Coverings of \([MO_n]\) and minimal orthomodular lattices

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Abstract. If \(T\) is an orthomodular lattice (OML), we denote by \([T]\) the equational class generated by \(T\). In this paper we characterize the finite OMLs \(T\) such that \([T]\) covers some \([MO_n]\). These OMLs \(T\) are the non-modular OMLs such that all proper sub-OMLs of \(T\) are modular. An OML satisfying that last property is called minimal. There exist infinitely many minimal OMLs provided by quadratic spaces over finite fields. We describe them and give a new way to represent their Greechie diagrams in two separate parts. Other methods to obtain finite minimal OMLs are given.

1. Introduction

This paper is concerned with the lattice, \(\mathcal{L}\), of all varieties of orthomodular lattices (OMLs). It is well known that the bottom of this lattice contains a chain of \([MO_n]\)s as depicted in Figure 1. (Recall that \(MO_n\) is the orthomodular lattice consisting only of the bounds, 0 and 1, and 2n atoms; it is depicted in Figure 3.)

We characterize the equational classes \([T]\), with \(T\) finite, that cover the classes \([MO_n]\) by showing that these classes are generated by minimal OMLs, i.e., by non-modular OMLs \(T\) having the property that each proper sub-OML of \(T\) is modular.

Here is an outline of the paper.

- In section 2, we introduce the notion of minimal OML and we give the initial properties and examples of such OMLs. (These are indicated by the empty boxes in Figure 1).
- In section 3, we prove that the equational class generated by a finite irreducible minimal OML covers some equational class \([MO_n]\) in the lattice of all equational classes of orthomodular lattices.
- In section 4, we obtain infinitely many finite minimal OMLs from quadratic spaces over finite fields. Descriptions of such OMLs are given in two parts, according to the characteristic of the field.
- In section 5, we study two ways to obtain new minimal OMLs, by “pasting a prong” and by “merger”.

Presented by R. Freese.
Received May 14, 2005; accepted in final form May 30, 2007.
2000 Mathematics Subject Classification: 06, 08 and 15.
Key words and phrases: Orthomodular lattice, equational class, exclusion, quadratic space.
The general reference for OMLs is [9] or [10]. We recall some definitions and properties.

(1) An ortholattice (OL) is a lattice $L$, with a least element 0 and a greatest element 1, equipped with an involutive anti-automorphism $x \mapsto x^\perp$, called an orthocomplementation, satisfying

$$\forall x \in L, \ x \lor x^\perp = 1 \text{ and } x \land x^\perp = 0.$$ 

A modular lattice is a lattice satisfying

$$x \leq z \Rightarrow x \lor (y \land z) = (x \lor y) \land z.$$ 

An orthomodular lattice (OML) is an ortholattice satisfying

$$x \leq z \Rightarrow x \lor (x^\perp \land z) = z.$$
(2) We say that two elements $x$ and $y$ in an OML $L$ commute (written $xCy$) if the sub-OML generated by $x$ and $y$ is boolean. This is equivalent to

$$y = (y \land x) \lor (y \land x^\perp).$$

An element $x$ in an OML $L$ is central if $x$ commutes with all elements of $L$. If $x$ is a central element of $L$, $L$ is isomorphic to $[0, x] \times [0, x^\perp]$. (In the interval algebra $[0, y]$ the orthocomplement is given by $t' = t^\perp \land y$.)

The center $C(L)$ of $L$ is the set of all the central elements of $L$. $C(L)$ is a boolean algebra, and $L$ is irreducible if and only if $C(L) = \{0, 1\}$. If $L$ is an OML, a maximal boolean sub-algebra of $L$ is called a block of $L$.

(3) We say that two elements $x$ and $y$ in an OML $L$ are orthogonal (written $x \perp y$) if $x \leq y^\perp$. Let $L$ be a finite OML and let $A$ be the set of the atoms of $L$. If $a$ and $b$ are atoms of $L$, a path from $a$ to $b$ in the graph $(A, \perp)$ is a sequence of atoms $a = a_0, a_1, \ldots, a_n = b$ such that $a_i \perp a_{i+1}$ for $0 \leq i \leq n - 1$; $n$ is called the length of the path.

A distance function $\delta$ in the graph $(A, \perp)$ is defined as follows: (i) $\delta(a, a) = 0$, (ii) if $a \neq b$, then $\delta(a, b)$ is the minimum of the lengths of all paths from $a$ to $b$, if there are such paths, and (iii) $\delta(a, b) = \infty$ otherwise. The distance between two subsets $M$ and $N$ of $A$ is defined as follows: for $M, N \subset A$, $\delta(M, N) = \inf\{\delta(a, b) \mid a \in M \land b \in N\}$. If $a \in A$, $\delta(a, N) = \delta(\{a\}, N)$. We shall speak of the distance between two blocks $A$ and $B$ of an orthomodular lattice $L$, or of the distance between an atom $a$ and a block $A$ of $L$, meaning $\delta(A \cap A, B \cap A)$ and $\delta(a, A \cap A)$, respectively.

(4) If $L$ is a finite OML, denote by $A$ the set of all the atoms of $L$ and by $B$ the set of all the blocks of $L$. Let $E = \{B \cap A \mid B \in B\}$, the hypergraph $G = (A, E)$ is called the Greechie diagram of $L$ (see [7]). $G$ is said to be cubic in case all the “lines” in $E$ have three points. If $G$ is cubic, a hypergraph $(A', E')$ is a partial sub-diagram of $G = (A, E)$ in case $A' \subset A$, $E' \subset E$, and $\{x, y, z\} \in E'$ for every $x, y, z \in A'$ with $\{x, y, z\} \in E$.

The well-known OMLs $D_{16}$ and the pentagon, herein denoted $N_2$ and $N_3$, respectively, are examples of Greechie cubic diagrams [7]. They are given in Figure 2; note that $D_{16}$ is a partial sub-diagram of the pentagon. A partial sub-diagram of $G$ is said of $D_{16}$-type (respectively, of pentagonal-type) if it is isomorphic, as hypergraph, to $D_{16}$ (respectively, to the pentagon).

(5) Let $(L_i)_{i \in I}$ be a family of disjoint OMLs and $L = \bigcup_{i \in I} L_i$; after the identification of the least elements and the greatest elements of the $L_i$, $L$, with the operations induced by the $L_i$, becomes an OML called the horizontal sum of
(L_i)_{i \in I}. For n \geq 2, MO_n denotes the horizontal sum of n four-element boolean algebras. MO_n is a modular OML with 2n + 2 elements.

2. Minimal OMLs related to an exclusion problem

In this section we discuss the notion of exclusion system S for comparable classes, \( C_1 \subset C_2 \) of algebras. Asking the question, “Which OMLs are members of every exclusion system for \( MOL \subset OML \)?” leads us to the notion of minimal OML; these are OMLs, T, which are non-modular such that every proper sub-algebra of which is modular or isomorphic to T. After showing that every exclusion system for \( MOL \subset OML \) contains an infinite minimal OML, we characterize the irreducible, finite, minimal OMLs.

**Definition 2.1.** Let \( C_1 \) and \( C_2 \) be two classes of algebras such that \( C_1 \subset C_2 \). An exclusion system for \( C_1 \subset C_2 \) is a class \( S \subset C_2 - C_1 \) such that, for all \( L \in C_2 \), \( L \notin C_1 \) if and only if there exists \( S \in S \) isomorphic to a subalgebra of \( L \).

Of course the whole class \( C_2 - C_1 \) is an exclusion system, but we are interested in exclusion systems with cardinality as small as possible. Consider the following classes of ortholattices:

- the class \( BA \) of boolean algebras,
- the class \( MOL \) of modular ortholattices,
- the class \( OML \) of orthomodular lattices,
• the class $\mathcal{OL}$ of ortholattices.

We have $BA \subset MOL \subset OML \subset OL$. We denote by $X$ the ortholattice given in Figure 4 and we denote by $\mathbf{2}$ the two-element boolean algebra. The following

![Figure 4. The OL \( X \).](image)

results are well known.

1. \( \{ X \} \) is an exclusion system for $OML \subset OL$
2. \( \{ MO_2, \mathbf{2} \times MO_2 \} \) is an exclusion system for $BA \subset OML$ and $BA \subset MOL$.

The proof of (1) is given in [9, p. 22]. For the proof of (2), we consider $BA \subset OML$. Let $L$ be a non-boolean OML; there exists in $L$ two non-commuting elements $x$ and $y$. The sub-OML of $L$ generated by $x$ and $y$ is a non-boolean quotient of the free OML, $F_2$, generated by two elements. By [2], $F_2$ is $2^4 \times MO_2$, so that the sub-OML is isomorphic to $MO_2$ or $2^i \times MO_2$ with $1 \leq i \leq 4$, and $\mathbf{2} \times MO_2$ is a sub-OML of $2^i \times MO_2$. The same proof works for $BA \subset MOL$. If $S$ is an exclusion system for $C_1 \subset C_2$ and $T$ is an exclusion system for $C_2 \subset C_3$, then $S \cup T$ is an exclusion system for $C_1 \subset C_3$. From this it follows that

3. \( \{ X, MO_2, \mathbf{2} \times MO_2 \} \) is an exclusion system for $BA \subset OL$.

