) equipping \(X\) with a field or a \(\sigma\)-field of subsets and imposing measure-theoretic conditions on the functions in \(F\); and (4) imposing conditions on the ranges of the functions in \(F\). We shall give some examples of these techniques, which of course may be applied in various combinations.

If \(X\) is a topological space, then \(C(X, \mathbb{R})\) denotes the subgroup of \(\mathbb{R}^X\) consisting of the continuous functions \(f: X \to \mathbb{R}\). The archimedean partially ordered abelian group \(C(X, \mathbb{R})\) is an \(\ell\)-group\(^{25}\), the infimum and supremum of continuous functions \(f, g \in C(X, \mathbb{R})\) being given by the pointwise minimum and maximum, respectively, of \(f\) and \(g\).

**EXAMPLE 69.** Let \(X\) be a compact Hausdorff space. Then the functions \(f \in C(X, \mathbb{R})\) are bounded, 1 is an order unit in \(C(X, \mathbb{R})\), \(C(X, \mathbb{R})\) is an \(\ell\)-unicommutative group with unit 1, the unit interval \(E(X, \mathbb{R})\) in \(C(X, \mathbb{R})\) is an MV-algebra\(^{26}\), and the subset \(P(X, \mathbb{R})\) of \(E(X, \mathbb{R})\) consisting of the sharp elements is a Boolean effect algebra, which in fact is the center of \(E(X, \mathbb{R})\). Again the triple \(P(X, \mathbb{R}) \subseteq E(X, \mathbb{R}) \subseteq C(X, \mathbb{R})\) is an analogue of the triple \(\mathcal{F}(\mathcal{S}) \subseteq \mathcal{E}(\mathcal{S}) \subseteq \mathcal{G}(\mathcal{S})\) of prototypes.

Let \(X\) be a compact Hausdorff space, and define \(M_1^+(X)\) to be the set of all probability measures on \(X\), i.e., the set of all nonnegative regular Borel measures \(\mu\) on \(X\) such that \(\mu(X) = 1\). Then each \(\mu \in M_1^+(X)\) determines a state \(\omega_\mu\) on the unigroup \(C(X, \mathbb{R})\) according to

\[
\omega_\mu(f) := \int_X f \, d\mu \text{ for all } f \in C(X, \mathbb{R}),
\]

\(^{25}\)Actually, \(C(X, \mathbb{R})\) is a lattice-ordered commutative linear associative algebra over \(\mathbb{R}\), and if \(X\) is compact and Hausdorff, then \(C(X, \mathbb{R})\) is a commutative lattice-ordered real Banach algebra under the supremum norm.

\(^{26}\)Actually, \(E(X, \mathbb{R})\) is a convex MV-algebra.
and \( \mu \mapsto \omega_{\mu} \) is an affine isomorphism of \( M_1^+ \) onto \( \Omega(C(X,\mathbb{R})) \). Thus, the state space of the archimedean \( \ell \)-unigroup \( C(X,\mathbb{R}) \) may be identified with \( M_1^+(X) \). (See [Goodearl, 1985, page 87].)

Recall that a compact Hausdorff space \( X \) is \textit{extremally disconnected} iff the closure of every open subset of \( X \) remains open, \( X \) is \textit{basically disconnected} iff the closure of every open \( F_\sigma \) subset of \( X \) remains open, and \( X \) is \textit{totally disconnected} iff the singleton subsets of \( X \) are the only nonempty connected subsets of \( X \). We note that

\[
\text{extremally disconnected} \implies \text{basically disconnected} \implies \text{totally disconnected}.
\]

In Example 69, if \( X \) is totally disconnected, then it is the Stone space of the Boolean algebra \( P(X,\mathbb{R}) \), and conversely, if \( B \) is a Boolean algebra and \( X \) is the Stone space of \( B \), then \( B \) is isomorphic to \( P(X,\mathbb{R}) \).

**THEOREM 70.** Let \( X \) be a compact Hausdorff space. Then:

(i) \( X \) is basically disconnected iff \( C(X,\mathbb{R}) \) is monotone \( \sigma \)-complete.

(ii) If \( X \) is totally disconnected, then \( X \) is basically disconnected iff \( P(X,\mathbb{R}) \) is a \( \sigma \)-complete Boolean algebra.

**Proof.** Part (i) follows from [Goodearl, 1985, Corollary 9.3]. Part (ii) is a consequence of the well-known fact that a Boolean algebra is \( \sigma \)-complete iff its Stone space is basically disconnected.

As in Example 55, suppose that \( \mathcal{A} \) is a unital \( C^* \)-algebra and \( G(\mathcal{A}) \) is the unigroup with unit 1 of self-adjoint elements in \( \mathcal{A} \). Then as a consequence of the Gelfand representation theorem [Kadison and Ringrose, 1983, Theorem 4.4.3], \( \mathcal{A} \) is commutative iff there is a compact Hausdorff space \( X \) such that \( G(\mathcal{A}) \) is isomorphic as a unital group to the archimedean \( \ell \)-unigroup \( C(X,\mathbb{R}) \). Conversely, if \( G(\mathcal{A}) \) is an \( \ell \)-group, then by a theorem of S. Sherman, \( \mathcal{A} \) is commutative [Sherman, 1951]. Thus, Example 69 is really the commutative (or, equivalently the \( \ell \)-group) version of Example 55.

An \( AW^* \)-\textit{algebra} is a \( C^* \)-algebra in which the right annihilator of every subset is a principal right ideal generated by a projection [Kaplansky, 1951],[Goodearl, 1985, page 313]. We omit the proof of the following theorem, which is obtained by combining known facts about \( AW^* \)-algebras and the theory of representations of Boolean algebra.

**THEOREM 71.** If \( X \) is a compact Hausdorff space, then the following conditions are mutually equivalent:

(i) \textit{The commutative \( C^* \)-algebra} \( C(X,\mathbb{C}) \) \textit{of continuous complex-valued functions on} \( X \) \textit{is an} \( AW^* \)-\textit{algebra}.

(ii) \( X \) \textit{is extremally disconnected}.
(iii) $X$ is totally disconnected and $P(X, \mathbb{R})$ is a complete Boolean algebra.

If $X$ is a convex subset of a real vector space, then a function $f : X \to \mathbb{R}$ is affine iff $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$ for all $x, y \in X$ and all real numbers $\lambda \in [0, 1]$.

**EXAMPLE 72.** If $X$ is a compact convex subset of a locally convex real vector space, then the set $\text{Aff}(X)$ of all continuous affine functionals $f : X \to \mathbb{R}$ is a linear subspace of $C(X, \mathbb{R})$ with $1 \in \text{Aff}(X)$, and thus $\text{Aff}(X)$ acquires the structure of an archimedean unigroup \(^{27}\) with unit $1$. However, unlike $C(X, \mathbb{R})$, $\text{Aff}(X)$ need not be lattice-ordered.

The archimedean unigroup $\text{Aff}(X)$ plays a crucial role in the study of interpolation groups [Goodearl, 1985, Chapter 7 and ff.]. If $G$ is an archimedean unigroup, then $G$ is embedded into $\text{Aff}(\Omega(G))$ under the evaluation mapping $g \mapsto \hat{g}$ given by $\hat{g}(\omega) := \omega(g)$ for all $\omega \in \Omega(G)$, and this embedding is the basis for the formulation of a faithful functional representation of monotone $\sigma$-complete unital $\ell$-groups [Foulis and Pulmannová, to appear, Section 6], [Goodearl, 1985, Chapter 9]. For this functional representation, the notion of a $g$-tribe \(^{28}\), which was introduced by A. Dvurečenskij in [Dvurečenskij, 2000], provides a generalization of the Loomis-Sikorski theorem for $\sigma$-complete Boolean algebras (see Example 74 and Theorem 75 below).

**DEFINITION 73.** Let $X$ be a nonempty set. A $g$-tribe on $X$ is a subgroup $T$ of $\mathbb{R}^X$ such that:

(i) Every function $f \in T$ is bounded.

(ii) $1 \in T$.

(iii) Whenever $(f_n)_{n=1,2,...}$ is a sequence in $T$ and there exists $f \in T$ with $f_n \leq f$ for all $n = 1, 2, ..., \text{ then } \bigvee \mathbb{R}^X \{f_n \mid n = 1, 2, ...\} \in T$.

**EXAMPLE 74.** Let $T$ be a $g$-tribe on the nonempty set $X$. Then, with the partial order induced from $\mathbb{R}^X$, $T$ is a monotone $\sigma$-complete unital $\ell$-group with $1$ as the unit. The unit interval $E_T$ in $T$ is a monotone $\sigma$-complete unital MV-algebra, and the set $P_T$ of sharp elements in $E_T$ is a $\sigma$-complete Boolean algebra. There is a $\sigma$-field of subsets $B_T$ of $X$ such that $P_T$ is the set of characteristic set functions $\chi_M$ for $M \in B_T$. Thus, the Boolean algebra $P_T$ is isomorphic to the $\sigma$-field $B_T$ under $\chi_M \leftrightarrow M$. Again we have an analogue $P_T \subseteq E_T \subseteq T$ of our prototypic triple $P(\mathfrak{S}) \subseteq E(\mathfrak{S}) \subseteq G(\mathfrak{S})$.

The following theorem [Dvurečenskij, 2000], [Dvurečenskij and Pulmannová, 2000, Theorem 7.1.24], [Foulis and Pulmannová, to appear, Theorem 6.7 and ff.],

\(^{27}\)Actually, $\text{Aff}(X)$ is a directed archimedean real vector space with order unit $1$ and also a Banach space under the restriction of the supremum norm on $C(X, \mathbb{R})$.

\(^{28}\)The "g" in "g-tribe" stands for "group."
can be regarded as a generalization of the Loomis-Sikorski representation theorem for $\sigma$-complete Boolean algebras.

**THEOREM 75.** Let $G$ be a monotone $\sigma$-complete unital $\ell$-group with unit $u$. Then there exists a $g$-tribe $T$ on the set $\partial_2\Omega(G)$ of extreme points of the state space $\Omega(G)$ and a surjective morphism of unital groups $\eta: T \to G$ such that $\eta$ preserves all existing countable suprema and infima.

**DEFINITION 76.** Let $B$ be a $\sigma$-field of subsets of a nonempty set $X$. Define $\mathcal{G}(X,B,\mathbb{R})$ to be the partially ordered abelian group with pointwise partial order and addition of all bounded $B$-measurable functions $f : X \to \mathbb{R}$.

A subset $M$ of the set $X$ determines and is determined by its characteristic set function $\chi_M$, which takes the value $1$ on $M$ and $0$ on $X \setminus M$. In [Zadeh, 1965], L. Zadeh introduced the notion of a fuzzy set, i.e., a function $e : X \to \mathbb{R}$ such that $0 \leq e(x) \leq 1$ for all $x \in X$, thus allowing for “grades of membership” of elements $x$ ranging from $e(x) = 1$ to $e(x) = 0$. Since then, a considerable literature on fuzzy sets has been developed and a number of connections with quantum logic have been explored (e.g., see [Beltrametti et al., 2000; Foulis, to appear(b); Gudder, 1998]).

