Order (2008) 25:55–68
DOI 10.1007/s11083-008-9077-5

Order Dimension of Orthomodular Amalgamations Over Trees

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Received: 10 September 2007 / Accepted: 26 February 2008 /
Published online: 29 March 2008

Abstract We consider the amalgamation of bounded involution posets over a strictly directed graph as applied to orthomodular lattices, orthomodular posets or orthoalgebras. In the finite setting, we show that the order dimension of the amalgamation does not exceed that of the amalgamated structures by more than one. We also present conditions under which equality obtains.

Keywords Amalgamation · Dimension · Involution poset · Bounded poset · Orthomodular lattice · Orthomodular poset · Orthoalgebra · Graph

Mathematics Subject Classifications (2000) 03G12 · 06A06 · 06A07 · 06C15 · 81P10

1 Introduction

Haviar and Hrnčiar [3] calculated the order dimension of certain atomic amalgams, the so-called loops. These atomic amalgams consist of finite Boolean algebras, called *blocks*, pasted together in such a way that a pair of blocks intersect either trivially in $\{\mathbf{0}, \mathbf{1}\}$ or the intersection consists of $\{\mathbf{0}, \mathbf{1}, a, a'\}$, where a is an atom of both blocks. They continued their work in [4] and calculated the order dimension of atomic amalgams of Boolean algebras that contain no loops as a subposet.

Boolean algebras, orthomodular lattices, orthomodular posets, and orthoalgebras are orthomodular structures listed in an increasing order of abstraction. See [5] for the appropriate definitions. In this paper, an amalgamation of certain bounded

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involution posets over strictly directed graphs is defined. It is shown that using this amalgamation on any of these orthomodular structures gives an orthoposet of the same type. We generalize the above results by proving that, when the graph is a tree and the orthoposets are orthomodular, the dimension of the amalgamation does not exceed the largest dimension of the amalgamated posets by more than 1. Furthermore, we introduce the notion of a tailored family of cycle-free sets and prove that, in the finite setting, when each of the amalgamated posets admits a tailored family, the dimension of the amalgamation equals the largest of the dimensions of the amalgamated posets.

2 Amalgamation over Strictly Directed Graphs

In this section we review the construction given in [1] of the amalgamation of orthoalgebras, orthomodular posets (OMPs) and orthomodular lattices over graphs. Let $G := (V, E)$ be a *strictly directed graph*, that is, G is a pair of disjoint sets (V, E) such that the set of *vertices* $V \neq \emptyset$, the set of *edges* $E \subseteq (V \times V)\backslash\Delta$, where $\Delta := \{(u, u) \mid u \in V\}$, and such that $(u, v) \in E$ implies $(v, u) \notin E$. Define $E^{-1} := \{(v, u) | (u, v) \in E\}$. For $\alpha = (u, v)$, define $\pi_1(\alpha) = u$ and $\pi_2(\alpha) = v$. A sequence of distinct vertices $v_0 v_1 v_2 \cdots v_n$, $n \geq 1$, such that $(v_i, v_{i+1}) \in E \cup E^{-1}$, is called a *path in G from u to v of length n*. In discussing a path in G we ignore the direction of the arrows, ignoring the usual convention. A path $v_0 v_1 v_2 \cdots v_n$ in Q with $n \geq 3$, and $(v_0, v_n) \in E \cup E^{-1}$ is called a *cycle*. If any two distinct vertices in G are joined by a path, then G is said to be *connected*. *All graphs considered in this paper are strictly directed connected graphs*.

A *bounded involution poset*, $\mathbf{Q} := (Q, \leq, ', \mathbf{0}, \mathbf{1})$, is a poset $(Q, \leq)$ together with a unary mapping $' : Q \to Q$ with (i) $x'' = x$, (ii) if $x \leq y$ then $y' \leq x'$ and (iii) Q contains a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$. We follow the usual convention in referring to Q in place of $\mathbf{Q}$. The elements of Q that are immediately above $\mathbf{0}$ are called *atoms* of Q. The set of all atoms of Q is denoted by $A(Q)$. For $x \in Q$, we define $x\!\uparrow := \{y \in Q \mid x \leq y\}$, and $x\!\downarrow := \{y \in Q \mid y \leq x\}$. For $P \subseteq Q$, $P' := \{x' \mid x \in P\}$. An atom a is *isolated* if $a\!\uparrow \cup\, a\!\downarrow = \{\mathbf{0}, a, \mathbf{1}\}$. Let $A^*(Q) := \{a \in A(Q) \mid a \text{ is not isolated}\}$.

A *pointed involution poset* $((L, \leq, '), (\iota, \tau))$, or simply $(L, (\iota, \tau))$ or L, is a bounded involution poset L with a distinguished ordered pair of distinct non-isolated atoms $(\iota, \tau) \in A^*(L) \times A^*(L)$; ι and τ are called the *initial* and *terminal* points of L, respectively.

It is shown in [1] that if $\{(L_\alpha, (\iota_\alpha, \tau_\alpha)) \mid \alpha \in E\}$ is a family of disjoint pointed involution posets indexed by the edges of the directed graph $G = (V, E)$, and if $L_\circ := \bigcup_{\alpha \in E} L_\alpha$, then we can define a mapping $\rho : E \to A^*(L_\circ) \times A^*(L_\circ) \subseteq L_\circ \times L_\circ$ by $\rho(\alpha) := (\iota_\alpha, \tau_\alpha)$ for every $\alpha \in E$, where $A^*(L_\circ) := \cup_{\alpha \in E} A^*(L_\alpha)$. We think of ρ as identifying the initial point of α with $\iota_\alpha \in L_\circ$ and the terminal point of α with $\tau_\alpha \in L_\circ$. Since the L_α's are disjoint, ρ is a one-to-one function. We now use ρ to make the corresponding identifications in $L_\circ$, that is, if $\pi_1(\alpha) = \pi_1(\beta)$ then ι_α is identified with ι_β, and so on. Technically, we construct an involution poset L as follows: Let $R_0 := \{(\mathbf{0}_\alpha, \mathbf{0}_\beta) \mid \text{for every } \alpha, \beta \in E\}$, $R_1 := \{(\tau_\alpha, \iota_\beta) \in L_\circ \times L_\circ \mid \pi_2(\alpha) = \pi_1(\beta)\}$, $R_2 := \{(\tau_\alpha, \tau_\beta) \in L_\circ \times L_\circ | \pi_2(\alpha) = \pi_2(\beta)\}$, $R_3 := \{(\iota_\alpha, \iota_\beta) \in L_\circ \times L_\circ | \pi_1(\alpha) = \pi_1(\beta)\}$, and $R_4 := \{(\iota_\alpha, \tau_\beta) \in L_\circ \times L_\circ \mid \pi_1(\alpha) = \pi_2(\beta)\}$. Define a relation $\equiv$ on $L_\circ \times L_\circ$ by $\equiv := \Delta \cup \bigcup_{i=0}^{4} (R_i \cup R_i' \cup R_i^{-1} \cup (R_i^{-1})')$, where $R_i' := \{(x', y') \mid (y, x) \in R_i\}$, and $R_i^{-1} := \{(x, y) \mid (y, x) \in R_i\}$. A tedious but elementary argument shows that $\equiv$ is an equivalence relation on $L_\circ \times L_\circ$. We denote the equivalence class of x by $[x]$. Let $L := L_\circ/\!\equiv$. Define $' : L \to L$ by $[x]' = [x^{'\alpha}]$ if $x \in L_\alpha$. It is not hard to check that $'$ is well-defined, and that $[x] \equiv [y]$ iff $[x]' \equiv [y]'$. *For convenience we write x' instead of $x^{'\alpha}$*, where $x \in L_\alpha$. Define a relation $\leq$ on L as follows: for $[x], [y] \in L$, write $[x] \leq [y]$ if there exists $\alpha \in E$ and $x_\alpha \equiv x,\ y_\alpha \equiv y$ with $x_\alpha \leq_{L_\alpha} y_\alpha$. Clearly $\leq$ is a partial order on L. We call L the *atomic amalgamation* of $L_\alpha,\ \alpha \in E$, over the strictly directed graph G via ρ, and write $(L; L_\alpha, G, \rho)$ to indicate that $L := L_\circ/\equiv$, where $L_\circ$ and $\equiv$ are defined as above using $L_\alpha,\ G$ and ρ. For $\alpha, \beta \in E$, we write $\alpha \sim \beta$ when, in L, $\iota_\alpha \equiv \iota_\beta$, $\iota_\alpha \equiv \tau_\beta$ or $\tau_\alpha \equiv \iota_\beta$. The mapping ρ induces a mapping $\rho_1 : V \to L$ defined by $\rho_1(v) := [\pi_1(\rho(\alpha))]$ for any $\alpha \in E$ with $\pi_1(\alpha) = v$. Define $r_0 := \rho_1(r(T))$.