We now discuss the case of $MOL \subset OML$. In [9, p. 347], G. Kalmbach presents the following problem: characterize modular ortholattices among OMLs $L$ by excluding a finite list of finite OMLs as sub-algebras of $L$. In other words, is there a finite exclusion system $S$ for $MOL \subset OML$ formed with only finite OMLs? That the answer to this question is negative follows from two of our results. In proposition 2.5 of this section, we show that there is no exclusion system for $MOL \subset OML$ formed with only finite OMLs. Later, in section 4, we show that there is no finite exclusion system for $MOL \subset OML$. We do not know of an interesting exclusion system for $MOL \subset OML$. We now turn our attention to a characterization of which OMLs belong to every exclusion system for $MOL \subset OML$.

**Definition 2.2.** A non-modular OML $L$, all of whose proper sub-OMLs are either modular or isomorphic to $L$, is called a *minimal* OML.

**Proposition 2.3.** An OML $L$ belongs (up to isomorphism) to every exclusion system for $MOL \subset OML$ if and only if $L$ is minimal.
Proof. Assume that $L$ is a minimal OML. If $S$ is an exclusion system for $\mathcal{MOL} \subset \mathcal{OML}$, there exists $S \in S$ such that $S$ is isomorphic to a sub-OML of $L$; as $S$ is non-modular and $L$ minimal, $S$ is isomorphic to $L$.

Assume that $L$ is not minimal and non-modular; then there exists a proper sub-OML, $T$, of $L$ such that $T$ is non-modular and not isomorphic to $L$. Let $S$ be the sub-class of $\mathcal{OML} - \mathcal{MOL}$ obtained by deleting all the OMLs isomorphic to $L$. As $T \in S$, $S$ is an exclusion system and, up to isomorphism, $L$ does not belong to $S$. □

Remark 2.4. Proposition 2.3 is more general; it works for all classes of algebras $C' \subset C$, allowing one to define minimal algebras of $C$ with respect to $C'$.

Proposition 2.5. Every exclusion system for $\mathcal{MOL} \subset \mathcal{OML}$ contains an infinite OML.

Proof. In [4], the authors construct an infinite non-modular OML $L$ such that all finite sub-OMLs of $L$ are modular. (Actually, $L$ is a sub-OML of the OML, $\mathcal{C}(H)$, of all the closed subspaces of a separate Hilbertian space $H$.) Let $S$ be an exclusion system for $\mathcal{MOL} \subset \mathcal{OML}$; then there exists $S \in S$ isomorphic to a sub-OML of $L$ and, as $S$ is not modular, it is infinite. □

Remarks 2.6. (1) We do not know if the infinite OML $L$ given in [4] is minimal, but this is not necessary for the previous proof.

(2) A specific example of an infinite minimal OML is given in [6]; it is obtained from a quadratic space over the field $\mathbb{Q}$.

For the following, denote by $\mathcal{M}_f$ the class of all finite minimal OMLs. Then an OML $L$ belongs to $\mathcal{M}_f$ if and only if $L$ is finite, non-modular and all proper sub-OMLs of $L$ are modular. Let $\text{Irr}(\mathcal{M}_f)$ denote the class of all irreducible OMLs in $\mathcal{M}_f$. Recall that a finite OML is irreducible if and only if it is simple [9, p. 79].

Proposition 2.7. Any reducible element of $\mathcal{M}_f$ has the form $T \times 2$ for some irreducible $T$ in $\mathcal{M}_f$. Conversely, for any irreducible $T$ in $\mathcal{M}_f$, $T \times 2$ is in $\mathcal{M}_f$.

Proof. If $L \in \mathcal{M}_f$ is not irreducible, we have, up to isomorphism, $L = T_1 \times T_2 \times \cdots \times T_n$ with $n \geq 2$ and, for $1 \leq i \leq n$, $T_i$ irreducible. Since $L$ is non-modular, we can assume that $T_1$ is non-modular. Then $T_1 \times 2$ is a non-modular sub-OML of $L$ and the minimality of $L$ implies $L = T_1 \times 2$ and $T_1$ minimal, i.e., $T_1 \in \text{Irr}(\mathcal{M}_f)$.

Consider $L = T \times 2$ with $T \in \text{Irr}(\mathcal{M}_f)$. Let $L'$ be a sub-OML of $L$; we have to prove that $L' = L$ or $L'$ modular. Let $p_1 : L \rightarrow T$ be the projection onto $T$. If $p_1(L') \neq T$, as $T$ is minimal, $p_1(L')$ is modular and, as $L' \subset p_1(L') \times 2$, $L'$ is modular. If $p_1(L') = T$, let $I = \{ u \in T \mid (u,0) \in L' \}$. Then $I$ is an ideal of $T$ such that if $u \notin T$ then $u^\perp \notin T$; this implies that $I$ is a p-ideal of $T$ and, as $T$ is simple, $I = T$ and $L' = T \times 2 = L$. □
Examples 2.8. Denote by $N_1$ the horizontal sum of $2^1$ and $2^2$; it is given in Figure 5. We have $N_1 \in \text{Irr}(M_f)$ and, by proposition 2.7, $N_1 \times 2 \in M_f$.

\begin{figure}
\centering
\includegraphics[width=0.3	extwidth]{figure5.png}
\caption{$N_1$.}
\end{figure}

The Greechie diagrams of $N_1$ and $N_1 \times 2$ are given in Figure 6.

\begin{figure}
\centering
\includegraphics[width=0.5	extwidth]{figure6.png}
\caption{$N_1$ and $N_1 \times 2$.}
\end{figure}

We denote by $H_3$ the class of finite OMLs all of whose blocks have precisely three atoms. These are the so-called cubic orthomodular lattices.

Lemma 2.9. (1) Let $T \in \text{Irr}(M_f)$ and let $e \in T$ with $e \neq 1$. Then $[0, e]$ is a modular OML.

(2) Let $L$ be a finite, irreducible OML such that $n(L) \geq 2^4$, where $n(L)$ denote the maximal cardinality of a block in $L$. Then $N_1$ is a sub-OML of $L$ or there exists $e \in L$, $e \neq 1$, such that $[0, e]$ is irreducible with $n([0, e]) \geq 2^4$.

(3) The only finite, irreducible, modular OMLs, different from $2$, are the $MO_n$ for $n \geq 2$.

(4) If $L \in H_3$, the modular sub-OMLs of $L$ are isomorphic to $2^i$, $1 \leq i \leq 3$, or $MO_n$, or $2 \times MO_n$.

Proof. (1) We may assume that $e \neq 0$. Let $C(e)$ be the sub-OML of $T$ formed of all the elements of $T$ commuting with $e$. Then $e$ is central in $C(e)$ and $C(e)$ is isomorphic to $[0, e] \times [0, e^\perp]$. Since $T$ is irreducible, $C(e)$ is a proper sub-OML of $T$; and since $T$ is minimal, $C(e)$ is modular and therefore $[0, e]$ is also modular.

(2) The proof of this result of G. Bruns and G. Kalmbach can be found in [3] and also in [9, p. 126].

(3) For the proof, see [9, p. 130].

(4) A modular sub-OML of $L$ has height $\leq 3$ and is a product of irreducible ones given in (3). □
Proposition 2.10. Every non-cubic irreducible element of $\mathcal{M}_f$ is isomorphic to $N_1$.

Proof. Let $T$ be an irreducible, minimal OML not isomorphic to $N_1$. The minimality of $T$ implies that $N_1$ is not a sub-OML of $T$. Assume there exists a block in $T$ with more than 8 elements. By part (2) of Lemma 2.9, there exists $e \in T$, $e \neq 1$, such that $[0,e]$ is irreducible with $n([0,e]) \geq 2^4$. By part (1) of the lemma, $[0,e]$ is modular, contradicting part (3) of the lemma because $n([0,e]) \geq 2^3$. Thus all the blocks of $T$ have 4 or 8 elements. Since $T$ is non-modular, some block $B$ of $T$ has 8 elements. If there is in $T$ a block with 4 elements, this block is in horizontal sum with $B$, and $N_1$ would be a sub-OML of $T$. Thus every block of $T$ has 8 elements.

The OMLs $N_2$ and $N_3$, given in Figure 2, are in $\text{Irr}(\mathcal{M}_f)$. So are the OMLs $N_4$ and $N_5$, given in Figure 7 by their Greechie diagrams; $N_5$ is also known as $G_{32}$.

![Figure 7. Examples of finite, irreducible and minimal OMLs.](image)

The following proposition is useful for finding elements of $\text{Irr}(\mathcal{M}_f)$. For an OML $L$, let $B$ denote the set of all the blocks of $L$ and recall that $A$ denotes the set of all atoms of $L$.

We denote by $\text{Is}(L)$ the class of all the OMLs isomorphic to the OML $L$.