**EXAMPLE 77.** Let $B$ be a $\sigma$-field of subsets of the nonempty set $X$. Then, $\mathcal{G}(X,B,\mathbb{R})$ is an archimedean $\ell$-unigroup with unit $1$, the unit interval $E(X,B,\mathbb{R})$ is a convex MV-algebra, and the set $\mathcal{P}(X,B,\mathbb{R})$ of sharp elements in $E(X,B,\mathbb{R})$, which coincides with the center of $E(X,B,\mathbb{R})$, consists of the characteristic set functions $\chi_M$ of measurable sets $M \in B$. Thus $\mathcal{P}(X,B,\mathbb{R})$ is a Boolean $\sigma$-effect algebra isomorphic to $B$. The functions $e \in E(X,B,\mathbb{R})$ may be interpreted as fuzzy sets in the sense of Zadeh, whereas the functions $\chi_M \in \mathcal{P}(X,B,\mathbb{R})$ correspond to the “sharp” sets $M \in B$. Again we have a triple $\mathcal{P}(X,B,\mathbb{R}) \subseteq E(X,B,\mathbb{R}) \subseteq \mathcal{G}(X,B,\mathbb{R})$ analogous to our prototypic triple $\mathcal{P}(\mathfrak{f}) \subseteq E(\mathfrak{f}) \subseteq \mathcal{G}(\mathfrak{f})$.

Suppose that $X$ is the phase space of a classical mechanical system $\mathcal{M}$ and $B$ is the $\sigma$-field of Borel subsets of $X$. Phase points $x \in X$ may be identified with physical states of the system $\mathcal{M}$, and (real) observables for $\mathcal{M}$ may be identified with $B$-measurable functions $f : X \to \mathbb{R}$. The idea is that the value of an observable $f$ when $\mathcal{M}$ is in state $x$ is $f(x)$. Thus, functions $f$ in the $\ell$-unigroup $\mathcal{G}(X,B,\mathbb{R})$ of Example 77 represent bounded observables. If $M \in B$, then the observable $\chi_M \in \mathcal{P}(X,B,\mathbb{R})$ corresponds to the “sharp” proposition asserting that the state $x$ of $\mathcal{M}$ belongs to the set $M$. Accordingly, observables $e \in E(X,B,\mathbb{R})$ are “fuzzy” or “unsharp” propositions concerning the state of $\mathcal{M}$.

In Example 77, $\mathcal{G}(X,B,\mathbb{R})$ is actually an order-unit normed linear space over $\mathbb{R}$, and the order-unit norm coincides with the supremum norm. Unlike the prototype $\mathcal{G}(\mathfrak{f})$, $\mathcal{G}(X,B,\mathbb{R})$ need not be a Banach space, however it can be completed to an order-unit Banach space $\bar{\mathcal{G}}(X,B,\mathbb{R})$ by appending all uniform limits of sequences of functions in $\mathcal{G}(X,B,\mathbb{R})$.

\footnote{We note that the Boolean unigroup $\mathcal{G}(X,B,\mathbb{Z})$ in Definition 65 is a subgroup of $\mathcal{G}(X,B,\mathbb{R})$ and $E(X,B,\mathbb{Z}) = \mathcal{P}(X,B,\mathbb{R})$.}
15 EFFECT-ORDERED RINGS

In Example 55 not only do we have \( P(A) \subseteq E(A) \subseteq G(A) \), but \( G(A) \) is an additive subgroup of the C*-algebra \( A \). In this section, we briefly explore the more general situation in which it is possible to realize a unital group \( G \) as an additive subgroup of a ring \( A \) that has some of the features of a C*-algebra.

**DEFINITION 78.** An effect-ordered ring is a ring \( A \) with unit 1 such that:

1. Under addition, \( A \) forms a partially ordered abelian group\(^{30}\) with positive cone \( A^+ \).
2. \( 1 \in A^+ \).
3. The additive subgroup \( G(A) := A^+ - A^+ \) of \( A \) is a unital group with positive cone \( G(A)^+ = A^+ \) and unit 1.
4. For all \( a, b \in A^+ \),
   - \( ab = ba \Rightarrow ab \in A^+ \),
   - \( aba = 0 \Rightarrow ab = ba = 0 \), and
   - \( (a - b)^2 \in A^+ \).

**EXAMPLE 79.** If \( A \) is a C*-algebra, then \( A \) is organized into an effect-ordered ring by defining \( A^+ := \{ aa^* \mid a \in A \} \), in which case \( G(A) \) in Definition 78 coincides with \( G(A) \) in Example 55.

**DEFINITION 80.** Let \( A \) be an effect-ordered ring. We denote the unit interval in the unital group \( G(A) \) by

\[
E(A) := \{ e \in G(A) \mid 0 \leq e \leq 1 \}.
\]

Also, we define

\[
P(A) := \{ p \in G(A) \mid p = p^2 \}
\]

and we refer to elements \( p \in P(A) \) as projections.

**EXAMPLE 81.** If \( X \) is a compact Hausdorff space, then \( A := C(X, \mathbb{R}) \) is a commutative C*-algebra, and is organized into an effect-ordered ring as in Example 79. As such, we have \( G(A) = C(X, \mathbb{R}) \), whence \( G(A) \) is an archimedean \( \ell \)-group, \( E(A) \) is a convex MV-algebra, and \( P(A) \) is the Boolean algebra of all characteristic set functions \( \chi_M \) of compact open subsets of \( X \).

**EXAMPLE 82.** Let \( B \) be a field of subsets of a nonempty set \( X \) and let \( G(X, B, \mathbb{Z}) \) be the archimedean unital \( \ell \)-group in Definition 65. Then, under pointwise multiplication, \( A := G(X, B, \mathbb{Z}) \) is a commutative effect-ordered ring with \( A^+ := G(X, B, \mathbb{Z})^+ \) and \( G(A) = G(X, B, \mathbb{Z}) \). Also, \( E(A) = P(A) = G(X, B, \mathbb{Z}) \) is the Boolean algebra of all characteristic set functions \( \chi_M \) of sets \( M \in B \).

\(^{30}\)Note that we do not assume that \( A \) is directed as a partially ordered additive group, nor do we assume that \( A^+ \) is closed under multiplication.
EXAMPLE 83. Let \( B \) be a \( \sigma \)-field of subsets of a nonempty set \( X \) and let \( G(X, B, \mathbb{R}) \) be the archimedean unital \( \ell \)-group in Example 77. Then, under pointwise multiplication \( \mathcal{A} := G(X, B, \mathbb{R}) \) is a commutative effect-ordered ring with \( \mathcal{A}^+ := G(X, B, \mathbb{R})^+ \) and \( G(A) = G(X, B, \mathbb{R}) \). Also, \( E(A) = E(X, B, \mathbb{R}) \) is the convex MV-algebra of (perhaps) fuzzy \( B \)-measurable subsets of \( X \), and \( P(A) = P(X, B, \mathbb{R}) \) is the set of sharp elements in \( E(A) \); indeed, \( P(A) \) is the Boolean \( \sigma \)-algebra of all characteristic set functions \( \chi_M \) of measurable sets \( M \in B \).

Standing Assumption: For the remainder of this section we assume that \( \mathcal{A} \) is an effect-ordered ring.

As a consequence of the following lemma, the effect-ordered ring \( \mathcal{A} \) provides a generalization of our prototypic triple \( \mathbb{P}(\mathcal{A}) \subseteq \mathbb{E}(\mathcal{A}) \subseteq G(\mathcal{A}) \).

LEMMA 84. If \( p \in P(A) \), then \( 1-p \in P(A) \), and we have

\[
0, 1 \in P(A) \subseteq E(A) \subseteq G(A) \subseteq A.
\]

Proof. Obviously, \( 0, 1 \in P(A) \). If \( p \in P(A) \), then \( p = p^2 \) and \( p \in G(A) \), so there exist \( a, b \in \mathcal{A}^+ \) such that \( p = a - b \). Therefore, by Definition 78 (4, iv), \( p = p^2 = (a - b)^2 \in \mathcal{A}^+ = G(A)^+ \). As \( 1, p \in G(A) \), we have \( 1 - p \in G(A) \) with \( (1-p)^2 = 1 - 2p + p^2 = 1 - p \), whence \( 1 - p \in P(A) \subseteq G(A)^+ \), and it follows that \( 0 \leq p \leq 1 \), i.e., \( p \in E(A) \).

In [Greechie et al., 1995, Definition 6.1] a weaker version of an effect-ordered ring \( \mathcal{A} \), called an effect ring, is defined in which it is not assumed that 1 is a generative order unit in \( G(A) \) and condition (4, iv) in Definition 78 is replaced by the weaker condition that, for all \( p \in G(A)^+ \), \( p = p^2 \Rightarrow 1 - p \in G(A)^+ \). Therefore, the properties of an effect ring developed in [Greechie et al., 1995] hold as well for an effect-ordered ring \( \mathcal{A} \).

LEMMA 85. Let \( e \in E(A) \) and \( p, q \in P(A) \). Then:

(i) \( e \leq p \Leftrightarrow e = pe \Leftrightarrow e = ep \Leftrightarrow e = pep \).

(ii) \( e \leq 1 - p \Leftrightarrow ep = 0 \Leftrightarrow pe = 0 \Leftrightarrow pep = 0 \).

(iii) If \( a, b \in G(A)^+ \), then \( ab = 0 \Rightarrow ba = 0 \).

(iv) \( pq \in P(A) \Leftrightarrow pq \in E(A) \Leftrightarrow pq = qp \).

Proof. (i) By [Greechie et al., 1995, Theorem 6.6 (iii)], we have \( e \leq p \Leftrightarrow e = pe = ep \Leftrightarrow e = pep \). Suppose \( e = pe \). Then \( (1-p)e = 0 \), so \( (1-p)e(1-p) = 0 \), and it follows from Definition 78 (4, iii) that \( e(1-p) = 0 \), i.e., \( e = ep \). Therefore, \( e = pe \Rightarrow e = ep = ep \Rightarrow e \leq p \), and similarly, \( e = ep \Rightarrow e \leq p \).

(ii) By (i), \( e \leq 1-p \Leftrightarrow e = (1-p)e \Leftrightarrow pe = 0 \), and similarly \( e \leq 1-p \Leftrightarrow ep = 0 \). If \( pe = 0 \), it is clear that \( pep = 0 \). Conversely, if \( pep = 0 \), then \( ep = pe = 0 \) by Definition 78 (4 iii).
(iii) $ab = 0 \Rightarrow aba = 0 \Rightarrow ba = 0$ by Definition 78 (4, iii).

(iv) By Lemma 84, $pq \in P(A) \Rightarrow pq \in E(A)$. Suppose $e := pq \in E(A)$. Then $pe = p^2q = pq = e$, so $e = ep$ by (i), and $e^2 = epq = eq = pq^2 = pq = e$, whence $pq = e \in P(A)$. Therefore, $pq \in P(A) \iff pq \in E(A)$. Now suppose $pq \in P(A)$, and again let $e := pq$. Then $e = e^2$, $e = pe$, and $e = eq$, whence $e^2 = e = pe = ep = eq = qe$ and $e \leq p, q$ by (i). Consequently, $p - e, q - e \in G(A)^+$ with $(p - e)(q - e) = e - e - e + e^2 = 0$, and it follows from (iii) that $0 = (q - e)(p - e) = qp - e - e + e = qp - e$, i.e., $pq = e = qp$. Conversely, if $pq = qp$, then $(pq)^2 = pqpq = p^2q^2 = pq$, so $pq \in P(A)$. 

THEOREM 86.

(i) $P(A)$ is a normal sub-effect algebra of $E(A)$.

(ii) $P(A)$ is an orthomodular poset (OMP) with $p \mapsto 1 - p$ as the orthocomplementation.

(iii) If $p \in E(A)$, then $p \in P(A) \iff p$ is sharp $\iff p$ is principal.

Proof. (i) By [Greechie et al., 1995, Theorem 6.6 (ii)], $P(A)$ is a sub-effect algebra of $P(A)$. To show that it is normal, suppose $d, e, f \in E(A)$ with $d + e + f \leq 1$, $p := d + e \in P(A)$, and $q := d + f \in P(A)$. Then $d \leq d + e = p$, so $d = pq$ by Lemma 85 (i). Also, $p + f = d + e + f \leq 1$, so $f \leq 1 - p \in P(A)$, whence $pf = 0$ by Lemma 85 (ii). Consequently, $pq = p(d + f) = d + 0 = d \in E(A)$, and it follows from Lemma 85 that $d = pq \in P(A)$.