For $u, v \in V$, the *distance* $d(u, v)$ is the length of a shortest path joining them, if such a path exists; otherwise $d(u, v) = \infty$; of course, $d(u, u) := 0$. A *tree* is a connected strictly directed graph with no cycles. A *rooted tree* T is a tree with a distinguished vertex, its root $r(T)$, with $d(r(T), \pi_1(\alpha)) \leq d(r(T), \pi_2(\alpha))$ for every $\alpha \in E$. Note that, in any rooted tree there is a unique path from the root to any vertex. *All trees considered in this paper are finite rooted trees*. We view a rooted tree T as a partially ordered set $(V, \leq_V)$ with the root $r(T)$ as the bottom element, where $u \leq_V v$ means there is a path from $r(T)$ to v passing through u. Observe that any two elements $u, v \in V$ have a meet $u \wedge_V v$ in V.

Let $\mathbf{L} := (L; L_\alpha, G, \rho)$ be the atomic amalgamation of pointed involution posets $L_\alpha, \alpha \in E$, over a tree T via ρ. Note that if each $(L_\alpha, \leq_\alpha, {}^{'\alpha})$ is an orthoposet, then $(L, \leq, ')$ is an orthoposet.

The results in [1] concerning closure within a class $\mathcal{C}$ of structures are here specified to trees; this yields the following proposition which is proved in [1] for any strictly directed graph satisfying one of certain "distancing conditions" each of which is satisfied in any tree.

Proposition 1 *Let $\mathcal{C}$ be the class of either all orthoalgebras, OMPs or orthomodular lattices. If each L_α is in $\mathcal{C}$, then so is* $\mathbf{L}$.

It is proved in [1] that $L_\alpha \simeq [L_\alpha] := \{[x] \mid x \in L_\alpha\}$ and that $[L_\alpha]$ is a subalgebra of L. Let $[A_\alpha]$ be the set of atoms of L_α. *Henceforth, for convenience and by abuse of notation, we write x, A_α and L_α for* $[x]$, $[A_\alpha]$ *and* $[L_\alpha]$, *respectively.*

We now present the main section of the paper in which the order dimension of L is computed from the order dimensions of the L_α's.

3 The Order Dimension of Amalgamated OMPs

In this section we assume that each L_α is a finite pointed OMP. Let $T = (V, E)$ be a tree with root $r(T)$, let $\{L_\alpha\}_{\alpha \in E}$ be a family of pointed OMPs, and for the rest of the paper, let $L = L_T$ be the atomic amalgamation, $(L; L_\alpha, T, \rho)$, of the L_α's over T via ρ. By Corollary 3.7 of [1], L is an OMP.

Let $P = (P, \leq)$ be a poset. The elements $x, y \in P$ are said to be *incomparable*, denoted $x \| y$, if neither $x \leq y$ nor $y \leq x$ holds; let $\mathrm{inc}(P) := \{(x, y) \in P \times P : x \| y\}$. An *alternating cycle* of length k in P is a sequence $(x_i, y_i),\ 1 \leq i \leq k$, of ordered pairs of $\mathrm{inc}(P)$ such that $y_i \leq x_{i+1}$ (mod k) in P. A subset $S \subseteq \mathrm{inc}(P)$ is *cycle-free* if it has no alternating cycle. A pair $(x, y) \in \text{inc}(P)$ is a *critical pair* if $x\uparrow \setminus \{x\} \subseteq y\uparrow$ and $y\downarrow \setminus \{y\} \subseteq x\downarrow$. Let $\text{crit}(P) := \{(x, y) \mid (x, y)$ is a critical pair of $P\}$. For any OMP P, by Corollary 1.2 of [2], $\text{crit}(P) = \{(x', y) \mid (x, y) \in A(P) \times A(P) \setminus \perp\}$, (recall that $x \perp y$ means $x \leq y'$). *For convenience we denote critical pairs of the form* (x', x) *by* $\hat{x}$. Let $\mathcal{R} := \{\preceq_1, \preceq_2, ..., \preceq_m\}$, where each $\preceq_i$ is a linear extension of $\leq$ for every $i \in \{1, 2, ..., m\}$; such an $\mathcal{R}$ is called a *realizer* of $\leq$ whenever $\bigcap_{i=1}^{m} \preceq_i = \leq$. The *order dimension* or (simply, *dimension*) of a poset P, denoted dim(P), is dim(P) $:= |\mathcal{R}|$ for any $\mathcal{R}$ which is a realizer of $\leq$ of minimal cardinality. In [6] it is shown that if $\mathbf{P} := (P, \leq)$ is a finite poset that is not a chain, then dim(P) is the least number of cycle-free sets covering $\text{crit}(P)$. If S is a cycle free set, then we sometimes write S_{xy} for S when $(x', y) \in S$; if $x = y$, we simply write $S_{\hat{x}}$ instead of S_{xx}.

For $\alpha, \beta \in E$, we write $\alpha \leq_E \beta$ to mean that there is a path from $r(T)$ to β passing through α, if also equality does not hold we write $\alpha <_E \beta$. Note that $\leq_E$ is a partial ordering on the set of edges of the rooted tree. Also note that in every path $v_0 v_1 \cdots v_n$ from the root to the vertex v_n, we have $(v_i, v_{i+1}) \in E$. We write $\alpha \|_E \beta$ and $\alpha \nparallel_E \beta$ for incomparability and comparability in the poset $(E, \leq_E)$, respectively.

We say that $T' := (V', E')$ is a *subtree* of T, written $T' \leq T$, whenever $V' \subseteq V$, $E' \subseteq E$ and (V', E') is a rooted tree. We do not insist that the root of the subtree be the same as the root of the tree. Note that every subtree $T' := (V', E')$ of T induces a sub-OMP $L_{T'}$ of L. Note that for any $u \in V$ there is a unique maximal subtree, denoted T_u, having u as a root; T_u is the subtree whose vertices are those vertices v of T with $u \leq_V v$. If $T' \leq T$ and $r(T') = \pi_1(\alpha)$ for exactly one α, then T' is called a *trunked tree with trunk* α or root $\pi_1(\alpha)$. Each $\alpha \in E$ determines a unique maximal trunked subtree, denoted by T_α, whose edges are $\{\beta \mid \alpha \leq_E \beta\}$. Let $L_{T_\alpha}^{\iota_\alpha}$ be the sub-OMP of $\mathbf{L}$ corresponding to the trunked tree T_α.

For each $v \in V$, we index the trunked trees T_β with root $v = \pi_1(\beta)$ as follows. Let $\{\beta_i^v\}_{i=1}^{k_v}$ be the family of edges such that $\pi_1(\beta_i^v) = v$; then $\{T_{\beta_i^v}\}_{i=1}^{k_v}$ is the indexed family of trunked trees with root v. We call this indexing the *standardized indexing of the family of edges* $\{\beta_i^v\}_{i=1}^{k_v}$ *at* v. Let $E_v := \{\beta_i^v\}_{i=1}^{k_v}$. This ordering is fixed throughout the paper. Define $I_1 := \{a \in A(L) \mid a = \rho_1(v),\ v \in V\}$, and $I_2 := \{\tau_\alpha \mid \alpha$ is a maximal element in the ordering $\leq_E$ on $E\}$. Let $I = I_1 \cup I_2 \subseteq L$.

For each $a \in I_1$, let E_a be the standardized indexing of edges in E for which $\pi_1(\alpha_i^a) = a$. Note that, in the tree $T = (V, E)$, the set E is the disjoint union of the E_a's with $a \in I_1$.

In [4], Haviar and Hrnčiar proved that the order dimension of the amalgamation of finite Boolean algebras $B_1, B_2, \ldots, B_k$ over a rooted tree is the $\max\{\dim(B_i)\}$, $i = 1, 2, \cdots, k$. In our upcoming theorems we generalize their result by calculating the dimension of the amalgamation of arbitrary finite pointed OMPs instead of finite Boolean algebras.

The key idea of the proof of the next theorem lies in extending and renaming of the cycle-free sets $\{S_i^\alpha\}_{i=1}^{n_\alpha}$ giving the dimension of the L_α's. Since, by assumption, $\max\{n_\alpha\}_{\alpha \in E} = n$; we begin by selecting an edge μ with $\dim(L_\alpha) = n$; thus there exist n cycle-free sets $\{S_i^\mu\}_{i=1}^{n}$ covering $\text{crit}(L_\mu)$. If necessary, for each $\alpha \in E$, we increase the cardinality of each family $\{S_i^\alpha\}_{i=1}^{n_\alpha}$ to n by defining $S_i^\alpha = \emptyset$ for $i = n_\alpha + 1, n_\alpha + 2, \cdots, n$. Next, we rename each $\{S_i\}_{i=1}^{n}$ so that $S_{\hat{\tau_\alpha}} = S_{\hat{\iota_\beta}}$ for every $\alpha, \beta \in E$ with $\tau_\alpha \sim \iota_\beta \sim a$, where $a \in I_1$ (defined below). Then we extend each S_i^μ to a relation R_i on L by defining $R_i := \bigcup_{\alpha \in E} S_i^\alpha$ for each $i = 1, 2, \cdots, n$. Finally, we extend each R_i to a relation S_i so that $\{S_i\}_{i=1}^{n}$ partitions $\text{crit}(L)$.