Proposition 2.11. Each $T \in \text{Irr}(\mathcal{M}_f)$ satisfies the following properties:

(1) If $T \notin \text{Is}(N_1)$, then for every block $B$ of $T$ and every atom $a$ of $T$, we have $\delta(a,B) \leq 2$.

(2) Let $G = (A, E)$ be the Greechie diagram of $T$. If $T \notin \text{Is}(N_1)$ and $T \notin \text{Is}(N_2)$, then every partial sub-diagram of $G$ of $D_{16}$-type is a partial sub-diagram of a partial sub-diagram of pentagonal-type of $G$.

Proof. (1) If not, then there exists an atom $a$ and a block $B$ such that the sub-algebra generated by $\{a\} \cup B$ is $\{0,1,a,a^\perp\} \cup B$, which is isomorphic to $N_1$.

(2) Let $B_1, B_2, B_3$ the three blocks of $T$ corresponding to a sub-diagram of $G$ of $D_{16}$-type. Since $T$ is minimal and different from $N_2$, $N_2$ is not a sub-OML of $T$. 


Therefore there exist atoms \(a\) in \(B_1\) and \(b\) in \(B_3\), with \(a \neq b\), such that \(c = (a \lor b)\perp\) is an atom of \(T\) not in \(N_2\). It follows that the sets \(\{a, c, (a \lor c)\perp\}\) and \(\{b, c, (b \lor c)\perp\}\) generate blocks distinct from \(B_1, B_2, B_3\); call them \(B_4\) and \(B_5\), respectively. Then \(B_1, B_2, B_3, B_4, B_5\) are the blocks of a partial sub-diagram of pentagonal-type of \(G\).

\[\Box\]

**Proposition 2.12.** Let \(T\) be a finite, irreducible, non-modular OML in \(\mathcal{H}_3\) such that

\[\forall a \in A, \forall B \in B, \delta(a, B) \leq 2.\]

\(T\) is minimal if and only if \(\forall a \in A, \forall B \in B\) such that \(\delta(a, B) = 2\), the sub-OML of \(T\) generated by \(a\) and \(B\) is \(T\) itself.

**Proof.** By part (4) of Lemma 2.9, the modular sub-OMLs of \(T\) are isomorphic to \(2^i\) for \(1 \leq i \leq 3\), \(MO_n\) or \(2 \times MO_n\). It follows that necessarily a non-modular sub-OML of \(T\) contains a block \(B\) and an atom \(a\) such that \(\delta(a, B) \geq 2\). \[\Box\]

**Remark 2.13.** The characterization of minimal OMLs given in proposition 2.12 has been used to write a computer program for testing minimality of finite OMLs. Using it, we have verified that \(N_i, i = 6, 7, 8\) are in \(\text{Irr}(\mathcal{M}_f)\); these are given in Figure 8.

![Figure 8](image_url). Examples of finite, irreducible and minimal OMLs.

### 3. The covering property

We now turn our attention to a characterization of the finite, irreducible, minimal OMLs. They are precisely the finite irreducible OMLs such that \([T]\) covers \([MO_n]\) for some \(n\). We also show that the mapping \(T \rightarrow [T]\) sending a finite, irreducible, minimal OML to its equational class is one-to-one, so that non-isomorphic finite, irreducible, minimal OMLs generate different varieties. Eventually, the following
result will imply that there are infinite sets of incomparable varieties of OMLs of the form \([T]\).

Recall that, if \(T\) is an OML, then the equational class \([T]\) generated by \(T\) is the smallest class of OMLs containing \(T\) and closed under sub-algebras, products and quotients.

**Proposition 3.1.** (1) Let \(T\) be a finite, non-modular, irreducible OML. \(T\) is minimal iff there exists \(n \geq 2\) such that \([T]\) covers \([MO_n]\).

(2) If \(T\) and \(T'\) are non-isomorphic OMLs in \(\text{Irr}(\mathcal{M}_f)\) then the classes \([T]\) and \([T']\) are incomparable.

For the proof, we use the following result of B. Jónsson [8]: let \(K\) be a finite set of finite algebras of the same type, let \([K]\) be the equational class generated by \(K\). Assume that the congruence lattice of every algebra in \(K\) is distributive. Then every subdirectly irreducible algebra in \([K]\) is a homomorphic image of a sub-algebra of an algebra in \(K\).

We recall that the congruence lattice of an OML is distributive ([9, p. 83]) and that, for a finite OML, the properties simple, subdirectly irreducible, and irreducible are equivalent ([9, p. 79]).

**Proof of Proposition 3.1.** (1) Assume \(T\) is minimal. Denote \(S_1, \ldots, S_k\) the modular sub-OMLs of \(T\). Since \(T\) is not boolean, by result (2) of section 2, one \(S_i\) is \(MO_2\) or \(2 \times MO_2\). Every \(S_i\) is a finite product of OMLs isomorphic to \(2\) or some \(MO_m\) with \(m \geq 2\), by part (3) of Lemma 2.9. Let \(n\) be the greatest \(m\) such that \(MO_m\) occurs in the decomposition of some \(S_i\). Then \(S_1, \ldots, S_k = [MO_n]\). In order to prove that \([T]\) covers \([MO_n]\), let \(C\) be an equational class such that \([MO_n] \subset C \subset [T]\) with \(C \neq [T]\). In order to prove that \(C = [MO_n]\) it suffices to prove that, for any subdirectly irreducible \(A\) in \(C\), we have \(A \in [MO_n]\).

Let \(A\) be a subdirectly irreducible OML in \(C\), then \(A \in [T]\) and, by Jónsson’s result, \(A\) is a quotient of some sub-OML \(T'\) of \(T\). If \(T' \neq T\), then minimality of \(T\) implies \(T'\) modular, hence \(T' \in \{S_1, \ldots, S_k\}\) and \(A \in [MO_n]\). If \(T' = T\), as \(T\) is a simple OML, then \(A = T\), contradicting the fact that \(C \neq [T]\).

In order to prove the converse, assume that, for some \(n \geq 2\), \([T]\) covers \([MO_n]\), and assume that \(T\) is not minimal. Then there exists a proper, non-modular sub-OML \(T'\) of \(T\) and we have \([MO_n] \subset [MO_n, T'] \subset [T]\). The covering property implies \([MO_n, T'] = [T]\) and, by Jónsson’s result, \(T\) is a quotient of a sub-algebra of \(MO_n\) or \(T'\). We obtain a contradiction in both cases, either because \(T\) is not modular or because the cardinality of \(T'\) is less than the cardinality of \(T\).

(2) Assume, for instance, that \([T'] \subset [T]\). By Jónsson’s result \(T'\) is a quotient of a sub-OML \(A\) of \(T\). If \(A \neq T\), \(A\) is modular and \(T'\) is not modular. If \(A = T\), then \(T' = T\), since \(T\) is simple. Hence we have a contradiction in both cases. \(\square\)
Remarks 3.2. (1) It is easy to see from the Greechie diagrams that \([N_1], [N_2], [N_3]\) cover \([MO_2]\). By [3] or [9, p. 121], there is no other \(T \in \text{Irr}(M_f)\) satisfying this property.

(2) \([N_4], [N_5], [N_6], [N_7]\) cover \([MO_3]\).

(3) \([N_8]\) covers \([MO_4]\).

Proposition 3.3. For any \(n \geq 2\), there are, up to isomorphism, finitely many \(T \in \text{Irr}(M_f)\) such that \([T]\) covers \([MO_n]\).

Proof. Assume \([T]\) covers \([MO_n]\) with \(T \in \text{Irr}(M_f), T \neq N_1\). Let \(a\) be an atom of \(T\) and denote by \(m\) the number of blocks of \(T\) passing through \(a\). Then \(MO_m \times 2\) is a sub-OML of \(T\) and \(MO_m\) belongs to \([T]\). As \([T]\) covers \([MO_n]\) we have \(m \leq n\).

Since \(T \in H_3\) (Prop. 2.10), the number of atoms \(x\) of \(T\) such that \(\delta(a, x) = 1\) is \(2m \leq 2n\). Hence the number of atoms \(y\) of \(T\) such that \(\delta(a, y) = 2\) is \(\leq (2n)^2\). Thus the number of atoms \(z\) of \(T\) such that \(\delta(a, z) \leq 2\) is \(\leq 4n^2 + 2n\). Let \(B\) be a block of \(T\) and let \(a_1, a_2, a_3\) be the atoms of \(B\). From part (1) of Proposition 2.11, for each atom \(x\) of \(T\), we have \(\delta(x, a_1) \leq 2\) or \(\delta(x, a_2) \leq 2\) or \(\delta(x, a_3) \leq 2\); hence \(3(4n^2 + 2n)\) is an upper bound for the number of atoms of \(T\), and there are only finitely many \(T\)'s.

While Proposition 3.3 sheds light on some of the covers of \([MO_n]\), we note that it does not characterize all of them. We have no idea whether or not there are other covers of \([MO_n]\).

4. Minimal OMLs obtained from quadratic spaces

Eight finite, irreducible, minimal OMLs, \((N_i)_{1 \leq i \leq 8}\), were given in the earlier sections. We now extend this list by generating two infinite sequences of such lattices. These are obtained from quadratic spaces over finite fields. Both sequences are associated with the sequence of prime numbers.