Part (ii) is [Greechie et al., 1995, Corollary 6.7], and part (iii) follows from [Greechie et al., 1995, Theorem 6.8].

The following definition is motivated by Definition 3.

DEFINITION 87. For each projection $p \in P(A)$, we define the compression mapping $J_p : G(A) \to A$ by $J_p(g) := pgp$ for all $g \in G(A)$.

LEMMA 88. If $p \in P(A)$, then:

(i) $J_p : G(A) \to G(A)$ is an order-preserving group homomorphism.

(ii) $J_p(1) = p$.

(iii) If $e \in E(A)$, then $e \leq p \iff J_p(e) = e$.

(iv) If $e \in E(A)$, then $J_p(e) = 0 \iff e \perp p$.

Proof. (i) Suppose $p \in P(A)$ and $g \in G(A)$. Then there exist $a, b \in A^+$ with $g = a - b$, and it follows that $J_p(g) = pgp = pap - pbp$. But, since $p, a, b \in A^+$, Definition 78 (4, ii) implies that $pap, pbp \in A^+$, whence $J_p(g) \in G(A)$. Therefore, $J_p : G(A) \to G(A)$, and it is obvious that $J_p$ is a group homomorphism. Also, Definition 78 (4 ii) implies that $J_p(G(A)^+) \subseteq G(A)^+$, so $J_p$ is order preserving.

Part (ii) is obvious, and parts (iii) and (iv) follow from Lemma 85 (i) and (ii).
THEOREM 89. Suppose that the unital group \( G(A) \) is archimedean and let \( J : G(A) \to G(A) \) be an order-preserving group homomorphism such that \( p := J(1) \in E(A) \) and, for all \( e \in E(A) \), \( e \leq p \Rightarrow J(e) = e \). Then \( p \in P(A) \) and \( J = J_p \).

**Proof.** See [Foulis, 2004a, Theorem 4.5].

LEMMA 90. Let \( p, q, r \in P(A) \) with \( p + q + r \leq 1 \). Then \( p + r, q + r \in P(A) \) and

\[
J_{p+r} \circ J_{q+r} = J_r.
\]

**Proof.** As \( p + r \leq p + q + r \leq 1 \), and \( P(A) \) is a sub-effect algebra of \( E(A) \), it follows that \( p + r \in P(A) \), and likewise \( q + r \in P(A) \). Also, \( p \leq 1 - r \), so \( pr = rp = 0 \) by Lemma 85 (ii). Likewise \( pq = qp = 0 \) and \( qr = rq = 0 \). Thus, \( (p + r)(q + r) = pq + pr + rq + r^2 = r \), and similarly \( (q + r)(p + r) = r \). Therefore, for all \( g \in G(A) \), \( J_{p+r}(J_{q+r}(g)) = (p + r)(q + r)(q + r)(p + r) = rgr = J_r(g) \).

16 ABSTRACTION FROM \( \mathbb{P}(\mathfrak{f}) \)—CB-GROUPS

We maintain our standing assumption that \( G \) is a unital group with unit \( u \neq 0 \) and unit interval \( E \). Motivated by Theorem 4 and the properties of compression operators on effect-ordered rings as per Section 15, we now begin a process of abstraction from the prototype \( \mathbb{P}(\mathfrak{f}) \) to a sub-effect algebra \( P \) of \( E \).

**DEFINITION 91.** A mapping \( J : G \to G \) is a **retraction** on \( G \) with **focus** \( p \) iff \( J \) is an order-preserving group endomorphism, \( p := J(u) \in E \), and for all \( e \in E \), \( e \leq p \Rightarrow J(e) = e \). By definition, a **compression** on \( G \) is a retraction \( J \) on \( G \) with focus \( p \) such that, for all \( e \in E \), \( J(e) = 0 \Rightarrow e \perp p \). A retraction \( J \) on \( G \) is **direct** iff \( g \in G^+ \Rightarrow J(g) \leq g \).

**LEMMA 92.** Every retraction \( J \) on \( G \) is idempotent, i.e., \( J = J \circ J \), and its focus \( J(u) \) is a principal, hence a sharp element of \( E \).

**Proof.** See [Foulis, 2004a, Lemma 2.3 (i) and (iii)].

By Theorem 4 and Theorem 9 (iv), every retraction \( J \) on our prototype \( G(\mathfrak{f}) \) is a compression and has the form \( J = J_P \) where \( P = J(1) \in \mathbb{P}(\mathfrak{f}) \) is the focus of \( J \). More generally, by Theorem 89 and Lemma 88 (iv), every retraction \( J \) on the unital group \( G(A) \) of an archimedean effect-ordered ring \( A \) is a compression and has the form \( J = J_p \) where \( p = J(1) \in P(A) \). As a consequence of [Foulis, 2004a, Theorem 2.8], every direct retraction on the unital group \( G \) is a direct compression on \( G \). By Theorem 12, the only direct compressions on the prototype \( G(\mathfrak{f}) \) are \( J_0 \) and \( J_1 \). On the other hand, interpolation unigroups generally have a good supply of direct compressions.

**THEOREM 93.** If \( G \) is an interpolation unigroup, then every retraction \( J \) on \( G \) is a direct compression, its focus \( p := J(u) \) is a sharp element of \( E \), and every sharp element of \( E \) is the focus of a uniquely determined direct compression on \( G \).
**Proof.** See [Foulis, 2003c, Theorem 3.5] and [Foulis, 2004a, Theorem 2.9].

Let $G$ be a unital group with unit $u$, let $J$ be a retraction on $G$, and define $J': G \to G$ by $J'(g) = g - J(g)$ for all $g \in G$. Since $J$ is idempotent, it follows that $J: G \to J(G)$ and $J': G \to J'(G)$ provide a representation of the abelian group $G$ as a direct product of the subgroups $J(G)$ and $J'(G)$ in the category of abelian groups, but not necessarily in the category of partially ordered abelian groups. However, if $J$ is a direct compression, then so is $J'$; moreover, $J(G)$ and $J'(G)$ are unital groups (with the partial order induced from $G$) with units $J(u)$ and $u - J(u)$, respectively, and $J: G \to J(G)$ and $J': G \to J'(G)$ provide a representation of $G$ as the direct product of $J(G)$ and $J'(G)$ as unital groups. For details of the basic theory of retractions and compressions on unital groups, see [Foulis, 2003c; T; Foulis, 2005; Foulis, to appear(a); Foulis, 2006; Foulis and Pulmannová, to appear].

**DEFINITION 94.** By a compression base for the unital group $G$, we mean a family $(J_p)_{p \in P}$ of compressions on $G$ indexed by a normal sub-effect algebra $P$ of $E$ such that (i) if $p \in P$, then $p = J_p(u)$ is the focus of $J_p$ and (ii) if $p, q, r \in P$ with $p + q + r \leq u$, then $J_{p+r} \circ J_{q+r} = J_r$. By a CB-group, we mean a unital group $G$ together with a specified compression base $(J_p)_{p \in P}$ for $G$. If $G$ is a CB-group with compression base $(J_p)_{p \in P}$, then elements $p \in P$ are called projections. A CB-group $G$ is said to be proper if every direct compression belongs to its compression base. A total CB-group is a CB-group $G$ such that every retraction on $G$ is a compression and belongs to the compression base.

**EXAMPLE 95.** With $(J_P)_{P \in \mathcal{P}(S)}$ as its compression base, the Hermitian group $G(S)$ is a total CB-group.

In the sequel, we always regard the Hermitian group $G(S)$ as a total CB-group as per Example 95. Thus, a proper CB-group $G$ with unit interval $E$ and compression base $(J_p)_{p \in P}$ provides an abstraction $P \subseteq E \subseteq G$ of the prototypic triple $\mathcal{P}(S) \subseteq E(S) \subseteq G(S)$.

**THEOREM 96.** Let $G$ be a CB-group with compression base $(J_p)_{p \in P}$. Then the normal sub-effect algebra $P$ of $E$ is an orthomodular poset (OMP) and, for all $p \in P$ and all $g \in G^+$, $J_p(g) = 0 \Leftrightarrow J_{u-p}(g) = g$. Moreover, $P$ has the property that every finite set of pairwise coexistent elements in $P$ is contained in a Boolean sub-effect algebra of $P$.\(^{31}\)

**Proof.** See [Foulis, to appear(a), Theorem 2] and [Foulis, 2005, Theorem 2.5].

**DEFINITION 97.** If $G$ is a CB-group with unit interval $E$ and compression base $(J_p)_{p \in P}$, then the triple $P \subseteq E \subseteq G$ consisting of the orthomodular poset $P$, the interval effect algebra $E$, and the CB-group $G$ is called a CB-triple.

If $P \subseteq E \subseteq G$ is a CB-triple, and if $E$ is regarded as a quantum logic, then elements $p \in P$ represent "sharp" propositions, and elements $e \in E \setminus P$ represent propositions that exhibit various degrees of "fuzziness."\(^{31}\)

\(^{31}\)This property of an OMP is usually called regularity [Harding, 1998].
THEOREM 98. If \( A \) is an effect ordered ring, then, with \( (J_p)_{p \in P(A)} \) as a compression base, \( G(A) \) is a proper CB group. Moreover, if \( G(A) \) is archimedean, then \( G(A) \) is a total CB group with compression base \( (J_p)_{p \in P(A)} \).

**Proof.** By Theorem 86 (i), \( P(A) \) is a normal sub-effect algebra of \( E(A) \), hence by Lemma 90, \( (J_p)_{p \in P(A)} \) is a compression base for \( G(A) \).

Suppose that \( J \) is a direct compression on \( G(A) \), let \( p := J(1) \) be the focus of \( J \), and let \( e \in E(A) \). Then \( 0 \leq e \leq 1 \) implies that \( 0 \leq J(e) \leq J(1) = p \leq 1 \), whence \( J(e) \in E(A) \) with \( J(e) \leq p \). Therefore, \( J(e) = pJ(e)p \) by Lemma 85 (i). Also, since \( J \) is direct and \( 0 \leq e, 1 - e \), we have \( J(e) \leq e \) and \( p - J(e) = J(1 - e) \leq 1 - e \), so \( 0 \leq e - J(e) \leq 1 - p \leq 1 \), and it follows that \( e - J(e) \in E(A) \) with \( e - J(e) \leq 1 - p \). Therefore, \( p(e - J(e))p = 0 \) by Lemma 85 (ii). Consequently, for all \( e \in E(A) \),

\[
J_p(e) = pep = p(J(e) + (e - J(e)))p = pJ(e)p + p(e - J(e))p = J(e) + 0 = J(e).
\]

But \( E(A) \) generates \( G(A) \) as a group, and since both \( J_p \) and \( J \) are group homomorphisms, it follows that \( J = J_p \). Therefore, every direct compression on \( G(A) \) belongs to the compression base \( (J_p)_{p \in P(A)} \), i.e., the compression base is proper. If \( G(A) \) is archimedean, then by Theorem 89, \( G(A) \) is a total CB-group. \( \blacksquare \)

COROLLARY 99. Let \( A \) be a unital \( C^* \)-algebra, let \( G(A) \) be the unigroup with unit 1 of self-adjoint elements in \( A \), and let \( P(A) \) be the set of all self-adjoint idempotents (i.e., projections) in \( A \) (Example 55). For each \( p \in P(A) \), define \( J_p: G(A) \to G(A) \) by \( J_p(g) := pgp \) for all \( g \in G(A) \). Then \( G(A) \) is a total CB-group with compression base \( (J_p)_{p \in P} \).

The following theorem shows that an arbitrary unital group \( G \) can be organized into a proper CB-group in at least one way. The proof, although straightforward, is a bit tedious, hence it is omitted.