Theorem 1 *Let* $\mathbf{L} = (L; L_\alpha, T, \rho)$ *be the atomic amalgamation of a family of finite pointed orthomodular posets* $\{L_\alpha\}_{\alpha\in E}$*, over a finite tree* $T := (V, E)$ *via* ρ*. Suppose that* $\dim(L_\alpha) = n_\alpha$ *for every* $\alpha \in E$ *with* $S_{\hat{\iota}_\alpha} \neq S_{\hat{\tau}_\alpha}$ *and* $\max\{n_\alpha\}_{\alpha\in E} := n$*. Then* $\dim(L) \leq n + 1$.

Proof By Corollary 3.7 of [1], L is an OMP, and by Corollary 1.2 of [2], $\mathrm{crit}(L) = \{(x', y) \mid (x, y) \in A(L) \times A(L) \setminus \perp\}$. Since $\dim(L_\alpha) = n_\alpha$ and $\max\{n_\alpha\}_{\alpha\in E} = n$, there exists $\mu \in E$ such that $\dim(L_\mu) = n_\mu = n$. Since L_μ is isomorphic to a subalgebra of L, $\dim(L) \geq n$.

Define a relation M on the set of atoms, $A(L)$, of L as follows: aMb if there exist distinct $\alpha, \beta \in E$ with $a \in A_\alpha$, $b \in A_\beta$ and $\pi_2(\alpha) \leq_V \pi_1(\beta)$. We write $a\|_M b$ if neither aMb nor $b\,Ma$ holds. Observe that $\|_M$ is a reflexive relation. Since $\dim(L_\alpha) = n_\alpha$ for every $\alpha \in E$, there exists a family $\{S_i^\alpha\}_{i=1}^{n_\alpha}$ of cycle-free sets covering $\mathrm{crit}(L_\alpha)$. For each α and each $i = n_\alpha + 1, n_\alpha + 2, \cdots, n$, define $S_i^\alpha = \emptyset$.

For $j \in \mathbf{N}$, define $E_j := \{\alpha \in E \mid d(r(T), \pi_2(\alpha)) = j\}$. By the hypothesis that $S_{\hat{\iota}_\alpha} \neq S_{\hat{\tau}_\alpha}$, for every $\alpha \in E$, and by renaming the S_i^α's, for each $\alpha \in E$, we may assume that, for $\alpha \in E_j$,

$$\hat{\iota}_\alpha \in S_1^\alpha \text{ and } \hat{\tau}_\alpha \in S_n^\alpha, \text{ if } j \text{ is odd; and}$$
$$\hat{\iota}_\alpha \in S_n^\alpha \text{ and } \hat{\tau}_\alpha \in S_1^\alpha, \text{ if } j \text{ is even.}$$

For $i = 1, 2, \cdots, n$, let $R_i := \bigcup_{\alpha\in E} S_i^\alpha$, and define the sets T_i, as well as T_{n+1}, as follows.

(A_1) If $x \in A_\alpha$, $y \in A_\beta$ with $\alpha <_E \beta$ and $\hat{x} \in R_i$, then $(x', y) \in T_i$ and $(y', x) \in T_{n+1}$.

(A_2) For $a \in I_1$, $\alpha_j^a, \alpha_k^a \in E_a$ with $x \in A(L^a_{T_{\alpha_j^a}})$, $y \in A(L^a_{T_{\alpha_k^a}})$, $x \neq y$ and $j < k$, we declare $(x', y) \in T_n$ if $\hat{a} \in R_1$; $(x', y) \in T_1$ if $\hat{a} \in R_n$; $(y', x) \in T_2$ if $y \in I$; and $(y', x) \in T_i$, for $i \in \{2, 3, \cdots, n-1\}$ if $y \notin I$ and $\hat{y} \in R_i$.

For $i = 1, 2, \ldots, n$, define $S_i := R_i \cup T_i$. Note that $\hat{x} \in \bigcup_{i=1}^n S_i \cup T_{n+1}$ for every critical pair of L of the form $\hat{x}$. Let $(x', y) \in \mathrm{crit}(L)$ be such that $x \neq y$. If x and y belong to the same A_α, for some $\alpha \in E$, then $(x', y) \in R_i \subseteq S_i$ for some $i \in \{1, 2, ..., n\}$. Suppose $x \in A_\alpha$, $y \in A_\beta$ and $\alpha \neq \beta$. If xMy or yMx then, by (A_1), $(x', y) \in \bigcup_{i=1}^n S_i \cup T_{n+1}$. If $x\|_M y$ then, by (A_2), $(x', y) \in \bigcup_{i=1}^n S_i$. Therefore $\bigcup_{i=1}^n S_i \cup T_{n+1} = \mathrm{crit}(L)$.

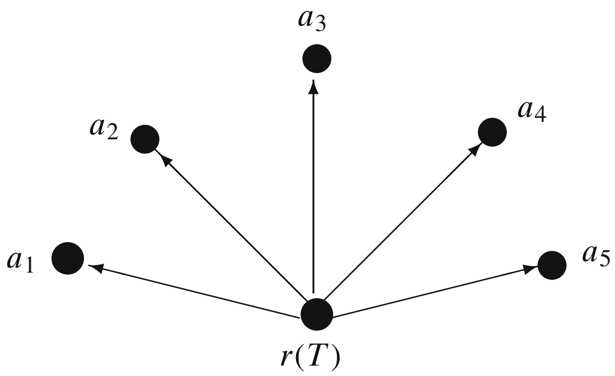
For $x \in A_\alpha$, $y \in A_\beta$, $z \in A_\gamma$, and $w \in A_\lambda$; with $\alpha \leq_E \beta <_E \gamma \leq_E \lambda$, or $\alpha, \beta \|_E \gamma, \lambda$, then it follows from ($A_1$) or ($A_2$), respectively, that $S_{xz} \neq S_{wy}$. Therefore, by construction, it follows that each S_i is a cycle-free set. Thus L is covered by $n + 1$ cycle-free sets $S_1, S_2, \cdots, S_n, T_{n+1}$ so that $\dim(L) \leq n + 1$. □

We note that the dimension of the amalgamation $\mathbf{L}$ may exceed the dimension of the amalgamated posets L_α's. The following example is adapted from [4]. Let

$$T := (\{r, a_1, a_2, a_3, a_4, a_5\}, \{\alpha_i = (r, a_i) \mid i = 1, 2, \cdots, 5\}),$$

let $r = r(T)$, see Fig. 1. For $i = 1, 2, \cdots, 5$, let L_i be a copy of D_{16} (a picket fence of three 8-element Boolean algebras in [2]) made into a pointed orthomodular lattice as indicated by Fig. 2, and let $\rho(\alpha_i) = (\iota_i, \tau_i)$.

Fig. 1 T



Let $\mathbf{L} = (L, L_i, T_0, \rho)$ be the atomic amalgamation of the L_i's over T (see Fig. 1) via ρ. The diagram for L is given in Fig. 3. In [3] it is shown that the dimension of each L_i is 3, and in [4] the dimension of L is 4.

A family $\mathcal{S}$ of cycle-free sets on a pointed involution poset $\mathbf{Q}$:=$(Q, (\iota, \tau))$ such that $|\mathcal{S}| = \dim(Q)$ is said to be a *tailored family for* Q if it satisfies the following conditions: for $w, x, y, z \in A(Q)$, ι and τ are the initial and terminal points of $\mathbf{Q}$, respectively, and $S \in \mathcal{S}$, we have

(t_1) $S_{\hat{\iota}}, S_{\hat{\tau}}, S_{\hat{x}}$ are pairwise distinct, for every $x \neq \iota, \tau$.

(t_2) $S_{\hat{\iota}} = S_{\iota x} = S_{x\iota}$, for every $x \not\perp \iota$; and $S_{\hat{\tau}} = S_{\tau x} = S_{x\tau}$, for every $x \not\perp \tau$ and $x \neq \iota$.

(t_3) $S_{\hat{x}} = S_{xy}$ or $S_{\hat{y}} = S_{xy}$ for every x, y, (Hence $S_{\hat{x}} = S_{\hat{y}}$ implies $S_{xy} = S_{yx} = S_{\hat{x}}$.)

(t_4) If $S_{wx} = S_{yz}$, $x \perp y$ and $w \not\perp z$ and $x, y \notin \{\iota, \tau\}$, then $S_{wx} = S_{wz}$.