This is the longest section of the paper. We break it up into three subsections, labeled 4.1–4.3. In turn, 4.1 has, after a short introduction, three parts labeled (a)–(c); 4.2 has six parts, labeled (a)–(f), a proposition summing up the most salient points discussed, followed by the four most tractable examples of the lattices discussed, in the case of characteristic not 2; finally, 4.3 discusses the case of characteristic 2.

4.1. The lattices \(L(E, \varphi)\) and \(T(E, \varphi)\). From a quadratic space \((E, \varphi)\) over a field \(K\), we construct an OML denoted \(T(E, \varphi)\). For infinitely many finite fields \(K\), \(T(E, \varphi)\) will provide an irreducible, finite, minimal OML.
(a) Let $K$ be any field and let $E$ be a 3-dimensional vector space over $K$. Let $\varphi: E \times E \to K$ be a non-singular bilinear form and $Q: E \to K$ the quadratic form associated to $\varphi$ \cite{1}.

When $\varphi(u,v) = 0$, we say that the vectors $u$ and $v$ are orthogonal (written $u \perp v$).

If $M$ is a subspace of $E$, denote by $M^\perp = \{ u \in E | \varphi(u,v) = 0, \forall v \in M \}$. Denote by $L(E,\varphi)$, or later simply $L$, the modular lattice of all the subspaces of $E$. $L(E,\varphi)$ is also a projective plane equipped with the polarity $M \mapsto M^\perp$; the points are the atoms $Ku$ and the lines the coatoms $(Ku)^\perp$ for $u \in E, u \neq 0$.

A nonzero vector $u \in E$ is called isotropic if $Q(u) = 0$; a nonzero subspace $M \in L(E,\varphi)$ is called isotropic if the restriction of $\varphi$ to $M$ is singular, i.e., $M \cap M^\perp \neq \{0\}$. Actually, the isotropic subspaces are the $K\omega$ and the $(K\omega)^\perp$ for $\omega$ isotropic vector in $E$. For an isotropic atom $K\omega$, we have $K\omega \subset (K\omega)^\perp$; so, in the general case, the polarity $M \mapsto M^\perp$ is not an orthocomplementation on the lattice $L(E,\varphi)$.

(b) Define $T(E,\varphi)$, or later simply $T$, to be the set of all non-isotropic subspaces of $E$. $T(E,\varphi)$ is not a sublattice of $L(E,\varphi)$, but it is a lattice for the order $\subseteq$ with the operations:

\[
M_1 \vee_T M_2 = \begin{cases} 
M_1 \vee_L M_2 & \text{if } M_1 \vee_L M_2 = M_1 + M_2 \text{ is not isotropic} \\
E & \text{if } M_1 \vee_L M_2 \text{ is isotropic}
\end{cases}
\]

\[
M_1 \wedge_T M_2 = \begin{cases} 
M_1 \wedge_L M_2 & \text{if } M_1 \wedge_L M_2 = M_1 \cap M_2 \text{ is not isotropic} \\
\{0\} & \text{if } M_1 \wedge_L M_2 \text{ is isotropic}
\end{cases}
\]

and $M \mapsto M^\perp$ is an orthocomplementation on $T(E,\varphi)$.

**Proposition 4.1.** $(T(E,\varphi), \subseteq, \perp)$ is an OML.

**Proof.** We have to prove that if $u$ and $v$ are nonzero and non-isotropic vectors of $E$ then $Ku \subset (Kv)^\perp \Rightarrow (Ku)^\perp \wedge_T (Kv)^\perp \neq \{0\}$. This is equivalent to $u \perp v \Rightarrow Ku + Kv$ is not isotropic.

Assume $u \perp v$ and $Ku + Kv$ is isotropic. Then there exists an isotropic vector $\omega$ such that $\omega \perp u$ and $\omega \perp v$. It follows that $u, \omega \in (Kv)^\perp \cap (K\omega)^\perp$. Since the subspace $(Kv)^\perp \cap (K\omega)^\perp$ is one dimensional, it follows that $u$ and $\omega$ are collinear, contradicting the fact that $\omega$ is isotropic and $u$ is not isotropic. \[\square\]

(c) In \cite{6} the authors prove that the OML $T(E,\varphi)$ is irreducible and they give a correspondence between some sub-fields of $K$ and some irreducible sub-OMLs of $T(E,\varphi)$. They also obtain the following results when $K$ is a finite field of characteristic $p$.

$T(E,\varphi)$ is, up to isomorphism, independant of the non-singular form $\varphi$; $T(E,\varphi)$ depends only on the field $K$. Moreover we have the following results:
If the characteristic of $K$ is $p \neq 2$, then $T(E, \varphi)$ is minimal if and only if $K$ is isomorphic to $\mathbb{F}_p$ with not only $p \neq 2$ but $p \neq 3$. When $K = \mathbb{F}_p$, we denote $T(E, \varphi)$ by $T_p$. Thus we have infinitely many minimal OMLs: $T_5, T_7, T_{11}, T_{13}, T_{17}, \ldots$.

If the characteristic of $K$ is 2, then $T(E, \varphi)$ is a horizontal sum of a 4-element boolean algebra with an OML, $T'$, and the following hold.

- For $K = \mathbb{F}_2$, $T(E, \varphi)$, denoted by $T_2$, is minimal; actually $T_2 = N_1$.
- For $K = \mathbb{F}_2^n$ with $n \geq 2$, we have $T'$ minimal if and only if $n$ is a prime number.
- For $K = \mathbb{F}_2^p$ with $p$ a prime, $T'$ is denoted by $T_2^p$. Thus we have infinitely many minimal OMLs: $T_2, T_4, T_8, T_{32}, \ldots$.

In order to describe the finite minimal OMLs $T_p$ and $T_{2p}$, let $K$ be a finite field of cardinality $q$ and let $E = K^3$ be the canonical 3-dimensional vector space over $K$. The non-singular bilinear form $\varphi: E \times E \to K$ and the quadratic form associated to $\varphi$, $Q: E \to K$, are defined, for $u = (x, y, z)$ and $u' = (x', y', z') \in E$, by $\varphi(u, u') = xx' + yy' + zz'$, $Q(u) = x^2 + y^2 + z^2$.

4.2. Description of the OML $T_p$ with $p$ a prime number, $p > 2$. We take $K = \mathbb{F}_p$ the finite field of cardinality $p$, where $p$ is a prime number different from 2.

As per our previous discussion, $L_p$ now denotes the modular lattice, $L(E, \varphi)$, of all subspaces of $E$, and $T_p$ now denotes the OML, $T(E, \varphi)$, of all non-isotropic subspaces of $E$.

(a) Note that the number of nonzero vectors of $E$ is $p^3 - 1$, and the number of atoms of $L$ is $\frac{p^3 - 1}{p - 1} = p^2 + p + 1$.

A nonzero vector $u = (x, y, z)$ is isotropic if $x^2 + y^2 + z^2 = 0$. In [1] we find that the number of isotropic vectors is $p^2 - 1$, so that the number of isotropic atoms of $L_p$ is $\frac{p^2 - 1}{p - 1} = p + 1$. Hence the number of atoms of $T_p$ is $(p^2 + p + 1) - (p + 1) = p^2$.

(b) As in Euclidean geometry, we can define a wedge-product on $E$, $(u, u') \mapsto u \wedge u'$, by setting, for $u = (x, y, z)$ and $u' = (x', y', z') \in E$,

$$u \wedge u' = (yz' - zy', zx' - zx', xy' - xy').$$

The wedge-product has the following properties:

1. $u \wedge u' = 0 \iff u$ and $u'$ are collinear $\iff Ku = Ku'$,
2. $u \wedge u' \perp u$ and $u \wedge u' \perp u'$,
3. $\varphi(u, u')^2 + Q(u \wedge u') = Q(u)Q(u')$.

By property (2), when $Ku \neq Ku'$, we obtain $Ku + Ku' = Ku \lor Ku' = [K(u \land u')]^\perp$.

(c) Let $Ku$ be an atom of $L_p$. Consider the plane $(Ku)^\perp$, and in this plane take two non-collinear vectors $v$ and $w$. Every atom in the plane $(Ku)^\perp$, other than $Kv$,
is uniquely represented by a vector $\lambda v + w$ with $\lambda \in K$. Moreover, we have,

$$\lambda v + w \text{ isotropic } \iff Q(\lambda v + w) = 0 \iff \lambda^2 Q(v) + 2\lambda \varphi(v, w) + Q(w) = 0.$$ 

If $v$ is non-isotropic, $Q(v) \neq 0$, we have a second degree equation with discriminant $\Delta = \varphi(v, w)^2 - Q(v)Q(w)$, and from property (3), $\Delta = -Q(v \wedge w)$. Noting that $v \wedge w$ and $u$ are collinear then, up to a square in $K$, we have $\Delta = -Q(u)$.

(1) If $Q(u) = 0$, there is just one isotropic atom in the plane $(Ku)^\perp$, and this atom is $Ku$.