THEOREM 100. The unital group \( G \) can be organized into a proper CB-group by taking the set \( (J_p)_{p \in P} \) of all direct compressions on \( G \), indexed by their own foci, as the compression base \( (J_p)_{p \in P} \). For the resulting CB-group, the OMP \( P \) of projections is a normal Boolean sub-effect algebra of the center of the unit interval \( E \) in \( G \), and if \( G \) is a unigroup, then \( P \) is the center of \( E \).

DEFINITION 101. A unital group \( G \), organized into a proper CB-group as in Theorem 100 is called a direct CB-group.

EXAMPLE 102. If \( G \) is an interpolation unigroup organized into a direct CB-group, then by Theorem 93, \( G \) is a total CB-group and \( P \) is the set of all sharp elements in \( E \).

THEOREM 103. If \( G \) is a proper CB-group, then \( G \) is a Boolean unigroup iff \( E = P \).

**Proof.** See [Foulis, 2003c, Theorem 6.5]. \( \blacksquare \)
The class of unital groups is organized into a category with unital morphisms as the morphisms (Definition 48). Likewise, to organize the class of proper CB-groups into a category, we use CB-morphisms as per the following definition.

**DEFINITION 104.** Let $G$ and $H$ be CB-groups with compression bases $(J^G_p)_{p \in P}$ and $(J^H_q)_{q \in Q}$, respectively. Then a **CB-morphism** from $G$ to $H$ is a unital morphism $\phi: G \to H$ such that $\phi(P) \subseteq Q$ and $\phi \circ J^G_p = J^H_{\phi(p)} \circ \phi$ for all $p \in P$.

If $\phi: G \to H$ is a bijective CB-morphism such that $\phi^{-1}: H \to G$ is also a CB-morphism, then $\phi$ is a **CB-isomorphism** from $G$ to $H$. Two CB-groups $G$ and $H$ are **isomorphic** (as CB-groups) iff there is a CB-isomorphism $\phi: G \to H$. A **CB-symmetry** (or **CB-automorphism**) of $G$ is a unital symmetry $\xi: G \to G$ that is also a CB-isomorphism.

The condition that a unital morphism between CB-groups is a CB-morphism is quite strong. For instance, if the unital group in Figure 4 of Section 13 is a CB-group, and if $\mathbb{R}$ and the Boolean unigroup in Figure 4 are organized into direct (and necessarily total) CB-groups as per Example 102, then, in general, neither $\alpha$ nor $\omega$ will be a CB-morphism.

As a consequence of Theorem 6, for the Hermitian group $G(\mathbb{F})$ (organized into a total CB-group as in Example 95), every unital symmetry $\xi: G(\mathbb{F}) \to G(\mathbb{F})$ is automatically a CB-symmetry. If $G$ is a CB-group with compression base $(J_p)_{p \in P}$, and if $\xi: G \to G$ is a CB-symmetry of $G$, then the restriction of $\xi$ to $P$ is an automorphism of the regular orthomodular poset $P$. Moreover, if $p \in P$, then $J_{\xi(p)} = \xi \circ J_p \circ \xi^{-1}$.

**Standing Assumption:** Henceforth, we assume that the unital group $G$ with unit $u \neq 0$ and unit interval $E$ is organized into a proper CB-group with compression base $(J_p)_{p \in P}$. The order-unit pseudonorm of $g \in G$ (Definition 53) is denoted by $\|g\|$.

**LEMMA 105.** Let $p \in P$ and $g \in G$. Then: (i) $0 \neq p \Rightarrow \|p\| = 1$. (ii) $\|J_p(g)\| \leq \|g\|$.

**Proof.** See [Foulis, 2004b, Theorem 3.3 (viii), (ix)].

**DEFINITION 106.** Let $p \in P$ and $g \in G$.

(i) $C(p) := \{g \in G \mid g = J_p(g) + J_{u-p}(g)\}$. If $g \in C(p)$, we say that $g$ is **compatible** with $p$.

(ii) $CPC(g) := \bigcap\{C(p) \mid p \in P$ and $g \in C(p)\}$.

If $\mathcal{A}$ is an effect-ordered ring organized into a CB-group as in Theorem 98, then for $p \in P(\mathcal{A})$ and $g \in G(\mathcal{A})$, we have $g \in C(p) \iff pg = gp$. Therefore, $h \in CPC(g) \iff h$ commutes with every projection that commutes with $g$. In particular, if $\mathcal{A}$ is a von Neumann algebra (or more generally, an AW*-algebra), then $h \in CPC(g)$ iff $h$ **double commutes** with $g$, i.e., iff $h$ commutes with every element in $\mathcal{A}$ that commutes with $g$. 


LEMMA 107. If $p \in P$ and $g \in G$, then $C(p)$ and $\text{CPC}(g)$ are subgroups of $G$ and, if $G$ is archimedean, then both $C(p)$ and $\text{CPC}(g)$ are closed in the order-unit-norm topology.

Proof. See [Foulis, 2004b, Corollary 3.2].

THEOREM 108. If $p, q \in P$, then the following conditions are mutually equivalent:

(i) $J_p \circ J_q = J_q \circ J_p$.
(ii) $J_p(q) = J_q(p)$.
(iii) $J_p(q) \leq q$.
(iv) $\exists \tau \in P, J_p \circ J_q = J_{\tau}.$
(v) $J_p(q) \in P$.
(vi) $q \in C(p)$.
(vii) $p \in C(q)$.
(viii) $p$ and $q$ are coexistent in $E$.
(ix) $p$ and $q$ are coexistent in $P$.


In view of Theorem 108, for $p, q \in P$ we usually write the condition $p \in C(q)$ in the form $pCq$.

THEOREM 109. Let $p \in P$. Then, with the partial order inherited from $G$, the subgroup $C := C(p) = C(u - p)$ can be organized into a CB-group with unit $u$, unit interval $E \cap C = \{e + f \mid e, f \in E, e \leq p, f \leq u - p\}$, and compression base $(J^c_q)_{q \in P \cap C}$, where, for each $q \in P \cap C$, $J^c_q$ is the restriction of $J_q$ to $C$.

Proof. See [Foulis, 2006, Theorem 3.4].

As mentioned in Section 3, the process of “dropping down” to a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ plays an important role in quantum mechanics. Every Hermitian operator $B \in \mathbb{G}(\mathcal{M})$ has a canonical extension to a Hermitian operator $\overline{B} \in \mathbb{G}(\mathcal{H})$ such that $\overline{B}(\mathcal{M}^\perp) = \{0\}$. Moreover, if $P \in \mathbb{P}(\mathcal{H})$ is the projection onto $\mathcal{M}$, then the mapping $B \mapsto \overline{B}$ is a unital isomorphism of $\mathbb{G}(\mathcal{M})$ onto the image $J_P(\mathbb{G}(\mathcal{H}))$ of $\mathbb{G}(\mathcal{H})$ under the compression $J_P$. Thus, abstracting from the prototype to the CB-group $G$, if $p \in P$, we regard the passage from $G$ to $J_p(G)$ as a generalization of the process of dropping down to a closed linear subspace of a Hilbert space.

THEOREM 110. Let $p \in P$ and let $H := J_p(G)$ be the image of $G$ under the compression $J_p$. Then $H$ is a subgroup of $G$, $H \subseteq C(p)$, and with the partial order inherited from $G$, $H$ can be organized into a CB-group as follows:

(i) The positive cone in $H$ is $H^+ = H \cap G^+$.
(ii) The unit for $H$ is $p$, and the unit interval in $H$ is $E_H = E \cap H = \{e \in E \mid e \leq p\}$.
(iii) The set of projections in $H$ is $P_H = P \cap H = \{q \in P \mid q \leq p\}$.
(iv) The compression base for $H$ is $(J^H_q)_{q \in P_H}$ where $J^H_q$ is the restriction to $H$ of $J_q$ for each $q \in P_H$.

Moreover, if $h \in H$ and $q \in P_H$, then $h$ is compatible with $q$ in the CB-group $H$ iff $h$ is compatible with $q$ in $G$. 
**Proof.** See [Foulis, to appear(a), Theorem 4].

In [Foulis, 2006, Theorem 3.5] it is shown that, if \( p \in P \), then \( C(p) \) is a direct product of \( J_p(G) \) and \( J_{u-p}(G) \) in the category of CB-groups.

**DEFINITION 111.** If \( g \in G \), we define

\[
P^{\pm}(g) := \{ p \in P \cap CPC(g) \mid g \in C(p) \text{ and } J_{u-p}(g) \leq 0 \leq J_p(g) \}.
\]

If \( P^{\pm}(g) \) is nonempty for every \( g \in G \), we say that the CB-group \( G \) has the **general comparability property**. A **comparability group** (respectively, a comparability unigroup) is defined to be a CB-group (respectively, a CB-unigroup) with the general comparability property. A **direct comparability group** (respectively, a **direct comparability unigroup**) is a direct CB-group (respectively, direct CB-unigroup) with the general comparability property.

A projection \( p \in P^{\pm}(g) \) "splits" \( g = J_p(g) + J_{u-p}(g) \) into a **positive part** \( J_p(g) \) and a **negative part** \( J_{u-p}(g) \). If \( A \) is a von Neumann algebra (or more generally, an AW*-algebra), then \( G(A) \), organized into a CB-group as in Corollary 99, is a comparability group. Indeed, if \( g \in G(A) \), then \( g \) has a **polar decomposition** \( g = |g| s = s|g| \) where \( |g| \in G(A)^{\pm} \cap CPC(g) \), \( |g|^2 = g^2 \), \( s^2 = s \in G(A) \cap CPC(g) \), \( s^2 \in P(A) \), \( sg = gs = |g| \), and \( p := \frac{1}{2}(s + s^2) \) is the smallest projection in \( P^{\pm}(g) \).

**THEOREM 112.** If \( G \) is a comparability group and \( p \in P \), then \( C(p) \) and \( J_p(G) \) are comparability groups.

**Proof.** See [Foulis, 2006, Theorem 4.7].

**THEOREM 113.** If \( G \) is a direct comparability group then \( G \) is an \( \ell \)-group.

**Proof.** See [Foulis, 2003c, Theorem 4.9].

If \( G \) is an interpolation unigroup, organized into a total direct CB-group as in Example 102, then \( G \) has the general comparability property iff it has the property of the same name studied in [Goodearl, 1985, Chapter 8].

**THEOREM 114.** Let \( G \) be an interpolation unigroup organized into a direct CB-group. If \( G \) is a comparability group, then \( G \) is an \( \ell \)-unigroup. On the other hand, if \( G \) is a monotone \( \sigma \)-complete \( \ell \)-unigroup, then \( G \) is a comparability group.\(^{32}\)

**Proof.** See Theorem 113 and [Goodearl, 1985, Theorem 9.9].

Every Boolean unigroup is a comparability group. Also, if \( B \) is a \( \sigma \)-field of subsets of a nonempty set \( X \), then the archimedean unital \( \ell \)-group \( G(X,B,\mathbb{R}) \) (Definition 76) is a comparability group.

**THEOREM 115.** Let \( G \) be a comparability group. Then: (i) If \( p \in E \), then \( p \in P \iff p \) is sharp \( \iff p \) is principal. (ii) \( G \) is unperforated and torsion free. (iii) \( G \) is archimedean iff for all \( a,b \in G^+ \), \( na \leq b \) for all positive integers \( n \) only if \( a = 0 \). (iv) If \( G \) is monotone \( \sigma \)-complete, then \( G \) is archimedean.