If such a family $\mathcal{S}$ exists we say that $\mathbf{Q}$ admits a tailored family. Note that every Boolean algebra admits a tailored family. Evidently, D_{16}, used in the above example, does not admit a tailored family. It is easy to see that, by (t_1), any tailored family for D_{16} requires at least 5 cycle-free sets (one for each of $\hat{\iota}$, $\hat{\tau}$ and three others for the

Fig. 2 L_i

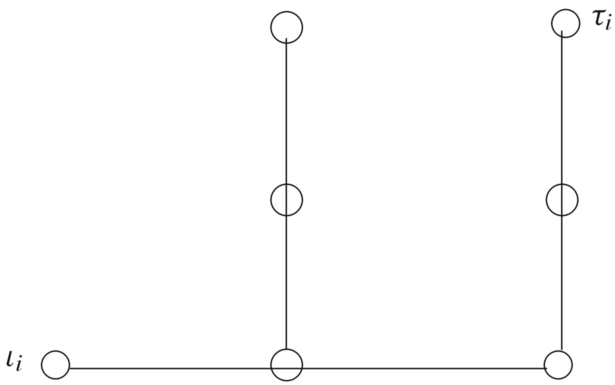
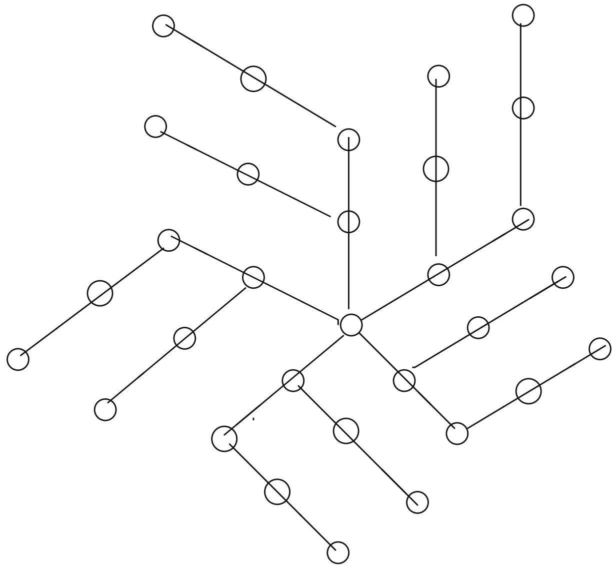


Fig. 3 $\mathbf{L} = (L; L_i, T, \rho)$



block containing neither ι nor τ). If, in Theorem 1, each L_α admits a tailored family, then we shall show in Theorem 2 that, in fact, $\dim(L) = n$.

We now give an example of a pointed OMP $(Q, (\iota, \tau))$ and a tailored family $\mathcal{S}$ covering $\text{crit}(Q) = \{(x', y) \mid x, y \in A(Q) \text{ with } x \not\perp y\}$. Let $Q = J_{18}$, the 18-element orthomodular poset originally discovered by Janowitz, be the atomic amalgamation of blocks, the atoms of which are $\{a, b, c\}$, $\{c, d, e\}$, $\{e, f, g\}$, and $\{g, h, a\}$, with $\iota = a$ and $\tau = e$. Choose the family of sets $\mathcal{S} = \{S_i\}_{i=1}^4$ as follows:

$$S_1 := \left\{\hat{a}, (a', d), (d', a), (a', e), (e', a), (a', f), (f', a)\right\},$$
$$S_2 := \left\{\hat{b}, \hat{d}, \hat{f}, \hat{h}, (b', d), (d', b), (b', f), (f', b), (b', g), (g', b), (b', h), (h', b), (d', f), (f', d), (d', g), (g', d), (d', h), (h', d), (f', h), (h', f)\right\},$$
$$S_3 := \left\{\hat{c}, \hat{g}, (c', f), (f', c), (c', g), (g', c), (c', h), (h', c)\right\},$$
$$S_4 := \left\{\hat{e}, (e', b), (b', e), (e', h), (h', e)\right\}.$$

One can easily check that, for each $i \in \{1, 2, 3, 4\}$, S_i is a cycle-free set, $\text{crit}(Q) = \bigcup_{i=1}^4 S_i$ and that $\mathcal{S}$ is a tailored family.

For $\alpha, \beta \in E$, we write $\alpha \vec{\|} \beta$ whenever $\alpha \|_E \beta$ and there exists $a \in V$ and $\beta_i^a, \beta_j^a \in E_a$, the standardized indexing of the edges at a, such that $\alpha = \beta_i^a$, $\beta = \beta_j^a$ with $i < j$.

Theorem 2 *Let $\mathbf{L} = (L; L_\alpha, T, \rho)$ be the atomic amalgamation of a family of finite pointed orthomodular posets $\{L_\alpha\}_{\alpha \in E}$, over a finite tree $T := (V, E)$ via ρ. If each L_α admits a tailored family $\{S_i^\alpha\}_{i=1}^{n_\alpha}$ and* $\max\{n_\alpha\}_{\alpha \in E} := n$*, then* $\dim(L) = n$.

Proof The proof is in three parts. First we define n sets, the S_i's, and show that they cover crit(L). Then, after listing some elementary facts in Lemma 1, we prove a crucial claim (Lemma 2) which states that no S_i has a cycle of length 2. Finally, we show that each S_i is cycle-free.

Let L_μ, M, and S_i^α's be as in the proof of Theorem 1. It is shown, in the proof of Theorem 1, that $\dim(L) \geq n$. We will show that $\dim(L) \leq n$ by constructing a family of n cycle-free sets $\{S_i\}_{i=1}^n$ with $\bigcup_{i=1}^n S_i = \text{crit}(L)$. For $a \in A_\alpha$ and $b \in A_\beta$ with $\alpha \neq \beta$, define $a \sqcap b := \iota_\gamma$ where γ is any edge of E with $\pi_1(\gamma) = \pi_1(\alpha) \wedge_V \pi_1(\beta)$. Note that here we really mean $[\iota_\gamma]$, but we are suppressing the brackets throughout the paper; thus $\sqcap$ is well defined.

For $i = 1, 2, \cdots, n$, let R_i, and T_i be defined as in the proof of Theorem 1; and let conditions (B_1) and (B_2) be identical to conditions (A_1) and (A_2) of Theorem 1 except that in (B_1) we replace T_{n+1} by T_i when $\hat{x} \in R_i$, that is, we place both (x', y) and (y', x) in T_i if $\hat{x} \in R_i$. Define $S_i := R_i \cup T_i$. Note that $\hat{x} \in \bigcup_{i=1}^n S_i$ for every critical pair of L of the form $\hat{x}$. Let $(x', y) \in \text{crit}(L)$ be such that $x \neq y$. If x and y belong to the same A_α, for some $\alpha \in E$, then $(x', y) \in R_i$, and hence $(x', y) \in S_i$ for some $i \in \{1, 2, ..., n\}$. Suppose $x \in A_\alpha$, $y \in A_\beta$ and $\alpha \neq \beta$. If xMy or yMx, then it follows, by (B_1), that $(x', y) \in \bigcup_{i=1}^n S_i$. If $x \|_M y$ then, by (B_2), $(x', y) \in \bigcup_{i=1}^n S_i$. Therefore $\bigcup_{i=1}^n S_i = \text{crit}(L)$. Now we show that each S_i is a cycle-free set. We shall need the following two lemmas. □

The straight-forward proof of the following lemma is a consequence of (B_2) and hence is omitted.

Lemma 1

(i) *If $x \in \{\iota_\alpha, \tau_\alpha\}$, and $y \in A_\beta$ with $\beta \overrightarrow{\|} \alpha$, then $S_{\hat{x}} \neq S_{zy}$ for any atom z satisfying zMx and $z \|_M y$.*

(ii) *If $x \in A_\alpha$, $x \notin \{\iota_\alpha, \tau_\alpha\}$ and $y \in A_\beta$ with $\alpha \overrightarrow{\|} \beta$, then $S_{\hat{x}} \neq S_{zy}$ for any atom z satisfying zMx and $z \|_M y$.*

(iii) *If $x \in A_\alpha$, $y, z \in A_\beta$ and $\alpha \overrightarrow{\|} \beta$, and $y \notin \{\iota_\beta, \tau_\beta\}$, then $S_{zx} = S_{\hat{y}}$.*

(iv) *If $x \in A_\alpha$, $y \in A_\beta$, $z \in A_\gamma$, and $w \in A_\lambda$ with $\alpha \nparallel_E \beta$, $\lambda \nparallel_E \gamma$ and $\alpha, \beta \|_E \gamma, \lambda$, then $S_{xz} \neq S_{wy}$. Note that α and γ may equal β and λ, respectively.*

(v) *If $\alpha \|_E \beta \|_E \gamma$ and $\alpha \|_E \gamma$ with $x \in A_\alpha$, $y \in A_\beta$, $z \in A_\gamma$ and $S_{xy} = S_{yz}$, then $S_{xz} = S_{xy}$. (It follows that if $u = x \sqcap y$ and $v = y \sqcap z$, then $S_{\hat{u}} = S_{\hat{v}}$.)*

(vi) *If $x \in A_\alpha$, $y \in A_\beta$, $z \in A_\gamma$, and $w \in A_\lambda$ with $\alpha, \beta \|_E \gamma, \lambda$, then $S_{xz} \neq S_{wy}$. Note that α and γ may equal β and λ, respectively.*

In what follows $\alpha, \beta, \gamma, \delta \in E$ and seemingly homeless elements are atoms (or primes of atoms) of L.