(2) If $-Q(u)$ is a nonzero square in $K$, there are two isotropic atoms in the plane $(Ku)^\perp$.

(3) If $-Q(u)$ is not a square in $K$, there are no isotropic vectors in the plane $(Ku)^\perp$.

If $v$ is isotropic, the equation becomes $2\lambda \varphi(v, w) + Q(w) = 0$. In this case, we have $\varphi(v, w) \neq 0$; otherwise, we would have $Q(w) = 0$, and $v$ and $w$ would be non-collinear vectors in the one dimensional subspace $(Kv)^\perp \cap (Kw)^\perp$. Then the equation has one root $\lambda_0$, and there are two isotropic atoms in the plane $(Ku)^\perp$; they are $Kv$ and $K(\lambda_0 v + w)$. Actually, in this case, we have $\Delta = \varphi(v, w)^2$, and so $-Q(u)$ is a nonzero square in $K$. Thus, there are two isotropic atoms in the plane $(Ku)^\perp$.

(d) A plane $(Ku)^\perp$ in $L_p$ has $\frac{p^2 - 1}{p - 1} = p + 1$ atoms. Thus the results obtained in (c) can be rephrased as follows:

If $Ku$ is an isotropic atom of $L_p$, then $Ku$ is orthogonal to itself and to $p$ atoms of $T_p$.

If $Ku$ is an atom of $T_p$, there are two cases, according to whether $-Q(u)$ is a square or not a square in $K$.

(1) If $-Q(u)$ is a square, $Ku$ is orthogonal to two isotropic atoms of $L_p$ and orthogonal to $p - 1$ atoms of $T_p$, then the degree of $Ku$ in $T_p$ (i.e., the number of blocks of $T_p$ passing through $Ku$) is $\frac{p - 1}{2}$. We say, in this case, that $Ku$ is a weak atom of $T_p$.

(2) If $-Q(u)$ is not a square, $Ku$ is orthogonal to $p + 1$ atoms of $T_p$, then the degree of $Ku$ in $T_p$ is $\frac{p + 1}{2}$. We say, in this case, that $Ku$ is a strong atom of $T_p$.

In $L_p$ there are $p + 1$ isotropic atoms; each isotropic atom is orthogonal to $p$ weak atoms of $T_p$; and each weak atom of $T_p$ is orthogonal to two isotropic atoms of $L_p$. Hence the number of weak atoms is $\frac{p(p + 1)}{2}$. It follows that the number of strong atoms is $p^2 - \frac{p(p + 1)}{2} = p(p - 1)\frac{p - 1}{2}$.

(e) If $Ku, Kw$ are the 3 atoms of a block $B$ of $T_p$, then the matrix of $Q$ in the ordered basis $(u, v, w)$ [1] is
In the canonical basis of $E = K^3$, the matrix of $Q$ is the identity matrix. We know that the determinant of a matrix of a quadratic form in an orthogonal basis is, up to a square, independent of the orthogonal basis, and so $Q(u)Q(v)Q(w)$ is a square in $K$. Moreover, we know that in the finite field $K$ a product of 2 non-squares is a square. Hence, there are two kinds of blocks in the OML $T_p$; these are depicted in Figure 9, where the signs $+$ or $-$ mean that $Q(u)$ is a square or $Q(u)$ is a non-square, respectively.

\[
\begin{pmatrix}
Q(u) & 0 & 0 \\
0 & Q(v) & 0 \\
0 & 0 & Q(w)
\end{pmatrix}.
\]

**Figure 9.** The two kinds of blocks in $T_p$.

Recall that in the finite field $\mathbb{F}_p$ with $p \neq 2$ we have

$-1$ square $\Leftrightarrow p \equiv 1 \pmod{4}$, and $-1$ non-square $\Leftrightarrow p \equiv 3 \pmod{4}$.

Hence, for describing the blocks of $T_p$, we have to distinguish two cases.

1. If $p \equiv 1 \pmod{4}$

   $Ku$ weak atom $\Leftrightarrow -Q(u)$ square $\Leftrightarrow Q(u)$ square;

   The two kinds of blocks are given in Figure 10, where $w$ and $s$ mean weak and strong, respectively.

   \[
   \begin{array}{ccc}
   w & w & w \\
   \hline
   w & s & s
   \end{array}
   \]

   **Figure 10.** Blocks in $T_p$, case $p \equiv 1 \pmod{4}$.

2. If $p \equiv 3 \pmod{4}$

   $Ku$ weak atom $\Leftrightarrow -Q(u)$ square $\Leftrightarrow Q(u)$ non-square;

   the two kinds of blocks are given in Figure 11, with $w$ and $s$ meaning the same as before.

   \[
   \begin{array}{ccc}
   s & s & s \\
   \hline
   s & w & w
   \end{array}
   \]

   **Figure 11.** Blocks in $T_p$, case $p \equiv 3 \pmod{4}$.

Since each block of $T_p$ has 3 atoms, 3 times the number of the blocks equals the sum of all the degrees of the atoms. There are $\frac{p(p+1)}{2}$ weak atoms, each
with degree $\frac{p+1}{2}$; and there are $\frac{p(p-1)}{2}$ strong atoms, each with degree $\frac{p+1}{2}$; and hence the number of blocks of $T_p$ is

$$\frac{1}{3}\left\{ \frac{p+1}{2} \cdot \frac{p-1}{2} + \frac{p(p-1)}{2} \cdot \frac{p+1}{2} \right\} = \frac{p(p^2-1)}{6}.$$  

(f) By lemma 2.9, the modular sub-OMLs of $T_p$ that are not boolean are isomorphic to $MO_k$ or $2 \times MO_k$, for some $k$. By the proof of proposition 3.1, $[T_p]$ covers $[MO_n]$ in case $n$ is the greatest $k$ such that $MO_k$ or $2 \times MO_k$ is isomorphic to a sub-OML of $T_p$.

A sub-OML of $T_p$ isomorphic to $2 \times MO_k$ has all its blocks passing through a fixed atom. Thus the greatest $k$ is the degree of a strong atom, it is $p+1/2$.

A sub-OML of $T_p$ isomorphic to $MO_k$ has atoms $Ku_1, \ldots, Ku_k$ such that, for $i \neq j$, $Ku_i \lor_T Ku_j = E$; this is equivalent to: for $i \neq j$, $(Ku_i)^\perp \cap (Ku_j)^\perp = K\omega_{ij}$ where $\omega_{ij}$ is an isotropic vector. Since each plane $(Ku_i)^\perp$ contains only two isotropic atoms, for $k \geq 4$ the condition is equivalent to: there exists an isotropic vector $\omega$ such that whenever $1 \leq i < j \leq k$ we have $(Ku_i)^\perp \cap (Ku_j)^\perp = K\omega$. The greatest such $k$ is $k = p$, because $(Kw)^\perp$ contains exactly $p$ atoms of $T_p$.

Thus for $p \geq 4$, the greatest $k$ such that $MO_k$ or $MO_k \times 2$ is isomorphic to a sub-OML of $T_p$ is $\max(\frac{p+1}{2}, p) = p$, and so, $[T_p]$ covers $[MO_p]$.

The following proposition sums up some of the results obtained in 4.2.

**Proposition 4.2.** For each prime number $p \neq 2$, the OML $T_p$ has the following properties:

1. The number of atoms is $p^2$. There are $\frac{p(p+1)}{2}$ weak atoms with degree $\frac{p-1}{2}$ and there are $\frac{p(p-1)}{2}$ strong atoms with degree $\frac{p+1}{2}$.

2. Each block of $T_p$ has three atoms and the number of blocks is $\frac{p(p^2-1)}{6}$. There are two kinds of blocks:
   - If $p \equiv 1 \pmod{4}$, then each block has three weak atoms or it has one weak and two strong atoms.
   - If $p \equiv 3 \pmod{4}$, then each block has three strong atoms or it has one strong and two weak atoms.

3. For $p \geq 5$, $T_p$ is a minimal OML and $[T_p]$ covers $[MO_p]$.

**Remark 4.3.** We have defined $T_p$ for $p$ a prime number, but in the same manner one can define the OML $T_q$ when $q$ is the cardinality of a finite field, i.e., $q = p^n$, with $p$ a prime number different from 2. The results (1) and (2) of Proposition 4.2 remain valid if $p$ is replaced by $q$; however, the result (3) holds only if $p$ is a prime number other than 2.

In the OML $T_p$, it is possible to be more specific about the number of blocks of each type passing through an atom. For that, we give the following definition.
Definition 4.4. A block $B$ in $T_p$ is called “of type 1” if its three atoms are of the same kind (three weak atoms or three strong atoms); $B$ is called “of type 2” if its three atoms are not of the same kind (one weak and two strong or one strong and two weak).

Remark 4.5. From the description of the blocks, given in Proposition 4.2 or in Figures 10 and 11, it is easy to see that

1. if $p \equiv 1 \pmod{4}$, then all the blocks passing through a strong atom are blocks of type 2;
2. if $p \equiv 3 \pmod{4}$, then all the blocks passing through a weak atom are blocks of type 2.