\(^{32}\) Also see Theorem 124 below.
Proof. For (i) and (iii), see [Foulis, 2005, Theorem 5.7, Lemma 3.5]. For (ii), see [Foulis, 2003c, Lemma 4.8]. For (iv), see [Foulis and Pulmannová, to appear, Theorem 4.4 (iv)]

DEFINITION 116. Let $G$ be a comparability group, let $g \in G$, and choose $p \in P^\pm(g)$. By [Foulis, 2005, Theorem 3.2], $J_p(g)$ and $J_{u-p}(g)$ are independent of the choice of $p$, so we can and do define

$$g^+ := J_p(g), \quad g^- := -J_{u-p}(g), \text{ and } \|g\| := g^+ + g^-.$$

If $A$ is a von Neumann algebra (or more generally an AW*-algebra), $g \in G(A)$, and $g = |g|s = s|g|$ is the polar decomposition of $g$, then $g^+ = \frac{1}{2}(|g| + g)$ and $g^- = \frac{1}{2}(|g| - g)$.

The properties of $g^+$, $g^-$, and $\|g\|$ for elements $g$ in a comparability group $G$ are developed in [Foulis, 2005, Section 4]. For instance, $g = g^+ - g^-$ with $0 \leq g^+, g^-, \|g\| \in C_0(g)$, $g^+ \wedge_{G^+} g^- = 0$, and so on.

17 THE RICKART PROJECTION PROPERTY AND RC-GROUPS

If $T$ is a bounded linear operator on the Hilbert space $\mathcal{H}$, define $T' \in P(\mathcal{H})$ to be the projection onto the null space $T^{-1}(0)$ of $T$. Then, for every bounded linear operator $S$ on $\mathcal{H}$, $TS = 0 \iff S = T'S$. In particular, if $A \in G(\mathcal{H})$ and $P \in P(\mathcal{H})$, then $P \leq A' \iff PA = AP = PAP = 0$; hence the CB-group $G(\mathcal{H})$ has the Rickart projection property as per the following definition.

DEFINITION 117. The CB-group $G$ has the Rickart projection property if there is a mapping $': G \to P$, called the Rickart mapping, such that, for all $g \in G$ and all $p \in P, p \leq g' \iff J_p(g) = 0$ and $g \in C(p)$.

A Rickart C*-algebra [Goodearl, 1985, page 312] is a C*-algebra $A$ such that, for every $a \in A$, there is a projection $a' \in P(A)$ such that the right annihilating ideal $\{b \in A \mid ab = 0\}$ of $a$ is the principal right ideal $a'A$ of $A$ generated by $a'$. For instance, every AW*-algebra, hence every von Neumann algebra, is a Rickart C*-algebra. If $A$ is a Rickart C*-algebra, then $P(A)$ is an orthomodular lattice (OML). An AW*-algebra is the same thing as a Rickart C*-algebra for which $P(A)$ is a complete OML.

EXAMPLE 118. If $A$ is a Rickart C*-algebra and $a \in A$, then $a^*a \in G(A)$ and $a' = (a^*a)'$, hence the mapping $': A \to P(A)$ is determined by its restriction to $G(A)$. Moreover, with this restriction as the Rickart mapping, $G(A)$ has the Rickart projection property.

LEMMA 119. The CB-group $G$ has the Rickart projection property if there is a mapping $\#': G \to P$ such that, for all $g \in G$ and all $p \in P$, $g^\# \leq p \iff J_p(g) = g$. Moreover, if $\#$ is such a mapping, then the corresponding Rickart mapping is given by $g' = u - g^\#$ for all $g \in G$. 
Proof. Suppose $G$ has the Rickart projection property and define $\# : G \to P$ by $g^\# := u - g'$ for all $g \in G$. Let $g \in G$ and $p \in P$. Then $g^\# \leq p \iff u - p \leq g' \iff g \in C(p)$ and $J_{u-p}(g) = 0 \iff g = J_p(g) + J_{u-p}(g)$ and $J_{u-p}(g) = 0$. Therefore, $g\# \leq p \Rightarrow J_p(g) = g$. Conversely, suppose $J_p(g) = g$. Then $J_{u-p}(g) = J_{u-p}(J_p(g)) = 0$, and it follows that $g\# \leq p$.

If $\# : G \to P$ satisfies $g^\# \leq p \iff J_p(g) = g$, then, by reversing the argument above, one sees that $G$ has the Rickart projection property.

The properties of the Rickart mapping $g \mapsto g'$ in a CB-group $G$ with the Rickart property are developed in [Foulis, 2005, Section 6]. For instance, for all $p, q \in P$, and all $g \in G$, $g'' := (g')' = u - g'$, $p' = u - p$, $g'' = 0 \iff g = 0$, $g''' = g'$, and so on. Furthermore, we have the following.

THEOREM 120. If the CB-group $G$ has the Rickart projection property, then $P$ is an orthomodular lattice and, for all $p, q \in P$, $p \land q = J_p((J_p(q'))')$.

Proof. See [Foulis, 2005, Theorem 6.4 (ii)].

DEFINITION 121. A comparability group with the Rickart projection property is called an RC-group. An RC-group that is also an $\ell$-group is called an RC-$\ell$-group. An archimedean RC-group (respectively, RC-$\ell$-group) is called an ARCl-group (respectively, an ARCl-$\ell$-group). An RC-unigroup (respectively, an ARCl-unigroup, an ARCl-$\ell$-unigroup) is an RC-group (respectively, an ARC-group, an ARCl-$\ell$-group) that is also a unigroup.\[33\]

THEOREM 122. If $G$ is an RC-group with unit interval $E$, then $G$ is an ARCl-group iff, the only $e \in E$ such that $ne \in E$ for every positive integer $n$ is $e = 0$.

Proof. See [Foulis, to appear (b), Theorem 5.10].

THEOREM 123. If $G$ is an archimedean comparability group and $P$ satisfies the chain conditions (i.e., there are no properly ascending or descending infinite chains in $P$), then $G$ is an ARCl-group and, for each $g \in G$, there exists a positive integer $N$ such that $g'' \leq N|g|$.

Proof. See [Foulis, 2005, Theorem 6.6].

THEOREM 124. If $G$ is a monotone $\sigma$-complete $\ell$-group, then $G$ is an ARCl-unigroup.

Proof. Assume the hypotheses. By Theorem 63, $G$ is archimedean, and by [Foulis and Pulmannová, to appear, Theorem 4.6 (ii)], $G$ is an RC-$\ell$-group. Also every unital $\ell$-group is a unigroup by Theorem 61.

THEOREM 125. Let $X$ be a compact Hausdorff space and organize $C(X, \mathbb{R})$ into an archimedean $\ell$-unigroup as in Example 69. Then the following conditions are mutually equivalent: (i) $X$ is basically disconnected. (ii) $C(X, \mathbb{R})$ is monotone $\sigma$-complete. (iii) $C(X, \mathbb{R})$ is an ARCl-unigroup.

\[33\] The authors do not know an example of an ARCl-group that is not a unigroup.
\textbf{Proof.} See [Foulis and Pulmannová, to appear, Theorem 4.8].

\textbf{THEOREM 126.} If $G$ is an $RC$-group (respectively, an $ARC$-group) and $p \in P$, then both $C(p)$ and $J_p(G)$ are $RC$-groups (respectively, $ARC$-groups).

\textbf{Proof.} See [Foulis, 2006, Theorem 5.5].

Suppose that $E$ is regarded as a quantum logic and $e \in E$. If $0 \neq p \in P$ and $p \leq e$, then the (possibly) fuzzy proposition $e$ has at least a sharp "part" $p$. If $e$ has no nonzero sharp part in this sense, then we say that $e$ is "blunt." This leads us to the following definition.

\textbf{DEFINITION 127.} An element $b \in G^+$ is blunt iff the only projection $p \in P$ such that $p \leq b$ is $p = 0$.

\textbf{LEMMA 128.} Let $G$ be an $RC$-group and let $b \in G^+$. Then: (i) $b$ is blunt iff $((u - b)^+)\prime = 0$. (ii) If $b$ is blunt, then $b \in E$.

\textbf{Proof.} See [Foulis, to appear(b), Theorem 5.6].

In the following theorem, we obtain a canonical decomposition of elements in the positive cone of an $RC$-group into sharp and blunt elements.

\textbf{THEOREM 129.} If $G$ is an $RC$-group and $0 \neq g \in G^+$, there is a uniquely determined finite descending sequence $p_1 \geq p_2 \geq \cdots \geq p_n$ of projections and a uniquely determined blunt element $b \in E$ such that $g = p_1 + p_2 + \cdots + p_n + b$ and $g \in \bigcap_{i=1}^{n} C(p_i)$. Moreover, $p_i \in CPC(g)$ for $i = 1, 2, \ldots, n$.

\textbf{Proof.} See [Foulis, to appear(b), Theorem 5.8, Theorem 5.9].

Recall that a Heyting algebra\footnote{Also called a Brouwer algebra or a Brouwer-Heyting algebra.} is a system $(L, \leq, 0, 1, \wedge, \vee, \supset)$ consisting of a bounded lattice $(L, \leq, 0, 1, \wedge, \vee)$ and a binary operation $\supset$ on $L$ called the Heyting implication connective such that, for all $x, y, z \in L, x \wedge y \leq z \iff x \leq (y \supset z)$ [Foulis, 2000a]. If the elements of the Heyting algebra $L$ are regarded as propositions and $y, z \in L$, then $y \supset z$ is interpreted as a proposition in $L$ asserting that $y$ "implies" $z$. A Heyting effect algebra is a lattice-ordered effect algebra $L$ equipped with a binary operation $\supset$ such that $(L, \leq, 0, 1, \wedge, \vee, \supset)$ is a Heyting algebra and $C(L) = \{ x \supset 0 \mid x \in L \}$.

\textbf{THEOREM 130.} Every Heyting effect algebra is an $MV$-algebra, hence it is the unit interval in an $\ell$-unigroup. If $G$ is an $\ell$-unigroup, then the unit interval $E$ in $G$ is a Heyting effect algebra iff $G$ is an $RCL$-unigroup.

\textbf{Proof.} See[Foulis, to appear(c), Theorems 5.2 and 7.10].

As is indicated by the theorems above and the examples that follow, many $CB$-groups $G$ for which $P$ and $E$ are good candidates for sharp and unsharp quantum logics are actually $ARC$-unigroups.
EXAMPLE 131. If $\mathcal{A}$ is an AW*-algebra, then $G(\mathcal{A})$ is an ARC-unigroup. If $\mathcal{A}$ is a commutative AW*-algebra, then $G(\mathcal{A})$ is an ARC $\ell$-unigroup.

As a special case of Example 131, the prototype $G(\mathcal{H})$ is an ARC-unigroup.

EXAMPLE 132. Every Boolean unigroup is an ARC $\ell$-unigroup.

EXAMPLE 133. If $\mathcal{B}$ is a $\sigma$-field of subsets of a nonempty set $X$, then $\mathcal{G}(X, \mathcal{B}, \mathbb{R})$ in Definition 76 is a monotone $\sigma$-complete ARC $\ell$-unigroup.

18 SPECTRAL THEORY IN AN ARC-GROUP

The development in the section is motivated by the following example.

EXAMPLE 134. Let $\mathcal{H}$ be a Hilbert space, let $A \in G(\mathcal{H})$, and let $M \mapsto P_M$ be the bounded PV-measure corresponding to the bounded observable $A$ (Definition 2). By definition, the spectral resolution for $A$ is the family $(P_\lambda)_{\lambda \in \mathbb{R}}$ of projections given by

$$P_\lambda := P_{(-\infty, \lambda]}$$

for $\lambda \in \mathbb{R}$.

It can be shown that, for all $\lambda \in \mathbb{R}$,

$$P_\lambda = ((A - \lambda 1)^+)'.$$

The spectral resolution $(P_\lambda)_{\lambda \in \mathbb{R}}$ for $A$ has the following properties (i)–(v), which determine it uniquely. For all $\lambda, \mu \in \mathbb{R}$:

(i) $\mu \leq \lambda \Rightarrow P_\mu \leq P_\lambda$.