Lemma 2 *Let* **L** *and $S := S_i$, for any fixed i, be as above. If $(x_1', y_1), (x_2', y_2) \in S$ with $y_1 \perp x_2$, then $(x_1', y_2) \in S$.*

Proof By Corollary 3.7 of [1], L is an orthomodular poset, and by Corollary 1.2 of [2], $\text{crit}(L) = \{(x', y) \mid (x, y) \in A(L) \times A(L) \setminus \perp\}$. Since $y_1 \perp x_2$, we have $y_1, x_2 \in A_\alpha$ for some fixed $\alpha \in E$. We have the following mutually exhaustive cases. We argue that $(x_1', y_2) \in S$ in each case not leading to a contradiction.

Throughout the proof we use "orthogonality" to mean "$y_1 \perp x_2$ and hence $S_{\hat{y}_1} \neq S_{\hat{x}_2}$." First assume that $x_1 \in A_\alpha$. Then we have the following two cases.

I. $y_2 \in A_\alpha$. If $x_1 = y_1$, then we may assume $x_2 \neq y_2$ (else, $S_{\hat{x}_1} = S_{\hat{x}_2}$ with $x_1 \perp y_2$.) Then either $\{x_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$ in which case t_3 implies $\hat{y}_2 \in S$ and also $(x_1', y_2) \in S$, or $\{x_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} \neq \emptyset$, in which case t_1 is contradicted.

Suppose $x_1 \neq y_1$. If $x_2 = y_2$, then an argument similar to that of the previous case shows that either $(x_1', y_2) \in S$, or a contradiction is obtained. Suppose $x_2 \neq y_2$. Note that if $\{x_1, y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$, then (t_4) applies with $w = x_1, x = y_1, y = x_2$ nd $z = y_2$, so that $(x_1', y_2) \in S$. We shall show that if $\{x_1, y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} \neq \emptyset$ we get a contradiction. Suppose this intersection is nonempty. We may assume $x_1 = \iota_\alpha$. Then $(x_1', y_1) \in S$ implies $\hat{\iota}_\alpha \in S$, using (t_2). Since $(x_2', y_2) \in S$, (t_3) implies $\hat{x}_2 \in S$ or $\hat{y}_2 \in S$ contradicting (t_1).

II. $y_2 \in A_\beta \backslash A_\alpha$ for some $\beta \neq \alpha$. First assume that $\alpha \not\parallel_E \beta$. If $x_1 \neq y_1$ then a consideration of whether or not $\{x_1, y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} \neq \emptyset$ yields seven cases; in each case, if $(x_1', y_2) \notin S$, then t_1, with either (B_1) or (B_2), yields a contradiction. Thus we may assume that $x_1 = y_1$. If $\alpha <_E \beta$, then orthogonality is contradicted. Thus we may assume that $\beta <_E \alpha$. If $x_2 = \iota_\alpha = \tau_\beta$, then orthogonality is contradicted, otherwise $(x_1', y_2) \in S$ by (B_1).

Now assume $\alpha \|_E \beta$. If $x_1 \neq y_1$, then a case analysis, first with $\alpha \vec{\|} \beta$ (later with $\beta \vec{\|} \alpha$) and considering the possibilities of $\{x_1, y_1\} \cap \{\iota_\alpha, \tau_\alpha\}$ being empty or not, yields $(x_1', y_2) \in S$ or leads to a contradiction of (B_2), (t_1), orthogonality, or Lemma 1(ii). Thus we may assume that $x_1 = y_1$. For $\alpha \vec{\|} \beta$, if $x_1 = \iota_\alpha \neq \iota_\beta$, or $x_2 = \iota_\alpha \neq \iota_\beta$ with $x_1 = \tau_\alpha$, or $x_1 = \tau_\alpha$ with $x_2 \neq \iota_\alpha$, then $(x_1', y_2) \in S$ by (B_2); if $x_1 = \iota_\alpha = \iota_\beta$, or $x_2 = \iota_\alpha \neq \iota_\beta$ with $x_1 \neq \tau_\alpha$, or $x_2 = \iota_\alpha = \iota_\beta$, or $x_2 = \tau_\alpha$ with $x_1 \neq \iota_\alpha$, then one of (B_2), orthogonality or Lemma 1(ii) is contradicted. The case $\beta \vec{\|} \alpha$ follows similarly.

Now assume that $x_1 \notin A_\alpha$. Then we have the following three cases.

I. $x_1 \in A_\beta,\ y_2 \in A_\alpha$ for some β. First assume $\alpha \not\parallel_E \beta$. If $x_2 = y_2$, we may assume $\beta <_E \alpha$ (else, orthogonality is contradicted.) Suppose $\{y_1, x_2\} \cap \{\iota_\alpha\} \neq \emptyset$; if $\iota_\alpha = \tau_\beta$, then either orthogonality or (t_1) is contradicted; if $\iota_\alpha \neq \tau_\beta$ then (B_1) implies $(x_1', y_2) \in S$. Now suppose that $\{y_1, x_2\} \cap \{\iota_\alpha\} = \emptyset$ then, by (B_1), $(x_1', y_2) \in S$. Thus we may assume that $x_2 \neq y_2$. Suppose that $\alpha <_E \beta$. If $\{y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$, then (B_1) and (t_3) imply that $(x_1', y_2) \in S$; if this intersection is not empty then orthogonality or (t_1) is contradicted. Thus we may assume that $\beta <_E \alpha$. Suppose $\{y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} \neq \emptyset$; if $\iota_\alpha = \tau_\beta$ then orthogonality or (t_1) is contradicted; if $\iota_\alpha \neq \tau_\beta$ then (B_1) implies $(x_1', y_2) \in S$. Now suppose $\{y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$ then (B_1) implies that $(x_1', y_2) \in S$.

Now assume $\alpha \|_E \beta$. If $x_2 \neq y_2$ then we may assume that $\alpha \vec{\|} \beta$. If $\{y_1, x_2, y_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$ or $y_1 = \iota_\alpha \neq \iota_\beta$ with $\{x_2, y_2\} \cap \{\iota_\alpha\} = \emptyset$ or $y_1 = \tau_\alpha$ with $\{x_2, y_2\} \cap \{\iota_\alpha\} = \emptyset$, then (B_2) implies $(x_1', y_2) \in S$; otherwise (t_1), (B_2) or Lemma 1(v) is contradicted. Therefore we may assume that $x_2 = y_2$. If $\alpha \vec{\|} \beta$ then (B_2) implies $(x_1', y_2) \in S$ unless $y_1 = \iota_\alpha \neq \iota_\beta$ with $x_2 = \tau_\alpha$ or $x_2 = \iota_\alpha \neq \iota_\beta$ or $x_2 = \tau_\alpha$ with $y_1 \neq \iota_\alpha$, in which case Lemma 1(v) is contradicted. If $\beta \vec{\|} \alpha$, then (B_2) implies $(x_1', y_2) \in S$ unless $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$ or $y_1 = \iota_\alpha \neq \iota_\beta$ with $x_2 \neq \tau_\alpha$ or $y_1 = \tau_\alpha$ with $x_1 \neq \iota_\alpha$, in which case Lemma 1(iii) is contradicted.

II. $x_1, y_2 \in A_\beta \backslash A_\alpha$ for some β. If $\alpha \not\parallel_E \beta$ then either $\alpha <_E \beta$ or $\beta <_E \alpha$; if $\alpha <_E \beta$ then orthogonality is contradicted; if $\beta <_E \alpha$ then $(x_1', y_2) \in S$. If $\alpha \|_E \beta$, then either Lemma 1(vi) or (B_2) is contradicted.

The following fact is used in the next case: because (V, E) is a tree, if $\alpha, \beta, \gamma \in E$ are distinct and $\gamma \not\parallel_E \alpha, \beta$, then one of the following holds: $\gamma <_E \alpha, \beta$ or $\beta <_E \alpha, \gamma$ or $\alpha <_E \beta, \gamma$.

III. $x_1 \in A_\beta \backslash A_\gamma$, $y_2 \in A_\gamma \backslash (A_\alpha \cup A_\beta)$ for some $\gamma \in E$. Note that α, β, γ are distinct. The proof of this last case is extensive. There are four main subcases according to which of the three edges α, β and γ are comparable.