In the two other cases, the answer is less evident, but one can prove that

1. when $p \equiv 1 \pmod{4}$, there are as many blocks of type 1 passing through a weak atom, as blocks of type 2;
2. when $p \equiv 3 \pmod{4}$, there are as many blocks of type 1 passing through a strong atom, as blocks of type 2.

It is easy to see that these results may be obtained from the following proposition.

Proposition 4.6. If $Ku$ is an atom of $T_p$, then there are as many strong atoms as weak atoms in the plane $(Ku)\perp$.

Proof. Note that each atom in the plane $(Ku)\perp$ belongs to exactly one block passing through the atom $Ku$.

Case (1): $p \equiv 1 \pmod{4}$ and $Ku$ is a strong atom. Every block $B$ passing through $Ku$ is of type 2, therefore $A \cap B \cap (Ku)\perp$ contains precisely 1 strong and 1 weak atom. Since $\{A \cap B \cap (Ku)\perp \mid B \in \mathcal{B}\}$ partitions $A \cap (Ku)\perp$, there are as many strong atoms as weak atoms in the plane $(Ku)\perp$.

Case (2): $p \equiv 3 \pmod{4}$ and $Ku$ is a weak atom. Every block passing through $Ku$ is again of type 2 and the result follows as in the previous case.

Case (3): $p \equiv 1 \pmod{4}$ and $Ku$ is a weak atom. As cited above in (c), in the plane $(Ku)\perp$ there are two isotropic atoms of $L_p$, $K\omega_1$ and $K\omega_2$, and the non-collinear vectors $\omega_1$ and $\omega_2$ provide a basis of the vector-space $(Ku)\perp$. Thus, each atom of $T_p$ in $(Ku)\perp$ is of the form $Kv$ with $v$ having the unique form $v = \lambda\omega_1 + \omega_2$ with $\lambda \in \mathbb{F}_p^* = \mathbb{F}_p - \{0\}$. We have $Q(v) = 2\lambda\varphi(\omega_1, \omega_2)$, with $\varphi(\omega_1, \omega_2) \neq 0$. The mapping $\lambda \rightarrow 2\lambda\varphi(\omega_1, \omega_2)$ is a bijection from $\mathbb{F}_p^*$ to itself, since $p > 2$. Since, in the finite field $\mathbb{F}_p$, there are as many nonzero squares as non-squares, we obtain that in $(Ku)\perp$ there are as many weak atoms ($Q(v)$ square) as strong atoms ($Q(v)$ non-square).

Case (4): $p \equiv 3 \pmod{4}$ and $Ku$ is a strong atom. Let $Ku$, $b$, and $c$ be the three atoms of a block in $(Ku)\perp$. From the description of the blocks given in Proposition
4.2 or in the figures 10 and 11, the atoms $b$ and $c$ are both strong or both weak. We consider the two cases.

Case (i). If $b$ and $c$ are strong atoms, then we have $b = Kv'$, $c = Kw'$, and $Q(v')$ and $Q(w')$ are squares. Then $Q(v') = \lambda'^2$ and $Q(w') = \mu'^2$, with $\lambda'$ and $\mu'$ in $F_p$. Setting $v = \lambda'^{-1}v'$ and $w = \mu'^{-1}w'$, we have $b = Kv$, $c = Kw$ and $Q(v) = Q(w) = 1$. If $Kx$ is an atom, different from $Kv$, in $( Ku)\perp$, as $\{v, w\}$ is an orthogonal basis of the vector-space $( Ku)\perp$, $x$ may be uniquely represented as $x = \lambda v + w$ with $\lambda \in F_p$ and $Q(x) = \lambda'^2Q(v) + Q(w) = \lambda'^2 + 1$. Now $Q(x)$ is a square $\iff \exists \mu \in F_p$ such that $\lambda'^2 + 1 = \mu^2$ $\iff \exists \mu$ such that $\mu^2 - \lambda'^2 = 1$. Moreover $\mu^2 - \lambda'^2 = (\mu - \lambda)(\mu + \lambda) = hk$, with $h = \mu - \lambda$ and $k = \mu + \lambda$, so that $\lambda = (k - h)/2$.

Then $hk = 1 \iff h = k^{-1} \iff \lambda = (k - k^{-1})/2$, with $k \in F_p^\ast$. Thus $Q(\lambda v + w)$ is a square $\iff \exists k \in F_p^\ast$ such that $\lambda = (k - k^{-1})/2$.

Now consider the function $f(k) = (k - k^{-1})/2$ from $F_p^\ast$ to $F_p$. We have $f(k) = f(k') \iff k - k^{-1} = k' - k'^{-1} \iff k - k' = k^{-1} - k'^{-1} \iff k - k' = (k'k)^{-1}(k' - k) \iff k' = k$ or $k' = -k^{-1}$. Note that $k \neq -k^{-1}$, because $-1$ is not a square in $F_p$. Thus, for $k \in F_p^\ast$, $k$ and $-k^{-1}$ are distinct and give the only value which $f$ maps to $f(k)$.

Therefore the number of $\lambda = f(k)$ for some $k$ is $(p - 1)/2$, so that the number of strong atoms $Kx$ distinct from $Kv$ is $(p - 1)/2$ and the number of strong atoms in $( Ku)\perp$ is $1 + (p - 1)/2 = (p + 1)/2$. Since the number of atoms in $( Ku)\perp$ is $p + 1$, the number of weak atoms in $( Ku)\perp$ is also $(p + 1)/2$.

Case (ii). If $b$ and $c$ are weak atoms, then we have $b = Kv'$, $c = Kw'$, and $Q(v')$ and $Q(w')$ are not squares. Since $-1$ is not a square in $F_p$, the set of the non-squares of $F_p$ is $\{-\lambda^2 | \lambda \in F_p\}$. Therefore there exist $\lambda, \mu$ with $Q(v') = -\lambda^2$ and $Q(w') = -\mu^2$. As in the previous case, it is possible to find vectors $v$ and $w$ such that $b = Kv$, $c = Kw$ and $Q(v) = Q(w) = -1$. If $Kx$ is an atom, different from $Kv$, in $( Ku)\perp$, we have $x = \lambda v + w$ and $Q(x) = -(\lambda^2 + 1)$.

Thus $Q(x)$ is not a square $\iff \exists \mu$ such that $-(\lambda^2 + 1) = -\mu^2$ $\iff \exists \mu$ such that $\lambda^2 + 1 = \mu^2$. Then, as in the previous discussion, we obtain that the number of weak atoms in $( Ku)\perp$ is $(p + 1)/2$, so that the number of strong atoms in $( Ku)\perp$ is also $(p + 1)/2$.

Before proceeding to the case of characteristic 2, we consider the OMLs $T_p$ presented by the four smallest odd primes. Note that only the latter three of these OMLs are minimal.

**Example:** $p = 3$, $K = F_3 = \{0, 1, -1\}$.

The atoms of $T_3$ are the $Ku$ with $u = (x, y, z)$ such that $x^2 + y^2 + z^2 \neq 0$.

We search the vectors $u$, up to collinearity, to obtain exactly one representative for each atom.

- If $x = 0$, $y = 0$, $z = 1$, then $u = (0, 0, 1)$. 

• If \( x = 0, \ y = 1, \ z^2 \neq -1 \), then \( z = 0, \ 1 \) or \(-1\), and \( u = (0,1,0), \ (0,1,1) \) or \((0,1,-1)\).
• If \( x = 1, \ y = 0, \ z^2 \neq -1 \), then \( u = (1,0,0), \ (1,0,1) \) or \((1,0,-1)\).
• If \( x = 1, \ y = 1, \ z^2 \neq -1 \), then \( z = 0 \) and \( u = (1,1,0)\).
• If \( x = 1, \ y = -1, \ z^2 \neq -1 \), then \( u = (1,-1,0)\).

We denote the 9 vectors representing the 9 atoms of \( T_3 \) as follows:

\[
\begin{align*}
v_1 &= (1,0,0) & v_2 &= (0,1,0) & v_3 &= (0,0,1) & v_4 &= (0,1,1) & v_5 &= (0,1,-1) \\
v_6 &= (1,0,1) & v_7 &= (1,0,-1) & v_8 &= (1,1,0) & v_9 &= (1,-1,0).
\end{align*}
\]

Denote by \( i \) the atom represented by the vector \( v_i \). By surveying the atoms orthogonal to each atom, we identify the blocks of \( T_3 \). The Greechie diagram of \( T_3 \) is given in Figure 12.

From the diagram, we remark that \( D_{16} \) (also called \( N_2 \) herein) is a sub-OML of \( T_3 \), so that \( T_3 \) is not minimal.

The method, given in the case \( p = 3 \), can be programmed. For any prime number \( p \), the computer program lists all the \( p^2 \) atoms and all the blocks of the OML \( T_p \).