(ii) $\bigwedge_{\mu < \lambda \in \mathbb{R}} P_\lambda = P_\mu$.

(iii) $\exists \alpha, \beta \in \mathbb{R}$, $P_\alpha = 0$ and $P_\beta = 1$.

(iv) $AP_\lambda = P_\lambda A$.

(v) If $\mu < \lambda$ and $Q := P_\lambda - P_\mu$, then $\mu Q \leq AQ \leq \lambda Q$.

Define

$$L := \sup\{\mu \mid P_\mu = 0\} \quad \text{and} \quad U := \inf\{\lambda \mid P_\lambda = 1\}.$$

Then properties (i)–(v) above imply that

$$A = \int_{L - 0}^{U} \lambda dP_\lambda,$$

where the integral is the limit in the sense of convergence in the uniform operator norm $\| \cdot \|$ of sums of Stieltjes type. Furthermore, the spectral bounds $L$ and $U$ satisfy

$$L = \sup\{\mu \mid \mu 1 \leq A\}, \quad U = \inf\{\lambda \mid A \leq \lambda 1\},$$

and $\|A\| = \max\{|L|, |U|\}$. 

Suppose \( A \in G(\mathfrak{g}) \) and \( (P_\lambda)_{\lambda \in \mathbb{R}} \) is the spectral resolution for the bounded observable \( A \). By property (ii) in Example 134, the spectral resolution \( (P_\lambda)_{\lambda \in \mathbb{R}} \) is "continuous from the right," but in general, it will fail to be "continuous from the left," i.e., it is quite possible to have \( \bigvee_{\lambda > \mu \in \mathbb{R}} P_\mu < P_\lambda \). In fact, if we define the "jump" at \( \lambda \in \mathbb{R} \) by

\[
D_\lambda := P_\lambda - \bigvee_{\lambda > \mu \in \mathbb{R}} P_\mu,
\]

then \( D_\lambda \neq 0 \) iff \( \lambda \) is an eigenvalue of \( A \). Furthermore, if \( \lambda \) is an eigenvalue of \( A \), then \( D_\lambda \) is the projection onto the \( \lambda \)-eigenspace of \( A \).

As usual, we denote by \( \mathbb{Q} \) the ordered field of rational numbers. The subfamily \( (P_\lambda)_{\lambda \in \mathbb{Q}} \) of \( (P_\lambda)_{\lambda \in \mathbb{R}} \) is called the rational spectral resolution for \( A \). By properties (i) and (ii) in Example 134, for any \( \mu \in \mathbb{R} \), we have

\[
P_\mu = \bigwedge_{\mu < \lambda \in \mathbb{Q}} P_\lambda,
\]

so the rational spectral resolution for \( A \) determines the spectral resolution for \( A \), hence it too determines \( A \).

Our purpose in this section is to generalize the notion of a rational spectral resolution to an arbitrary ARC-group and to show that the generalization acquires most of the important properties of the prototype.

Standing Assumption: In this section, we assume that \( G \) is an ARC-group. We maintain our assumptions that \( u \neq 0 \) is the unit in \( G \), \( E \) is the unit interval in \( G \), \( (J_p)_{p \in P} \) is the compression base for \( G \), and \( \| \cdot \| \) is the order-unit norm on \( G \). Also, for simplicity, we denote the state space \( \Omega(G) \) simply by \( \Omega \). If \( (p_i)_{i \in I} \) is an indexed family of projections in \( P \), and if we write \( p = \bigwedge_{i \in I} p_i \), we mean that the infimum \( \bigwedge_{i \in I} p_i \) of the family exists in \( P \) and equals \( p \). A similar convention applies to the supremum \( \bigvee_{i \in I} p_i \).

**DEFINITION 135.** If \( g \in G \), then the spectral lower and upper bounds \( L_g \) and \( U_g \), respectively, for \( g \) are defined by

\[
L_g := \sup \{ m/n \mid m, n \in \mathbb{Z}, 0 < n, \text{ and } mu \leq ng \}
\]

and

\[
U_g := \inf \{ m/n \mid m, n \in \mathbb{Z}, 0 < n, \text{ and } ng \leq mu \}.
\]

**LEMMA 136.** Let \( g \in G \). Then:

(i) \( -\infty < L_g \leq U_g < \infty \).

(ii) \( [L_g, U_g] = \{ \omega(g) \mid \omega \in \Omega \} \).

(iii) \( \|g\| = \max\{ |L_g|, |U_g| \} \).

**Proof.** See [Goodearl, 1985, Proposition 4.7].
**DEFINITION 137.** Let \( g \in G \). If \( \lambda \in \mathbb{Q} \), choose \( m, n \in \mathbb{Z} \) with \( 0 < n \) such that \( \lambda = m/n \) and define
\[
p_{\lambda} := ((ng - mu)^+)' \quad \text{and} \quad d_{\lambda} := (ng - mu)'.
\]
By [Foulis, 2004b, Lemma 4.1], \( p_\lambda \) and \( d_\lambda \) are well defined. The family \( (p_\lambda)_{\lambda \in \mathbb{Q}} \) is called the *rational spectral resolution* for \( g \) and \( (d_\lambda)_{\lambda \in \mathbb{Q}} \) is called the *family of rational eigenprojections* for \( g \).

As is shown by the following theorem, the rational spectral resolution of \( g \in G \) has properties analogous to properties (i)--(v) in Example 134.

**THEOREM 138.** Let \( g \in G \) and let \( (p_\lambda)_{\lambda \in \mathbb{Q}} \) be the rational spectral resolution for \( g \). Then, for all \( \lambda, \mu \in \mathbb{Q} \):

(i) \( \mu \leq \lambda \Rightarrow p_\mu \leq p_\lambda \).

(ii) \( \bigwedge_{\mu < \lambda \in \mathbb{Q}} p_\lambda = p_\mu \).

(iii) \( \mu < L_g \Rightarrow p_\mu = 0 \) and \( \lambda > U_g \Rightarrow p_\lambda = u \).

(iv) \( g \in C(p_\lambda) \).

(v) Suppose \( \mu < \lambda \), let \( q := p_\lambda - p_\mu \), and choose an integer \( n > 0 \) such that \( n\mu, n\lambda \in \mathbb{Z} \). Then \( n\mu q \leq nJ_q(g) \leq n\lambda q \).

**Proof.** Parts (i),(iii),(iv), and (v) follow from parts (i),(iii),(v),(vi), and (viii) of [Foulis, 2004b, Theorem 4.1]. Part (ii) follows from [Foulis, 2004b, Theorem 4.4 (i)]\(^{35}\).

Additional properties of the rational spectral resolution of \( g \in G \) are assembled in the next theorem.

**THEOREM 139.** Let \( g \in G \) and let \( (p_\lambda)_{\lambda \in \mathbb{Q}} \) be the rational spectral resolution for \( g \). Then, for all \( \lambda, \mu \in \mathbb{Q} \):

(i) \( p_\lambda, d_\lambda \in P \cap CPC(g), \ g \in C(d_\lambda), \) and \( p_\lambda C d_\lambda \).

(ii) \( \mu < \lambda \Rightarrow d_\mu \leq p_\mu \leq (d_\lambda)' \).

(iii) \( L_g \mu \Rightarrow 0 < p_\mu \) and \( \lambda < U_g \Rightarrow p_\lambda < u \).

(iv) \( L_g = \sup\{\mu \in \mathbb{Q} \mid p_\mu = 0\} \) and \( U_g = \inf\{\lambda \in \mathbb{Q} \mid p_\lambda = u\} \).

(v) If \( \omega \in \Omega, \mu < \lambda, q \in P, \) and \( q \leq p_\lambda - p_\mu, \) then \( \mu \omega(q) \leq \omega(J_q(g)) \leq \lambda \omega(q) \).

(vi) \( \bigvee_{\lambda > \mu \in \mathbb{Q}} p_\mu = p_\lambda - d_\lambda \).

\(^{35}\)By [Foulis, 2004b, Example 4.1], part (ii) of Theorem 138 depends on the assumption that \( G \) is archimedean.
Proof. Parts (i)–(v) follow from parts (i),(iv),(v),(vi),(vii), and (ix) of [Foulis, 2004b, Theorem 4.1], and part (vi) follows from [Foulis, 2004b, Theorem 4.4 (ii)].

The following theorem is a generalized version of the fact that, if \((P_\lambda)_{\lambda \in \mathbb{R}}\) is the spectral resolution for a Hermitian operator \(A\), then \(A = \int_{L-0}^U \lambda \, dP_\lambda\).

**THEOREM 140.** Let \(g \in G\) and let \((P_\lambda)_{\lambda \in \mathbb{Q}}\) be the rational spectral resolution for \(g\). Suppose that \(\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{Q}\) with \(\lambda_0 < L_g < \lambda_1 < \cdots < \lambda_{N-1} < U_g < \lambda_N\) and choose \(\gamma_i \in \mathbb{Q}\) with \(\lambda_{i-1} \leq \gamma_i \leq \lambda_i\) for \(i = 1, 2, \ldots, N\). Define \(u_i := p_{\lambda_i} - p_{\lambda_{i-1}}\) and \(\Delta \lambda_i := \lambda_i - \lambda_{i-1}\) for \(i = 1, 2, \ldots, N\). Let \(\epsilon := \max\{\Delta \lambda_i \mid i = 1, 2, \ldots, N\}\). Choose \(n > 0\) such that \(n\lambda_j, n\gamma_i \in \mathbb{Z}\) for \(j = 0, 1, 2, \ldots, N\) and \(i = 1, 2, \ldots, N\). Then:

(i) \(\sum_{i=1}^{N} u_i = u\).

(ii) \(\sum_{i=1}^{N} n\lambda_{i-1} u_i \leq ng \leq \sum_{i=1}^{N} n\lambda_i u_i\).

(iii) If \(s := \sum_{i=1}^{N} n\Delta \lambda_i u_i\), then \(-s \leq ng - \sum_{i=1}^{N} n\gamma_i u_i \leq s\).

(iv) \(|ng - \sum_{i=1}^{N} n\gamma_i u_i| \leq n\epsilon\).

(v) If \(\omega \in \Omega\), then \(|\omega(g) - \sum_{i=1}^{N} \gamma_i \omega(u_i)| \leq \epsilon\).

**Proof.** See [Foulis, 2004b, Theorem 4.2].

**COROLLARY 141.** Each \(g \in G\) is uniquely determined by its own rational spectral resolution.

**Proof.** Suppose that \(g, h \in G\) and that both \(g\) and \(h\) have the same rational spectral resolution \((p_\lambda)_{\lambda \in \mathbb{Q}}\). Let \(\omega \in \Omega\). By part (v) of Theorem 140, \(|\omega(g) - \omega(h)| \leq 2\epsilon\), and since we can choose \(\lambda_0, \lambda_1, \ldots, \lambda_N\) so that \(\epsilon\) is arbitrarily small, it follows that \(\omega(g) = \omega(h)\), whence \(\omega(h - g) = \omega(g - h) = 0 \geq 0\). Since \(\omega \in \Omega\) is arbitrary and \(G\) is archimedean, Theorem 52 implies that \(h - g, g - h \in G^+\), whence \(g = h\).

Let \((P_\lambda)_{\lambda \in \mathbb{R}}\) be the spectral resolution for the bounded observable \(A \in G(S)\). As usual, if \(\tau \in \mathbb{R}\), then by definition \(\tau\) belongs to the resolvent set of \(A\) iff there is an open interval \(I \subseteq \mathbb{R}\) such that \(\tau \in I\) and \(P_\lambda\) is constant on \(I\). Recall that the spectrum of \(A\), in symbols \(\text{spec}(A)\), is the complement in \(\mathbb{R}\) of the resolvent set of \(A\). We now proceed to generalize these notions to the ARC-group \(G\).