III.1. $\alpha \|_E \beta \|_E \gamma$ and $\alpha \|_E \gamma$. First we consider the case: $\iota_\alpha = \iota_\beta = \iota_\gamma$. Assume $y_1, x_2 \neq \iota_\alpha$; if $\beta \vec{\|} \alpha \vec{\|} \gamma$ or $\gamma \vec{\|} \alpha \vec{\|} \beta$, then ($B_2$) implies $(x_1', y_2) \in S$; otherwise (B_2) is contradicted. Thus we may assume that $\{y_1, x_2\} \cap \{\iota_\alpha\} \neq \emptyset$. If $y_1 = \iota_\alpha$, then (t_2) implies $\hat{y}_1 \in S$; hence $S_{x_2 y_2} = S_{\hat{y}_1}$ with $y_1 = x_2 \sqcap y_2$ contradicting (B_2). If $x_2 = \iota_\alpha$, then (t_2) implies $\hat{x}_2 \in S$. Thus $S_{x_1 y_1} = S_{\hat{x}_2}$ with $x_2 = x_1 \sqcap y_1$ contradicting (B_2).
Next we consider the case: $\iota_\alpha = \iota_\beta \neq \iota_\gamma$. We consider the subcases (i) $\alpha, \beta \vec{\|} \gamma$, (ii) $\gamma \vec{\|} \alpha, \beta$ and (iii) $\alpha \vec{\|} \gamma \vec{\|} \beta$ or $\beta \vec{\|} \gamma \vec{\|} \alpha$. In (iii), ($B_2$) is contradicted; hence we may assume (i) or (ii) holds. Assume $y_1, x_2 \neq \iota_\alpha$; if (i) holds, then ($B_2$) is contradicted; if (ii) holds then (B_2) assures that $(x_1', y_2) \in S$ when $\alpha \vec{\|} \beta$, and yields a contradiction otherwise. Now assume $\{y_1, x_2\} \cap \{\iota_\alpha\} \neq \emptyset$. Suppose $y_1 = \iota_\alpha$; if (i) holds, then (B_2) implies $(x_1', y_2) \in S$; if (ii) holds then (B_2) is contradicted. Now suppose $x_2 = \iota_\alpha$. When (i) holds, if $\alpha \vec{\|} \beta$, then (B_2) is contradicted, else (B_2) implies $(x_1', y_2) \in S$; when (ii) holds, (B_2) implies $(x_1', y_2) \in S$.
Next we consider the case: $\iota_\alpha = \iota_\gamma \neq \iota_\beta$. We consider the subcases (i) $\alpha, \gamma \vec{\|} \beta$, (ii) $\beta \vec{\|} \alpha, \gamma$ and (iii) $\alpha \vec{\|} \beta \vec{\|} \gamma$ or $\gamma \vec{\|} \beta \vec{\|} \alpha$. In (iii), ($B_2$) is contradicted; hence we may assume (i) or (ii) holds. Suppose that $x_2 \neq \iota_\alpha$. Assume (i) holds, if $\alpha \vec{\|} \gamma$ then (B_2) is contradicted, else (B_2) implies $(x_1', y_2) \in S$. Assume (ii) holds, if $\alpha \vec{\|} \gamma$ then (B_2) implies $(x_1', y_2) \in S$, else (B_2) is contradicted. Now suppose $x_2 = \iota_\alpha$; if (i) holds then (t_2) implies $\hat{x}_2 \in S$, contradicting (B_2); if (ii) holds then (B_2) implies $(x_1', y_2) \in S$.
Next we consider the case: $\iota_\alpha \neq \iota_\beta = \iota_\gamma$. We consider the cases (i) $\beta, \gamma \vec{\|} \alpha$, (ii) $\alpha \vec{\|} \beta, \gamma$ and (iii) $\beta \vec{\|} \alpha \vec{\|} \gamma$ or $\gamma \vec{\|} \alpha \vec{\|} \beta$. In (i) and (ii), ($B_2$) is contradicted; in (iii) (B_1) implies $(x_1', y_2) \in S$.
Finally we consider the case: $\iota_\alpha, \iota_\beta, \iota_\gamma$ are distinct. If $\beta \vec{\|} \alpha \vec{\|} \gamma$ or $\gamma \vec{\|} \alpha \vec{\|} \beta$, then ($B_2$) implies $(x_1', y_2) \in S$, else Lemma 1(vi) in contradicted.

III.2.A. $\alpha \not\parallel_E \beta$ and $\alpha, \beta \|_E \gamma$. Then $\alpha, \beta \vec{\|} \gamma$, call it $(*)$, or $\gamma \vec{\|} \alpha, \beta$, call it $(**)$. We first assume that $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$; in case of $(*)$, (B_1) implies $\hat{y}_1 \in S$, contradicting Lemma 1(ii), and in case of $(**)$, (B_2) implies $(x_1', y_2) \in S$. We now explore the possibilities when the intersection $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\}$ is nonempty.
First we consider the case: $\iota_\alpha = y_1$. Suppose $(*)$ holds. If $\alpha <_E \beta$, then (B_2) implies $(x_1', y_2) \in S$ when $\iota_\alpha \neq \iota_\gamma$; otherwise ($B_1$) implies $\hat{y}_1 \in S$, contradicting (B_2) because this would imply that $y_1 = x_2 \sqcap y_2$ with $S_{\hat{y}_1} = S_{x_2 y_2}$; thus we may assume that $\beta <_E \alpha$. By Lemma 1(ii), $x_1 \in \{\iota_\alpha, \tau_\alpha\}$ so that $(x_1', y_2) \in S$, by (B_2). Next, suppose $(**)$ holds. If $\alpha <_E \beta$, then (B_1) implies $\hat{y}_1 \in S$, contradicting Lemma 1(i); thus we may assume $\beta <_E \alpha$; then $\tau_\beta \neq \iota_\alpha$ (else Lemma 1 (ii) is contradicted); if $x_1 \notin \{\iota_\beta, \tau_\beta\}$, then ($B_2$) implies $(x_1', y_2) \in S$; if $x_1 \in \{\iota_\beta, \tau_\beta\}$ then Lemma 1(ii) is contradicted.
Next we consider the case: $\iota_\alpha = x_2$. Suppose $(*)$ holds. If $\alpha <_E \beta$, then we may assume $\iota_\alpha = \iota_\gamma$ (else (B_1) implies $\hat{y}_1 \in S$). By (t_2), $\hat{x}_2 \in S$

contradicting orthogonality; if $y_1 = \tau_\alpha$ then $\hat{y}_1 \in S$, by (t_2) when ($\tau_\alpha = \iota_\beta$) or by ($B_1$) (when $\tau_\alpha \neq \iota_\beta$). By ($B_2$), $(x'_1, y_2) \in S$; if $y_1 \neq \tau_\alpha$, then (B_1) implies $\hat{y}_1 \in S$ contradicting Lemma 1(ii). Now suppose that $\beta <_E \alpha$; if $x_1 \in \{\iota_\beta, \tau_\beta\}$, then ($B_2$) and Lemma 1(ii) imply $(x'_1, y_2) \in S$.

Next, suppose (∗∗) holds. If $\beta <_E \alpha$ then $x_1 \notin \{\iota_\beta, \tau_\beta\}$ by Lemma 1(i), so that $(x'_1, y_2) \in S$ by (B_1). Therefore we may assume that $\alpha <_E \beta$. If $\iota_\alpha = \iota_\gamma$, then ($t_2$) implies $\hat{x}_2 \in S$; but by (t_2) in case $y_1 = \tau_\alpha = \iota_\beta$ or by (B_2) otherwise, we get $\hat{y}_1 \in S$ contradicting orthogonality. Thus we may assume $\iota_\alpha \neq \iota_\gamma$; if $y_1 = \tau_\alpha$ then, by (t_2) (in case $y_1 = \tau_\alpha = \iota_\beta$) or by ($B_2$) (otherwise), we get $\hat{y}_1 \in S$ contradicting Lemma 1(i); if $y_1 \neq \tau_\alpha$, then (B_2) implies $(x'_1, y_2) \in S$.

Thus we may assume that $\iota_\alpha \neq y_1, x_2$.

Now we consider the case: $\tau_\alpha = y_1$. Suppose (∗) holds. If $\alpha <_E \beta$, then (B_2) implies $(x'_1, y_2) \in S$; thus we may assume $\beta <_E \alpha$; then (B_1) implies $\hat{x}_1 \in S$; if $x_1 \in \{\iota_\beta, \tau_\beta\}$, then ($B_2$) implies $(x'_1, y_2) \in S$, else Lemma 1(ii) is contradicted. Next, suppose (∗∗) holds. If $\alpha <_E \beta$ then, by (t_2) (in case $\tau_\alpha = \iota_\beta$) or by ($B_1$) (otherwise), we have $\hat{y}_1 \in S$ contradicting Lemma 1(i); thus we may assume that $\beta <_E \alpha$; then (B_1) implies $\hat{x}_1 \in S$; if $x_1 \in \{\iota_\beta, \tau_\beta\}$, then Lemma 1(i) is contradicted; otherwise (B_2) implies $(x'_1, y_2) \in S$.