**Example 4.7.** \( p = 5, \ K = \mathbb{F}_5 = \{0,1,2,-1,-2\} \).

The 25 atoms of \( T_5 \) are represented by the following vectors:

\[
\begin{align*}
v_1 &= (1,0,0) & v_2 &= (0,1,0) & v_3 &= (0,0,1) & v_4 &= (0,1,1) \\
v_5 &= (0,1,-1) & v_6 &= (1,0,1) & v_7 &= (1,0,-1) & v_8 &= (1,1,0) \\
v_9 &= (1,-1,0) & v_{10} &= (1,2,-2) & v_{11} &= (1,1,-1) & v_{12} &= (1,-2,2) \\
v_{13} &= (1,-1,1) & v_{14} &= (1,-1,-1) & v_{15} &= (1,2,-1) & v_{16} &= (1,1,1) \\
v_{17} &= (1,2,2) & v_{18} &= (1,1,2) & v_{19} &= (1,-2,-2) & v_{20} &= (1,2,1) \\
v_{21} &= (1,-2,1) & v_{22} &= (1,-1,-2) & v_{23} &= (1,-1,2) & v_{24} &= (1,1,-2) \\
v_{25} &= (1,-2,-1)
\end{align*}
\]

The Greechie diagram of \( T_5 \) is given in Figure 13.

\([T_5]\) covers \([MO_5]\); for instance, the atoms of a sub-OML of \( T_5 \) isomorphic to \( MO_5 \) are given by 2, 10, 19, 22 and 24.

**Example 4.8.** \( p = 7 \).

A suitable encoding of the 49 atoms of \( T_7 \) is presented in Table 1. With this numbering we can generate a diagram for the structure of the 56 blocks of \( T_7 \) as
follows. By the discussion given in 4.2, part (e), the blocks of $T_7$ are of two types, those with three strong atoms and those with one strong atom and two weak atoms. This allows us to separate the usual Greechie diagram into two diagrams, making the presentation relatively readable; the vertices of the first diagram are the strong atoms and the vertices of the second are the weak atoms. Figure 14 is the diagram of the strong atoms; it is an ordinary Greechie diagram.

Figure 15 is a graph representing orthogonality of the weak atoms. The vertices are labeled by the weak atoms, but there is more information in that the edges are labeled by the strong atom which, when added to the orthogonal pair of weak atoms, yields the three atoms of a block of the second type. Juxtaposing these two figures, so interpreted, provides the usual Greechie diagram for $T_7$. Note that each strong atom is in 4 blocks, 2 of type 1 and 2 of type 2, whereas each weak atom is in 3 blocks of type 2.

The atoms 4, 13, 22, 24, 30, 38 and 45 generate a sub-OML of $T_7$ showing that $[T_7]$ covers $[MO_7]$.

**Example 4.9.** $p = 11$. The Greechie diagram for $T_{11}$ has 121 atoms and 220 blocks. (The encoding is defined in Table 2.) As with the presentation of $T_7$, we present it by presenting first the Greechie diagram of the strong atoms, then presenting the orthogonality graph of the weak atoms, with the edges of the latter labeled by the
strong atoms. However, in order to increase readability, we have split the second graph into two figures. Thus Figure 16 represents the Greechie diagram of the strong atoms; in it, every atom is strong and each block consists of 3 strong atoms. Figures 17 and 18, taken together (with one overlaid upon the other), represent the orthogonality graph of the weak atoms; the labels on an edge indicate the strong atom which is orthogonal to the vertices labeling the edge. (Note that only some of the edges are labeled in Figure 17, and these are precisely the edges not labeled in Figure 18.)

The atoms 3, 16, 27, 38, 49, 70, 82, 92, 93, 103 and 115 generate a sub-OML of \( T_{11} \) showing that \( [T_{11}] \) covers \( [MO_{11}] \).

The encoding for the 49 atoms of \( T_7 \) is given in Table 1.

The encoding for the 121 atoms of \( T_{11} \) is given in Table 2.

We have finished our discussion of odd characteristic and now proceed to the alternative.

### 4.3. Description of \( T \) in case of characteristic 2

We take \( K = \mathbb{F}_q \) with \( q = 2^p \), where \( p = 1 \) or \( p \) is a prime number. Recall the notations \( E = K^3 \), \( \varphi(u, u') = xx' + yy' + zz' \), \( Q(u) = x^2 + y^2 + z^2 \). \( L \) is the modular lattice of all the subspaces of \( E \), and \( T \) is the OML of all the non-isotropic subspaces of \( E \).
Throughout this section, let $u_0 = (1, 1, 1)$. For $u = (x, y, z) \in E$, we have

\[ Q(u) = 0 \iff x^2 + y^2 + z^2 = 0 \iff (x + y + z)^2 = 0 \] (characteristic of $K = 2$)

\[ \iff x + y + z = 0 \iff u \in (Ku_0)^\perp \]

In the plane $(Ku_0)^\perp$ there are $q^2 - 1$ nonzero vectors, so that in $L$ there are $\frac{q^2 - 1}{q - 1} = q + 1$ isotropic atoms. In $E$ there are $q^3 - 1$ nonzero vectors, so that in $L$ there are $q^2 + q + 1$ atoms and in $T$ there are $q^2 + 1$ atoms (note that $(Ku_0)^\perp$ is an atom in $T$). We argue that $T$ is a horizontal sum one of whose summands is the 4-element boolean algebra $T_0 = \{\{0\}, E, Ku_0, (Ku_0)^\perp\}$. If $Ku$ is an atom of $T$, $Ku \neq Ku_0$, then we have $Ku \lor L Ku_0 = (K\omega)^\perp$ with $\omega$ a nonzero vector in $(Ku_0)^\perp$. Then $\omega$ is isotropic and $Ku \lor T Ku_0 = E$. If $(Kv)^\perp$ is a coatom of $T$, since $v$ is not isotropic, $v \notin (Ku_0)^\perp$, then $Ku_0$ is not an atom in $(Kv)^\perp$ and $(Kv)^\perp \lor T Ku_0 = E$. Thus $T$ is the horizontal sum of $T_0$ and the OML $T' = T - \{Ku_0, (Ku_0)^\perp\}$. 
Example 4.10. $q = 2$, $K = \mathbb{F}_2 = \{0, 1\}$.

The atoms of $T$ are represented by the vectors $u = (x, y, z)$ such that $x + y + z \neq 0$. We obtain $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$. The Greechie diagram of $T$ is given in Figure 19.

We denote this $T$ by $T_2$. Actually $T_2 = N_1$; it is a minimal OML. For $q = 2^p$ with $p$ a prime number, $T'$ is denoted by $T_q$, and from [6], $T_q$ is a minimal OML. Since the number of atoms of $T$ is $q^2 + 1$, the number of atoms of $T_q$ is $q^2 - 1$.

Let $Ku$ be an atom of $T_q$. The plane $(Ku)^\perp$ contains $q + 1$ atoms of $L$. Since $(Ku)^\perp \cap (Ku_0)^\perp$ is one-dimensional, there is only one isotropic atom in $(Ku)^\perp$, so $(Ku)^\perp$ contains $q$ atoms of $T_q$. These $q$ atoms are pairwise orthogonal, so that the number of blocks of $T_q$ passing through $Ku$ (that is, the degree of $Ku$) is $q^2 / 2$. Thus,
the number of blocks of $T_q$ is

$$\frac{1}{3} \left[ \frac{q}{2} (q^2 - 1) \right] = \frac{q(q^2 - 1)}{6}.$$

**Example 4.11.** $q = 4$, $K = F_4 = \{0, 1, \alpha, \alpha^2\}$ with $\alpha^2 = \alpha + 1$.

The 15 atoms of $T_4$ are represented by the vectors in Table 3. The Greechie diagram of $T_4$, given in Figure 20, is here represented in two different ways. We remark that $T_4 = N_5 = G_{32}$ and $[T4]$ covers $[MO_3]$. For instance, one set of atoms

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**Figure 17.** $p=11$ weak atom graph (with some of the edges labeled).
Figure 18. $p=11$ weak atom graph (with the rest of the edges labeled).

Table 1

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generating a sub-OML isomorphic to $MO_3$ is $\{5,6,9\}$. These atoms are orthogonal to the isotropic vector $w = (\alpha, \alpha^2, 1)$. 

Table 2

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Table 3

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As $\frac{q^2}{2}$ is the degree of each atom of $T_q$, the greatest $k$ such that $MO_k \times 2$ is a sub-OML of $T_q$ is $\frac{q}{2}$.

If $Ku_1, \ldots, Ku_k$ are the atoms of a sub-OML of $T_q$ isomorphic to $MO_k$, necessarily we have that for all $i$ and $j$ with $i \neq j$, $(Ku_i)^\perp \cap (Ku_j)^\perp$ is an isotropic atom. Since each $(Ku_i)^\perp$ contains only one isotropic atom, the previous condition is equivalent to the following: there exists an isotropic vector $\omega$ such that $\forall i$, $1 \leq i \leq k$, $Ku_i \subset (K\omega)^\perp$. For each isotropic vector $\omega$, $(K\omega)^\perp$ contains $q + 1$ atoms of $L$, one is $K\omega$, another one is $Ku_0$, and the other $q - 1$ are atoms of $T_q$. Then, we obtain that $q - 1$ is the greatest $k$ such that $MO_k$ is isomorphic to a sub-OML of $T_q$. Since $\max(\frac{q^2}{2}, q - 1) = q - 1$, we obtain, from the proof of the proposition 3.1, that $[T_q]$ covers $[MO_{q-1}]$.