**DEFINITION 142.** If \(g \in G\) and \((P_\lambda)_{\lambda \in \mathbb{Q}}\) is the rational spectral resolution of \(g\), we say that a real number \(\tau \in \mathbb{R}\) belongs to the resolvent set of \(g\) iff there is an open interval \(I \subseteq \mathbb{R}\) with \(\tau \in I\) such that \(\lambda, \mu \in \mathbb{Q} \cap I \Rightarrow p_\lambda = p_\mu\). The spectrum of \(g\), in symbols \(\text{spec}(g)\), is defined to be the complement in \(\mathbb{R}\) of the resolvent set.
of \( g \). Let \((d_{\lambda})_{\lambda \in \mathbb{Q}}\) be the family of rational eigenprojections of \( g \). If \( \alpha \in \mathbb{Q} \) and \( d_{\alpha} \neq 0 \), we say that \( \alpha \) is a rational eigenvalue of \( g \).

**Lemma 143.** If \( g \in G \), then \( \text{spec}(g) \) is a closed nonempty subset of the closed interval \([L_g, U_g]\) with \( L_g = \inf \text{spec}(g) \) and \( U_g = \sup \text{spec}(g) \).

**Proof.** See [Foulis, 2004b, Theorem 5.1].

Let \( g \in G \). By part (ii) of Theorem 139, if \( \lambda, \mu \in \mathbb{Q} \) with \( \lambda \neq \mu \), then the eigenprojections \( d_{\lambda} \) and \( d_{\mu} \) are orthogonal, corresponding to the fact that, in the prototype \( G(5) \), eigenvectors for distinct eigenvalues are orthogonal.

**Lemma 144.** Let \( g \in G \), let \( \alpha \in \mathbb{Q} \), and suppose the \( \alpha \) is not an accumulation point of \( \text{spec}(g) \) (i.e., there is an open interval \( I \subseteq \mathbb{R} \) such that \( \alpha \in I \) and every \( \tau \in I \) with \( \tau \neq \alpha \) belongs to the resolvent set of \( g \)). Then \( \alpha \in \text{spec}(g) \) iff \( \alpha \) is a rational eigenvalue of \( g \).

**Proof.** See [Foulis, 2004b, Theorem 5.1 (ii)].

**Theorem 145.** Let \( g \in G \). Then:

(i) Every rational eigenvalue of \( g \) belongs to \( \text{spec}(g) \).

(ii) Every rational isolated point of \( \text{spec}(g) \) is a rational eigenvalue of \( g \).

**Proof.** Part (i) follows from [Foulis, 2004b, Theorem 5.2 (i)] and part (ii) is a direct consequence of Lemma 144.

**Theorem 146.** Let \( g \in G \) and let \((d_{\lambda})_{\lambda \in \mathbb{Q}}\) be the family of rational eigenprojections for \( g \). Suppose that \( \alpha_1, \alpha_2, ..., \alpha_N \) are distinct rational numbers and choose a positive integer \( n \) such that \( n\alpha_i \in \mathbb{Z} \) for \( i = 1, 2, ..., N \). For \( i = 1, 2, ..., N \), define \( u_i := d_{\alpha_i} \). Then:

(i) If \( \text{spec}(g) = \{\alpha_1, \alpha_2, ..., \alpha_N\} \), then \( u_i \neq 0 \) for \( i = 1, 2, ..., N \), \( \sum_{i=1}^{N} u_i = u \), and \( ng = \sum_{i=1}^{N} n\alpha_i u_i \).

(ii) If \( 0 \neq u_i \in P \) for \( i = 1, 2, ..., N \), \( \sum_{i=1}^{N} v_i = u \), and \( ng = \sum_{i=1}^{N} n\alpha_i v_i \), then \( v_i = u_i \) for \( i = 1, 2, ..., N \) and \( \text{spec}(g) = \{\alpha_1, \alpha_2, ..., \alpha_N\} \).

**Proof.** See [Foulis, 2004b, Theorems 5.3 and 5.4].

Of course, one would like to cancel the positive integer \( n \) from both sides of the inequalities and equations in part (v) of Theorem 138, parts (ii), (iii), and (iv) of Theorem 140, and parts (i) and (ii) of Theorem 146, but this is not feasible unless \( G \) is the additive group of a partially ordered rational vector space. However, as is shown in [Foulis, 2006, Section 7], the ARC-group \( G \) can always be embedded as a subgroup of the additive group of a partially ordered rational vector space.
which is an additive ARC-group and for which the projections are the same as the projections in \( G \). Thus, for \( g \in G \), the spectral resolution is the same whether calculated in \( G \) or in \( D \), and in \( D \) the desired cancellations of \( n \) can be made. Although \( G \) and \( D \) have the same state space \( \Omega \), the unit interval in \( G \) can be a proper subset of the unit interval in \( D \). For details, see the cited reference.

**THEOREM 147.** Let \( g \in G \). Then:

(i) \( g \in G^+ \iff \text{spec}(g) \subseteq [0, \infty) \).

(ii) \( g \in E \iff \text{spec}(g) \subseteq [0, 1] \).

(iii) \( g \in P \iff \text{spec}(g) \subseteq \{0, 1\} \).

**Proof.** See [Foulis, 2004b, Theorem 5.5].

The next definition is motivated by the following observations: Let \( A \in \mathcal{G}(\mathcal{F}) \) and let \( \mathcal{M} = A(\mathcal{F}) \) be the (norm) closure of the range of \( A \). Then \( A'' \) is the projection onto \( \mathcal{M} \). Also: (i) As a bounded linear operator on \( \mathcal{F} \), \( A \) has an inverse \( A^{-1} \) iff \( 0 \notin \text{spec}(A) \). (ii) \( A(\mathcal{F}) \) is (norm) dense in \( \mathcal{F} \) iff \( A'' = 1 \). (iii) The range \( A(\mathcal{F}) \) of \( A \) is (norm) closed iff the restriction \( A|_{\mathcal{M}} \) of \( A \) to \( \mathcal{M} \) has an inverse \( (A|_{\mathcal{M}})^{-1} \) as a bounded linear operator on \( \mathcal{M} \).

**DEFINITION 148.** Let \( g \in G \). Then:

(i) \( g \) is nonsingular iff \( 0 \notin \text{spec}(g) \).

(ii) \( g \) is range dense iff \( g'' = u \).

(iii) \( g \) is range closed iff either \( g = 0 \) (i.e., \( g'' = 0 \)) or \( g \neq 0 \) and \( g \) is a nonsingular element of the RC-group \( J_{g''}(G) \).

**THEOREM 149.** If \( g \in G^+ \), then the following conditions are mutually equivalent:

(i) \( g \) is nonsingular.

(ii) \( 0 < L_g \).

(iii) \( 0 < \omega(g) \) for all \( \omega \in \Omega \).

(iv) \( g \) is an order unit in \( G \).

(vi) There is a positive integer \( N \) such that \( u \leq Ng \).

**Proof.** See [Foulis, 2004b, Theorem 5.6].

**THEOREM 150.** Let \( g \in G \). Then:

(i) If \( g \neq 0 \), then \( g \) is range dense in the RC-group \( J_{g''}(G) \).
(ii) \( g \) is range closed iff \( \exists \epsilon \in \mathbb{R} \) with \( \epsilon > 0 \) such that both open intervals \((-\epsilon, 0)\) and \((0, \epsilon)\) are contained in the resolvent set of \( g \).

(iii) \( g \) is nonsingular iff it is both range dense and range closed.

(iv) If \( g \in G^+ \), then \( g \) is range closed iff there is a positive integer \( N \) such that \( g'' \leq N g \).

**Proof.** See [Foulis, 2004b, Theorem 5.7].

**Lemma 151.** Let \((p_\lambda)_{\lambda \in \mathbb{Q}}\) be the rational spectral resolution for \( g \in G \), let \( q \in P \) and let \((q_\lambda)_{\lambda \in \mathbb{Q}}\) be the rational spectral resolution for \( q \). Then:

(i) \( g \in C(q) \iff p_\lambda Cq \) for all \( \lambda \in \mathbb{Q} \).

(ii) If \( \lambda \in \mathbb{Q} \), then \( q_\lambda = 0 \) for \( \lambda < 0 \), \( q_\lambda = u - q \) for \( 0 \leq \lambda < 1 \), and \( q_\lambda = u \) for \( 1 \leq \lambda \).

(iii) \( g \in C(q) \iff p_\lambda Cq_\mu \) for all \( \lambda, \mu \in \mathbb{Q} \).

**Proof.** Part (i) follows from [Foulis, 2004b, Theorem 4.3], Part (ii) follows from Theorem 146 (i) and Theorem 147 (iii), and part (iii) follows from parts (i) and (ii).

With \( g := p \in P \), part (iii) of Lemma 151 ensures that the extended definition of compatibility in the following definition is consistent with the previously defined notion of compatibility for pairs of projections.

**Definition 152.** Let \( g, h \in G \) and let \((p_\lambda)_{\lambda \in \mathbb{Q}}\) and \((q_\lambda)_{\lambda \in \mathbb{Q}}\) be the spectral resolutions of \( g \) and \( h \), respectively. Then, by definition, \( g \) and \( h \) are compatible, in symbols \( gCh \), iff \( p_\lambda Cq_\mu \) for all \( \lambda, \mu \in \mathbb{Q} \).

**Lemma 153.** Let \( g, h, k \in G \), let \((r_\lambda)_{\lambda \in \mathbb{Q}}\) be the rational spectral resolution for \( k \), and let \( p \in P \). Then:

(i) \( gCh \iff hCg \).

(ii) \( g \in C(p) \iff gCp \).

(iii) \( gCk \iff gCr_\lambda \) for all \( \lambda \in \mathbb{Q} \).

(iv) If \( gCk \) and \( hCk \), then \((g + h)Ck \).

**Proof.** Part (i) is obvious, and part (ii) follows from Lemma 151 (iii) and Definition 152.

(iii) Let \((p_\lambda)_{\lambda \in \mathbb{Q}}\) be the rational spectral resolution for \( g \). Then by Definition 152 and Lemma 151 (i), \( gCk \iff p_\mu C r_\lambda \), \( \forall \mu, \lambda \in \mathbb{Q} \iff g \in C(r_\lambda) \), \( \forall \lambda \in \mathbb{Q} \); hence by (ii), \( gCk \iff gCr_\lambda \), \( \forall \lambda \in \mathbb{Q} \).

(iv) Suppose \( gCk \) and \( hCk \). Then, by (iii) and (ii), \( g, k \in C(r_\lambda) \), \( \forall \lambda \in \mathbb{Q} \). For each \( \lambda \in \mathbb{Q} \), \( C(r_\lambda) \) is a subgroup of \( G \); hence \( g + h \in C(r_\lambda) \), \( \forall \lambda \in \mathbb{Q} \), and it follows from (ii) and (iii) that \((g + h)Ck \).
By a standard theorem for Hermitian operators, if $A, B \in G(\mathfrak{g})$, then $A$ and $B$ are compatible in the sense of Definition 152 iff $AB = BA$. More generally, if $A$ is an AW*-algebra and $g, h \in G(A)$, then $gCh$ iff $gh = hg$ in $A$.

Let $\mathcal{A}$ be an AW*-algebra. Then every maximal commutative subset of $G(\mathcal{A})$ has the form $G(C)$ where $C$ is a maximal commutative *-subalgebra of $A$. Moreover, if $C$ is a maximal commutative *-subalgebra of $\mathcal{A}$, then $C$ is a commutative AW*-algebra, $G(C)$ is a subgroup of $G(\mathcal{A})$, and $G(C)$ is an ARC $\ell$-unigroup. Therefore, $G(\mathcal{A})$ is covered by subgroups each of which is an ARC $\ell$-group. We now proceed to show that the same is true for every ARC-group $G$.