Finally we consider the case: $\tau_\alpha = x_2$. Suppose $\alpha <_E \beta$. If (∗) holds, then (B_1) implies $\hat{y}_1 \in S$. Suppose $\iota_\alpha \neq \iota_\gamma$; if $y_1 = \iota_\alpha$, then $(x'_1, y_2) \in S$, else Lemma 1(ii) is contradicted. Thus we may assume $\iota_\alpha = \iota_\gamma$. If $y_1 = \iota_\alpha$, then $y_1 = x_2 \sqcap y_2$ with $S_{\hat{y}_1} = S_{x_2 y_2}$ contradicting (B_2); otherwise Lemma 1(ii) is contradicted. Thus we may assume that (∗∗) holds. In this case (B_1) implies $\hat{y}_1 \in S$. If $y_1 \neq \iota_\alpha$, then $(x'_1, y_2) \in S$ by (B_2) and Lemma 1(i). The proof of the case $\beta <_E \alpha$ is similar to that of the previous case ($\tau_\alpha = y_1$).

III.2.B. $\alpha \nparallel_E \gamma$ and $\alpha, \gamma \parallel_E \beta$. Then either $\alpha, \gamma \vec{\parallel} \beta$, call it (⋆), or $\beta \vec{\parallel} \alpha, \gamma$, call it (⋆⋆). First we consider the case $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$. Suppose (⋆) holds. If $\alpha <_E \gamma$ then, by (B_2), $(x_1, y_2) \in S$. Thus we may assume $\gamma <_E \alpha$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then Lemma 1(v) is contradicted, else (B_2) implies $(x'_1, y_2) \in S$. Next, suppose (⋆⋆) holds. If $\alpha <_E \gamma$, then (B_1) implies $\hat{x}_2 \in S$, contradicting Lemma 1(iii). Thus we may assume $\gamma <_E \alpha$; if $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then ($B_2$) implies $(x'_1, y_2) \in S$ (else, by (B_1), $\hat{x}_2 \in S$, contradicting Lemma 1(iii)).

Next we consider the case: $\iota_\alpha = y_1$. Suppose (⋆) holds. Assume that $\alpha <_E \gamma$. Suppose $\iota_\alpha \neq \iota_\beta$; if $x_2 \neq \tau_\alpha$, then (B_2) implies $(x'_1, y_2) \in S$; if $x_2 = \tau_\alpha$ then, by (t_1) (in case $\tau_\alpha = \iota_\gamma$), or, by ($B_2$) (otherwise), we have $\hat{x}_2 \in S$, contradicting Lemma 1(v). Thus we may assume $\iota_\alpha = \iota_\beta$. Then ($t_2$) implies $\hat{y}_1 \in S$. By (t_2) in case $\tau_\alpha = \iota_\gamma$ or by (B_2), we get $\hat{x}_2 \in S$ contradicting orthogonality. Thus we may assume that $\gamma <_E \alpha$; if $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then $(x'_1, y_2) \in S$; otherwise $\hat{x}_2 \in S$ contradicting Lemma 1(iii). Next, suppose (⋆⋆) holds. If $\alpha <_E \gamma$, then (t_2) implies $\hat{y}_1 \in S$. By (t_2) (in case $\tau_\alpha = \iota_\gamma$) or by ($B_2$) (otherwise), we get $\hat{x}_2 \in S$ contradicting orthogonality; otherwise (B_2) implies $\hat{y}_2 \in S$ contradicting Lemma 1(iii) unless $y_2 \in \{\iota_\gamma, \tau_\gamma\}$ in which case (B_2) implies $(x'_1, y_2) \in S$.

Now we consider the case: $\iota_\alpha \neq x_2$. Suppose $\alpha, \gamma \overrightarrow{\|} \beta$. We may assume that $\gamma <_E \alpha$ (else, (B_2) implies $\hat{x}_1 \in S$ contradicting Lemma 1(v)). If $y_1 \in \{\iota_\gamma, \tau_\gamma\}$ or $x_2 = \iota_\gamma$, then Lemma 1(v) is contradicted, otherwise (that is, if neither $y_1 \in \{\iota_\gamma, \tau_\gamma\}$ nor $x_2 = \iota_\gamma$ hold) $(x_1', y_2) \in S$. Next, suppose $\beta \overrightarrow{\|} \alpha, \gamma$. Assume $\alpha <_E \gamma$; if $\iota_\alpha \neq \iota_\beta$, then ($B_2$) implies $(x_1', y_2) \in S$; and if $\iota_\alpha = \iota_\beta$, then ($B_1$) implies $\hat{x}_2 \in S$ with $x_2 = x_1 \sqcap y_1$ and $S_{\hat{x}_2} = S_{x_1 y_1}$, contradicting ($B_2$). Thus we may assume that $\gamma <_E \alpha$. Then (B_1) implies $\hat{y}_1 \in S$ or $\hat{x}_2 \in S$ in case $x_2 = \tau_\alpha$ contradicting Lemma 1(iii).

We consider the case: $\tau_\alpha = y_1$ and $\iota_\alpha \neq x_2$. Suppose $\alpha, \gamma \overrightarrow{\|} \beta$. We may assume $\gamma <_E \alpha$ (else, by (B_2), $(x_1', y_2) \in S$). Then (B_1) implies $\hat{y}_2 \in S$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then Lemma 1(v) is contradicted; if $y_2 \notin \{\iota_\gamma, \tau_\gamma\}$ then $(x_1', y_2) \in S$. Next, suppose $\beta \overrightarrow{\|} \alpha, \gamma$. If $\alpha <_E \gamma$, then (B_1) implies $\hat{x}_2 \in S$ contradicting Lemma 1(iii). Thus we may assume that $\gamma <_E \alpha$. Suppose $\iota_\gamma \neq \iota_\beta$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then $(x_1', y_2) \in S$, else Lemma 1(iii) is contradicted. Suppose $\iota_\gamma = \iota_\beta$. If $y_2 \notin \{\iota_\gamma, \tau_\gamma\}$, then Lemma 1(iii) is contradicted. If $y_2 = \iota_\gamma$, then (B_1) implies $\hat{y}_2 \in S$ and $y_2 = x_1 \sqcap y_1$ with $S_{\hat{y}_2} = S_{x_1 y_1}$ contradicting (B_2). If $y_2 = \tau_\gamma$, then (B_2) implies $(x_1', y_2) \in S$.

Finally we consider the case: $\tau_\alpha = x_2$ and $\iota_\alpha \neq y_1$. Suppose $\alpha, \gamma \overrightarrow{\|} \beta$. If $\alpha <_E \gamma$ then, by (t_2) in case $\tau_\alpha = \iota_\gamma$ or by (B_1) otherwise, we get $\hat{x}_2 \in S$ contradicting Lemma 1(v). Thus we may assume $\gamma <_E \alpha$. In this case (B_1) implies $\hat{y}_1 \in S$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then Lemma 1(v) is contradicted, else (B_2) implies $(x_1', y_2) \in S$. Next, suppose $\beta \overrightarrow{\|} \alpha, \gamma$. If $\alpha <_E \gamma$, then as above, by (t_2) or by (B_2), we get $\hat{x}_2 \in S$. By (B_2), $(x_1', y_2) \in S$. Thus we may assume $\gamma <_E \alpha$. In this case (B_1) implies $\hat{y}_2 \in S$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then ($B_2$) implies $(x_1', y_2) \in S$, else Lemma 1(iii) is contradicted.

III.2.C. $\beta \nparallel_E \gamma$ and $\beta, \gamma \|_E \alpha$. If $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$, or $\tau_\alpha \in \{y_1, x_2\}$ or $\iota_\alpha \notin \{y_1, x_2\}$, then Lemma 1(vi) is contradicted. For the remainder of this subcase, by symmetry, we may only consider $\beta <_E \gamma$.

Now suppose that $\iota_\alpha = y_1$; if $\iota_\beta = \iota_\alpha$ then (t_2) implies $\hat{y}_1 \in S$ contradicting (B_2); if $\iota_\beta \neq \iota_\alpha$ then Lemma 1(vi) is contradicted. Assume that $\iota_\alpha = x_2$. Suppose $\beta, \gamma \overrightarrow{\|} \alpha$; if $\iota_\alpha = \iota_\beta$ then (t_2) implies $\hat{y}_1 \in S$ contradicting (B_2); if $\iota_\alpha \neq \iota_\beta$ then Lemma 1(vi) is contradicted. Thus we may assume $\alpha \overrightarrow{\|} \beta, \gamma$; if $\iota_\alpha = \iota_\beta$ then (B_2) implies $\hat{x}_2 \in S$ contradicting Lemma 1(v); if $\iota_\alpha \neq \iota_\beta$ then Lemma 1(vi) is contradicted.