The following proposition sums up the main results obtained in 4.3.

**Proposition 4.12.** For $q = 2^p$, with $p$ a prime number, the minimal OML $T_q$ has the following properties.

1. The number of atoms is $q^2 - 1$ and each atom has degree $\frac{q}{2}$.
2. Each block has three atoms and the number of blocks is $\frac{2(q^2-1)}{p}$.
3. $[T_q]$ covers $[MO_{q-1}]$. 

5. Other finite minimal OMLs

5.1. Adding a prong. In section 4, we found two infinite sequences of finite, irreducible, minimal OMLs from quadratic spaces:

1. \( T_p \) for every prime number \( p \neq 2, 3 \); and
2. \( T_q \) for every \( q = 2^p \) with \( p = 1 \) or \( p \) a prime number.

They are, however, not the only finite irreducible minimal MOLs. If we consider the examples given in section 2, then \( N_i \) for \( 1 \leq i \leq 8 \), we see that \( N_1 = T_2 \), \( N_5 = T_4 \), and \( N_2 \) is obtained from \( T_3 \) by deleting a block. But the other \( N_i \) for \( i \neq 1, 2, 5 \) do not seem to be linked to quadratic spaces.

We remark that we obtain \( N_6 \) from \( N_5 \) by adding the diagram of \( 2 \times MO_3 \) pasted by 3 atoms; we obtain \( T_8 \) from \( T_7 \) in the same manner. We obtain \( N_3 \) from \( N_2 \) by adding the diagram of \( 2 \times MO_2 \) pasted by 2 atoms; \( N_4 \) is obtained from \( N_3 \) in the same manner.

More generally, for \( n \geq 2 \), the act of pasting the diagram of \( 2 \times MO_n \) to that of a finite, irreducible, minimal OML to obtain an other one, is called adding a prong. One is depicted in Figure 21. The following proposition shows that the only finite, irreducible, minimal OMLs, obtained from another such OML by adding a prong, are those that we already know.

**Proposition 5.1.** The only \( T \in \text{Irr}(M_f) \), obtained from another OML \( T' \in \text{Irr}(M_f) \) by adding a prong, are \( N_3 \), \( N_4 \), \( N_6 \), and \( N_8 \). These are obtained from \( N_2 \), \( N_3 \), \( N_5 \), and \( N_7 \), respectively.

**Proof.** Let \( T' \in \text{Irr}(M_f) \) and assume that there exists \( T \in \text{Irr}(M_f) \) obtained from \( T' \) by adding a prong \( 2 \times MO_n \). Denote \( x_1, \ldots, x_n \) the atoms common to \( T' \) and the prong, \( y_1, \ldots, y_n \) the free atoms of the prong (free means that the degree is 1, i.e., \( \text{deg}(y_i) = 1 \)), and \( a \) the atom of the prong with \( \text{deg}(a) = n \).

![Figure 21. The prong of T.](image)

Since there are no triangles and no squares in the Greechie diagram of \( T \), we have, in \( T' \), \( \forall i \neq j \), with \( i \neq j \), \( \delta(x_i, x_j) = 3 \) (see [7]). Let \( B \) be a block of \( T' \) not
passing through any \( x_i \). Since \( T \in \text{Irr}(M_f) \), by Proposition 2.11, we have, in \( T \), \( \forall i \ \delta(y_i, B) = 2 \) and then we have in \( T' \), \( \forall i \ \delta(x_i, B) = 1 \). Since two different \( x_i \) cannot be linked to the same atom of \( B \), it follows that \( n = 2 \) or \( n = 3 \).

Case (1): \( n = 2 \). We have three kinds of blocks in \( T' \): blocks passing through \( x_1 \), blocks passing through \( x_2 \), and the other blocks \( B \) which satisfy \( \delta(x_1, B) = \delta(x_2, B) = 1 \). A block like \( B \) is linked to one block passing through \( x_1 \), is linked to one block passing through \( x_2 \), and its third atom \( t \) is free. Since, by Proposition 2.11, \( \delta(t, C) \leq 2 \) for any other block \( C \) like \( B \), there exists at most 2 blocks like \( B \). So we obtain only \( T' = N_2 \) or \( T' = N_3 \), which are given in Figure 22.

![Figure 22. \( N_2 \) and \( N_3 \).](image)

Case (2): \( n = 3 \). There are 4 kinds of blocks in \( T' \); these are given in Figure 23. The 3 former kinds are blocks passing through \( x_1 \), \( x_2 \) or \( x_3 \). The blocks of the fourth kind are blocks \( B \) which satisfy \( \delta(x_1, B) = \delta(x_2, B) = \delta(x_3, B) = 1 \). A block like \( B \) is linked at \( a_1 \) to one block passing through \( x_1 \), is linked at \( a_2 \) to one block passing through \( x_2 \), and is linked at \( a_3 \) to one block passing through \( x_3 \). From an atom, like \( a_1 \), there passes at most one block like \( B \), because there are no triangles and no squares in \( T' \).

![Figure 23. \( T' \).](image)

Fix an \( i \), say \( i = 1 \). If \( \text{deg}(x_1) = 1 \), the only possibilities for \( T' \) are given in Figure 24; the first is the non-minimal OML \( T' = T_3 \), and the second is also not minimal because it contains \( N_1 \) as a sub-OML.

Thus we can assume that \( \forall i \ \text{deg}(x_i) \geq 2 \). In this case, from each atom \( y \) of \( T' \) different from \( x_1 \), \( x_2 \), and \( x_3 \), there passes one block like \( B \), because \( \delta(y, C) \leq 2 \) for every block \( C \) in \( T' \). So, from each atom of \( T' \) different from \( x_1 \), \( x_2 \), and \( x_3 \),
Figure 24. Possibilities for $T'$, when $\text{deg}(x_1) = 1$.

Figure 25. $T'$, when $\text{deg}(x_i) \geq 2$.

deg(x_1) = \text{deg}(x_2) = \text{deg}(x_3)$.

If $\text{deg}(x_1) = 2$, we obtain only $T' = N_5$, given in Figure 26. If $\text{deg}(x_1) = 3$,

we obtain only $T' = N_7$, given in Figure 27. In this example, we can verify the property given in part (2) of Proposition 2.11. $B, C, D$ are the blocks of a $D_{16}$-type partial diagram and this diagram is a partial diagram of a pentagonal diagram, one side of the pentagon is the block $G$ and one vertex is the atom $z$. $B, C, E$ are the blocks of a $D_{16}$-type partial diagram and this diagram is a partial diagram of a pentagonal diagram, one side of the pentagon is the block $F$ and one vertex is the atom $y$. Since there passes only one block of the fourth kind through $y$ and $z$, there is no other possibility for another block, like $D$ or $E$, passing through $x_1$. This shows that the case $\text{deg}(x_1) \geq 4$ is impossible. \qed
5.2. Merger. (a) We now describe a type of double pronged procedure. Refer to Figure 21. In this procedure, instead of one OML, we have two OMLs; in one the atoms $x_i$ are embedded, and in the other, the atoms $y_i$ are embedded. We illustrate with an example. Take two copies of $N_3$, take 5 new atoms and 10 new blocks passing pairwise through these atoms for linking each atom of one copy to an atom of the other copy, taking care not to introduce either triangles or squares. In this way, we obtain a minimal OML, given in Figure 28. Actually this minimal OML is isomorphic to $T_5$.

(b) Another example is the minimal OML, called $N_{10}$ and given in Figure 30, obtained by the merger of 2 copies of $N_5$; in this case we use 5 new atoms and
15 new blocks. $N_{10}$ is interesting because it has 35 atoms and 35 blocks and it is self-dual; this means that there exists a duality between atoms and blocks, a mapping $\psi$ of points to lines and lines to points such that a point $\chi$ is on a line $\mathcal{L}$ iff the point $\psi(\mathcal{L})$ is on the line $\psi(\chi)$. These are the only known examples of minimal OMLs obtained by a merger of other minimal OMLs.

**Conclusion.** The exclusion problem for the classes $\mathcal{MOL} \subset \mathcal{OML}$ led us to the study of minimal OMLs. To obtain all the finite minimal OMLs, it is sufficient to know all the irreducible ones, because the reducible ones are of the form $T \times 2$ with $T$ an irreducible minimal OML. In section 4, we found two infinite sequences of finite, irreducible minimal OMLs, from quadratic spaces. In section 5, we studied
two ways to obtain new finite minimal OMLs. We proved that we have already discovered all the possible irreducible minimal OMLs obtained by adding a prong to another such OML. The merger operation gave us two new finite irreducible minimal OMLs. An open question is: Are there other finite, irreducible minimal OMLs?

We suggest to the reader the interesting question of how to characterize the minimal OMLs that arise from linear spaces.

REFERENCES


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