**DEFINITION 154.** A C-block in $G$ is a maximal set of pairwise compatible elements in $G$.

By Zorn's lemma, every subset of $G$ consisting of pairwise compatible elements can be extended to a C-block. In particular, every element $g \in G$ belongs to a (not necessarily unique) C-block, hence $G$ is covered by its own C-blocks.

**THEOREM 155.** If $H$ is a C-block in $G$, then $H$ is a subgroup of $G$, and with the partial order inherited from $G$, $H$ can be organized into an ARC-group as follows:

(i) The positive cone in $H$ is $H^+ = H \cap G^+$.

(ii) The unit for $H$ is $u$, and the unit interval in $H$ is $E_H = E \cap H$.

(iii) The set of projections in $H$ is $P_H = P \cap H$.

(iv) The compression base for $H$ is $(J_q^H)_{q \in P_H}$ where $J_q^H$ is the restriction to $H$ of $J_q$ for each $q \in P_H$.

Also, $P_H$ is a maximal Boolean sub-effect algebra of the orthomodular lattice $P$ and $H$ is an ARC $\ell$-group. Furthermore, if $G$ is monotone $\sigma$-complete, then so is $H$, and the Boolean algebra $P_H$ is $\sigma$-complete. Conversely, if $B$ is a maximal Boolean sub-effect algebra of $P$, there is a uniquely determined C-block $H$ in $G$, namely $H = \bigcap\{C(b) \mid b \in B\}$, such that $B = P_H$.

**Proof.** See Theorem [Foulis and Pulmannová, to appear, 5.10].

As a consequence of Theorem 155, every ARC-group is covered by subgroups each of which is an ARC $\ell$-group. In other words, every ARC-group has the structure of a collection of ARC $\ell$-groups "pasted together" in a coherent manner. Thus, the ARC $\ell$-groups are the "building blocks" for ARC-groups. In [Foulis and Pulmannová, to appear, Chapter 6], a functional representation is given for ARC $\ell$-groups (also see Theorem 75).

**19 RETROSPECTIVE**

The development in Sections 7–18 provides our rational for the following:
Scholium. For the general study of quantum logics, we consider the notion of a CB-triple

\[ P \subseteq E \subseteq G \]

where \( G \) is a proper CB-group with unit interval \( E \) and compression base \( (J_p)_{p \in P} \), to be an appropriate abstraction of the prototypic triple

\[ \mathbb{P}(5) \subseteq \mathbb{E}(5) \subseteq G(5). \]

Observables are unital morphisms from a Boolean unigroup into \( G \), states are unital morphisms from \( G \) to \( \mathbb{R} \), and symmetries are CB-automorphisms of \( G \) (Figure 4, Section 13). In particular, since ARC-groups and ARC-unigroups admit a practical spectral theory and a resulting notion of compatibility that generalizes the commutativity condition for bounded observables on \( 5 \), we regard ARC-groups and ARC-unigroups, their unit intervals, and their projection lattices as especially promising candidates for generalizations of \( G(5) \), \( \mathbb{E}(5) \), and \( \mathbb{P}(5) \), respectively.

In this exposition we have paid scant attention to some topics, and outright omitted other topics that are of considerable relevance to contemporary quantum physics and to quantum logic. Some topics have been downplayed or suppressed simply to keep the exposition within reasonable bounds. Other topics were omitted because the appropriate ideas have not yet been suitably incorporated into the theory of CB-groups. In an effort to indicate promising directions for future research, we conclude this article with an annotated list of some of these downplayed or omitted topics along with a few pertinent references.

Among the topics that were only briefly mentioned and that invite further study are:

- **The action of symmetry groups on the CB-triple** \( P \subseteq E \subseteq G \) [Foulis, 2000b; Foulis and Wilce, 2000]. Because of the ubiquity of group-theoretic methods in contemporary physics, the main thesis of this article, as per the Scholium above, is reinforced by the observation that one can formulate an auspicious mathematical basis for a theory of symmetries on CB-groups. (See Section 4 and Definition 104.) In [Foulis, 2000b; Foulis and Wilce, 2000], a start has been made on a comprehensive theory of symmetry groups acting on CB-groups. By combining \( \sigma \)-observables with symmetries on CB-groups, the important idea of a system of covariance becomes available, and there appear to be good prospects for a generalization of Mackey’s imprimitivity theorem [Mackey, 1993] in this context.

- **Phase infrastructure.** As we mentioned in Section 2, for the Hermitian group \( G(5) \), the underlying Hilbert space \( 5 \) carries phase information that is lost in the passage from normalized state vectors \( \psi \in 5 \) to the expectation measures \( \omega_\psi \) on \( G(5) \). For orthoalgebras and effect algebras, an analogous infrastructure is provided by so-called test spaces and E-test spaces, respectively [Barnum et al., 2005, Sections 2 and 3]. In the spirit of the present article, the test groups introduced in [Foulis et al., 1996] provide an appropriate infrastructure for unital groups, and
every unigroup is an image under an order-preserving group homomorphism of a lattice-ordered test group. Presumably, enrichment of test groups by operators analogous to projections on a Hilbert space should provide a suitable infrastructure for CB-groups, but this has yet to be studied.

- **CB-groups for coupled systems.** Suppose that \( S \) is a composite system with component systems \( S_1 \) and \( S_2 \). If \( G_1 \) and \( G_2 \) are the CB-groups for \( S_1 \) and \( S_2 \), respectively, how is the CB-group \( G \) for \( S \) related to \( G_1 \) and \( G_2 \)? In Hilbert-space-based quantum physics, if \( \mathcal{H}_1, \mathcal{H}_1, \) and \( \mathcal{H}_2 \) are the Hilbert spaces for \( S, S_1, \) and \( S_2 \), respectively, then \( \mathcal{H} \) is the Hilbert-space tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), and there is a canonical bilinear mapping \( (A_1, A_2) \mapsto A_1 \otimes A_2 \) from \( G(\mathcal{H}_1) \times G(\mathcal{H}_2) \) to \( G(\mathcal{H}) \).

For arbitrary unital groups \( G_1 \) and \( G_2 \), there is a notion of a tensor product of \( G_1 \) and \( G_2 \) [Foulis et al., 1994, Section 9], but, even for Hermitian groups \( G(\mathcal{H}_1) \) and \( G(\mathcal{H}_2) \), this tensor product is not the right thing.\(^{36}\) For Hermitian groups \( G(\mathcal{H}) \), it is the infrastructure of the Hilbert spaces \( \mathcal{H} \) that enables the formulation of physically meaningful tensor products, and an analogous infrastructure for unital and CB-groups (see above) is probably required to obtain appropriate generalizations thereof. For an authoritative article addressing the problem of representing compound physical systems, especially in the context of quantum information and communication theory, see [Barnum et al., 2005].

- **The special case in which the unital group (or CB-group, or RC-group) is the additive group of an order-unit normed space \( U \) in separating order and norm duality with a base-normed space \( V \).** Any archimedean unital group \( G \) can be embedded into the additive group of an order-unit normed space \( U \) in duality with a base-normed space \( V \) having the state space \( \Omega(G) \) as its cone base [Cook and Foulis, 2004]. In [Bugajski et al., 2000] it is shown that every convex \( \sigma \)-effect algebra can be realized as the unit interval in a Banach order-unit normed space \( U \). There is an obvious connection between the theory of order-unit-normed and base-normed spaces \( U \) and \( V \) in spectral duality [Alfsen and Scultz, 1979] and the theory of RC-groups, but the precise relationship between the two theories needs to be worked out in detail.

- **The functional representation of monotone \( \sigma \)-complete ARC\(\ell \)-groups.** This topic is especially significant owing to the fact that every monotone \( \sigma \)-complete ARC\(\ell \)-group is covered by such subgroups [Goodearl, 1985, Corollary 16.15], [Foulis and Pulmannová, to appear, Chapter 6].

- **Dilation.** The Naimark (or Nagy-Naimark) dilation theorem mentioned in Section 6 plays an important role in stochastic (or phase space) quantum mechanics [Schroek, 1996, Chapter 2], and it is fairly clear how to formulate a more general notion of dilation in the context of CB-groups. However, this matter has not yet been studied seriously.

\(^{36}\)It enables the representation of discontinuous and nonmeasurable \( R \)-valued bihomomorphisms, hence it is much too large.
• Jordan algebras The prototype $G(5)$ is a Jordan algebra [Topping, 1965] as well as an RC-group. Which RC-groups can be organized into Jordan algebras and how much of the theory of Jordan algebras of self-adjoint operators can be carried over to those that can be so organized?

• Wigner’s theorem on symmetry transformations. Wigner’s theorem, which was used in the proof of Theorem 6, has an extremely interesting generalization, originally due to R. Wright [Wright, 1977a], which has largely escaped the attention of the quantum logic community.\(^{37}\) Wright’s theorem could and should be studied further in an effort to better understand just what makes it work and possibly to obtain analogous results for more general CB-groups. Here is a statement of Wright’s theorem.

THEOREM 156. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces with $\dim(\mathcal{H}) > 2$ and $\dim(\mathcal{K}) > 0$. Suppose that $\phi: \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{K})$ is a $\sigma$-effect-algebra morphism. Then $\phi$ can be extended uniquely to a linear Jordan isometry $\Phi: \mathbb{G}(\mathcal{F}) \to \mathbb{G}(\mathcal{K})$. Furthermore, there exist orthogonal closed linear subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$, there exist Hilbert spaces $\mathcal{F}_1$, $\mathcal{F}_2$ (possibly of dimension 0), and there exists a unitary map $U_1: \mathcal{F} \otimes \mathcal{F}_1 \to \mathcal{M}$ and an anitunitary map $U_2: \mathcal{F} \otimes \mathcal{F}_2 \to \mathcal{N}$ such that

$$
\Phi(A) = U_1(A \otimes 1_1)U_1^{-1} + U_2(A \otimes 1_2)U_2^{-1} \quad \text{for all } A \in \mathbb{G}(\mathcal{F}).
$$

As a consequence of Wright’s theorem, every $\sigma$-effect-algebra morphism $\phi: \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{K})$ preserves arbitrary suprema and infima and has the following properties for all $P, Q \in \mathbb{P}(\mathcal{F})$: (i) $P \perp Q \iff \phi(P) \perp \phi(Q)$. (ii) $P = Q \iff \phi(P) = \phi(Q)$. (iii) $PQ = QP \iff \phi(P)\phi(Q) = \phi(Q)\phi(P)$. That an arbitrary $\sigma$-effect-algebra morphism should have these features is an extremely strong property of the projection lattices of Hermitian groups, and can hardly be expected to hold for a very large class of RC-groups.\(^{38}\)

Among the topics of significance in contemporary physics or quantum logic, and that have obvious connections with the theory of CB-groups, but were entirely omitted from this exposition, are:

• Quantum information theory and quantum computation. [Barnum, 2002; Barnum et al., 2005] and [Dalla Chiara et al., 2004, Chapter 17].

• Causal logic of relativistic physics [Casini, 2002; Cegla and Jadczyk, 2979].

• Stochastic or phase-space quantum mechanics [Schroeck, 1994; Schroeck, 1996].

• The representation, logic, and statistics of sequential measurements [Foulis, 2002; Gudder and Greechie, 2002].

\(^{37}\)However, see [Beltrametti et al., 2000, Section IV].

\(^{38}\)In his Ph.D. thesis [Wright, 1977b], Wright was able to extend his theorem to the class of von Neumann algebras acting on separable Hilbert spaces and containing no direct summand of type $L_2$. 
• The consistent histories approach to quantum theory [Griffiths, 1984].

• Topological effect algebras [Wilce, 2005].

• The property (or attribute) lattice of a physical system [Aerts et al., 2000], [Foulis, 1999, Section 6].

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