III.3.A. $\alpha \|_E \beta$ and $\gamma <_E \alpha, \beta$. First we consider the case: $\{y_1, x_2\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$. If $\iota_\alpha = \iota_\beta = \tau_\gamma$, then ($B_1$) implies $\hat{y}_2 \in S$. If $\iota_\gamma = y_2$, then (B_1) implies $(x_1', y_2) \in S$; otherwise Lemma 1(i) or Lemma 1(iii) is contradicted depending on whether $\alpha \overrightarrow{\|} \beta$ or $\beta \overrightarrow{\|} \alpha$, respectively. The result follows similarly if $\iota_\alpha = \tau_\gamma$ and $\iota_\beta \neq \tau_\gamma$ or $\iota_\beta = \tau_\gamma$ and $\iota_\alpha = \tau_\gamma$. If $\iota_\alpha = \iota_\beta \neq \tau_\gamma$, then ($B_1$) implies $\hat{y}_2 \in S$. For $\alpha \overrightarrow{\|} \beta$, we may assume $y_2 \notin \{\iota_\gamma, \tau_\gamma\}$, else Lemma 1(i) is contradicted. Hence, by (B_1), $(x_1', y_2) \in S$. For $\beta \overrightarrow{\|} \alpha$, we may assume $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, else Lemma 1(iii) is contradicted. By (B_1), $(x_1', y_2) \in S$. Finally if $\iota_\alpha, \iota_\beta, \iota_\gamma$ are distinct, then (B_1) implies $\hat{y}_2 \in S$. Suppose $\alpha \overrightarrow{\|} \beta$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then Lemma 1(i) is contradicted, else $(x_1', y_2) \in S$. Thus we may assume that $\beta \overrightarrow{\|} \alpha$. If $y_2 \in \{\iota_\gamma, \tau_\gamma\}$, then ($B_1$) implies $(x_1', y_2) \in S$, else Lemma 1(iii) is contradicted.

Next we consider the case: $\iota_\alpha = y_1$. If $\iota_\alpha = \iota_\beta = \tau_\gamma$, or $\iota_\alpha = \tau_\gamma$ and $\iota_\beta \neq \tau_\gamma$ or $\iota_\beta = \tau_\gamma$ and $\iota_\alpha \neq \tau_\gamma$, then (B$_1$) implies that $\hat{y}_2 \in S$. It follows from (t$_2$) that $\hat{y}_1 \in S$, contradicting (t$_1$). If $\iota_\alpha = \iota_\beta \neq \tau_\gamma$, then (B$_1$) implies $\hat{y}_2 \in S$. By (t$_2$) we get $\hat{y}_1 \in S$. Hence $y_1 \in \{\iota_\gamma, \tau_\gamma\}$, else (t$_1$) is contradicted. Thus (B$_1$) implies that $(x'_1, y_2) \in S$.

Now we consider the case: $\iota_\alpha = x_2$. If $\iota_\alpha = \iota_\beta = \tau_\gamma$. Then (t$_2$) implies $\hat{x}_2 \in S$ contradicting (B$_2$). Suppose $\iota_\alpha = \tau_\gamma$ and $\iota_\beta \neq \tau_\gamma$. If $\alpha \overrightarrow{\|} \beta$, then (t$_2$) implies $\hat{x}_2 \in S$ contradicting Lemma 1(v); otherwise the result follows as in the case ($\iota_\alpha = \iota_\beta = \tau_\gamma$) above. If $\iota_\beta = \tau_\gamma$ and $\iota_\alpha \neq \tau_\gamma$, then (t$_2$) implies $\hat{x}_2 \in S$. If $\beta \overrightarrow{\|} \alpha$, then (B$_2$) is contradicted, else Lemma 1(v) is contradicted. If $\iota_\alpha = \iota_\beta \neq \tau_\gamma$ or if $\iota_\alpha, \iota_\beta, \iota_\gamma$ are distinct, then the result follows as in the first case ($\{y_1, x_1\} \cap \{\iota_\alpha, \tau_\alpha\} = \emptyset$.)

Thus we may assume $\iota_\alpha \notin \{y_1, x_2\}$. The cases corresponding to $\tau_\alpha \in \{y_1, x_2\}$ are similar.

III.3.B. $\alpha \|_E \gamma$ and $\beta <_E \alpha, \gamma$. The proof of this case is similar to that of part A.

III.3.C. $\beta \|_E \gamma$ and $\alpha <_E \beta, \gamma$. In this case (B$_1$) implies $\hat{y}_1, \hat{x}_2 \in S$, contradicting orthogonality.

III.4. $\alpha \not\|_E \beta \not\|_E \gamma$ and $\gamma \not\|_E \alpha$. Then we have one of the following subcases

(i) $\alpha <_E \beta <_E \gamma$,
(ii) $\alpha <_E \gamma <_E \beta$,
(iii) $\beta <_E \alpha <_E \gamma$
(iv) $\beta <_E \gamma <_E \alpha$,
(v) $\gamma <_E \alpha <_E \beta$,
(vi) $\gamma <_E \beta <_E \alpha$.

Suppose (i) or (ii) holds; then by (t$_2$) (in case $x_2 = \tau_\alpha = \iota_\beta$ or $y_1 = \tau_\alpha = \iota_\beta$) or by (B$_1$), we get $\hat{y}_1, \hat{x}_2 \in S$ contradicting orthogonality. Suppose (iii) holds; if $x_1 \in \{\iota_\beta, \tau_\beta\}$ and $x_2 \notin \{\iota_\alpha, \tau_\alpha\}$, then (t$_2$) is contradicted, otherwise $(x'_1, y_2) \in S$. Suppose (iv) holds; then (B$_1$) implies $\hat{x}_1 \in S$; since $\beta <_E \gamma$, (B$_1$) implies $(x'_1, y_2) \in S$. Suppose (v) holds; if $x_2 \neq \iota_\alpha$ or $\tau_\gamma \neq \iota_\alpha$, then (B$_1$) implies $\hat{y}_2 \in S$; since $\gamma <_E \beta$, (B$_1$) implies $(x'_1, y_2) \in S$; we may assume $x_2 = \iota_\alpha = \tau_\gamma$; if $y_1 \neq \tau_\alpha$, then (B$_1$) and (t$_2$) imply $\hat{y}_1, \hat{x}_2 \in S$ contradicting orthogonality. Finally, suppose (vi) holds; then (B$_1$) implies $\hat{y}_2 \in S$; since $\gamma <_E \beta$, (B_1) implies $(x'_1, y_2) \in S$. This proves Case III and completes the proof of Lemma 2. □

We now return to the conclusion of the proof of Theorem 2. Having proved, in Lemma 2, that no S_i has an alternating cycle of length 2, we now prove by induction that each S_i is a cycle-free set.

Let P_n be the statement that if the set $X := \{(x'_1, y_1), (x'_2, y_2), \cdots, (x'_n, y_n)\} \subseteq S_i$ with $y_k \perp x_{k+1}$, $k = 1, 2, \cdots, n-1$, then $y_n \not\perp x_1$. (Thus P_n states that there are no n-element alternating cycles in S_i; so that S_i is cycle-free if P_n is true for each n.) Note that P_1 is trivially true and P_2 is true by Lemma 2.

Suppose that P_{k-1} is true. We prove that P_k is true. Let X be defined as above. By the induction hypothesis, $(x'_1, y_{k-1}) \in S_i$. Thus $(x'_1, y_{k-1}), (x'_k, y_k) \in S_i$ with $y_{k-1} \perp x_k$. It follows from Lemma 2, that $(x'_1, y_k) \in S_i$, and hence $x_1 \not\perp y_k$. Hence P_k is true.

Thus P_n is true for every n, and so S_i is a cycle-free set. This completes the proof of Theorem 2.

In conclusion, we have shown that when the graph is a tree and the orthoposets are orthomodular then the dimension of the amalgamation does not exceed the largest dimension of the amalgamated posets by more than 1, and equality is achieved when each of the amalgamated posets admits a tailored family of cycle-free sets covering the set of critical pairs of the amalgamated posets. Whether or not having the full strength of a tailored family is necessary for these conclusions is an open question. It would be interesting to identify the class of orthomodular posets that admit a tailored family. In [2] we identified the critical pairs in an orthomodular poset **P** as precisely $\{(x', y) \mid x, y \in A(P) \setminus \perp\}$. The proofs of our Theorems 1 and 2 are based on this characterization. We showed in [2] that this characterization fails for orthoalgebras. It seems that an extension of our results to orthoalgebras, and possibly effects algebras, would require some such characterization of critical pairs.

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