Distributivity conditions and the order-skeleton of a lattice

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Abstract. We introduce "π-versions" of five familiar conditions for distributivity by applying the various conditions to 3-element antichains only. We prove that they are inequivalent concepts, and characterize them via exclusion systems. A lattice L satisfies $D_{0\pi}$ if $a \wedge (b \vee c) \leqslant (a \wedge b) \vee c$ for all 3-element antichains $\{a, b, c\}$. We consider a congruence relation $\sim$ whose blocks are the maximal autonomous chains and define the order-skeleton of a lattice L to be $\widetilde{L} := L/{\sim}$. We prove that the following are equivalent for a lattice L: (i) L satisfies $D_{0\pi}$, (ii) $\widetilde{L}$ satisfies any of the five π-versions of distributivity, (iii) the order-skeleton $\widetilde{L}$ is distributive.

1. Introduction

Distributive lattices are perhaps the most familiar class of lattices. They may be defined via any of the following ternary relations on L:

$D(a, b, c)$ means $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
$D^*(a, b, c)$ means $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
$D_m(a, b, c)$ means $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, and
$D_0(a, b, c)$ means $a \wedge (b \vee c) \leqslant (a \wedge b) \vee c$.

In fact, a lattice L is distributive in case any one, and hence all, of the following equivalent conditions hold:

(i) $D(a, b, c)$ for all $a, b, c \in L$, (ii) $D^*(a, b, c)$ for all $a, b, c \in L$,
(iii) $D_m(a, b, c)$ for all $a, b, c \in L$, (iv) $D_0(a, b, c)$ for all $a, b, c \in L$.

Recall that elements a, b of a lattice L are *incomparable*, written as $a \parallel b$, if they are not comparable. An *antichain* in L is a subset of L in which any two distinct elements are incomparable. We denote by π_L the set of antichains in L and by π_L^n the set of n-element antichains in L, where $n \geq 1$.

In [8], one of us found that a π-version of distributivity proved to be of some importance in the study of when certain mappings are residuated. This motivated us to consider the π-version of each of the properties (i)-(iv), by replacing, in each case, "for all $a, b, c \in L$" with " for all $\{a, b, c\} \in \pi_L^3$."

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A lattice L is *π-meet-distributive* (resp. *π-join-distributive*) if $D(a,b,c)$ (resp. $D^*(a,b,c)$) holds for all $\{a,b,c\} \in \pi_L^3$. A lattice L is *π-distributive* if it is both π-meet-distributive and π-join-distributive. A lattice L satisfies the *π-median law* if $D_m(a,b,c)$ holds for all $\{a,b,c\} \in \pi_L^3$. A lattice L satisfies $D_{0\pi}$ if $D_0(a,b,c)$ holds for all $\{a,b,c\} \in \pi_L^3$. We have resisted considering π-semi-distributivity because it is equivalent to semi-distributivity as defined in [4].

We observe that for a modular lattice, the five π-versions of distributivity are all equivalent to distributivity. But, in general, they are not equivalent to each other as we show in Section 3.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of algebras such that $\mathcal{C}_1 \subset \mathcal{C}_2$; an *exclusion system* for $\mathcal{C}_1 \subset \mathcal{C}_2$ is a class $\mathcal{S} \subset \mathcal{C}_2 - \mathcal{C}_1$ such that, for $L \in \mathcal{C}_2$, $L \notin \mathcal{C}_1$ iff there exists $S \in \mathcal{S}$ isomorphic to a subalgebra of L. We denote by $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{M}$ the classes of lattices, distributive lattices, and modular lattices, respectively. Let $\mathcal{D}_{0\pi}$, $\mathcal{D}_{m\pi}$, $\mathcal{D}_{\wedge\pi}$, $\mathcal{D}_{\vee\pi}$, and $\mathcal{D}_\pi$ be the classes of lattices satisfying $D_{0\pi}$, the π-median law, π-meet-distributivity, π-join-distributivity, and π-distributivity, respectively. Recall that N_5 is the 5-element non-modular lattice and M_3 is the 5-element modular non-distributive lattice. It is well known that $\{M_3, N_5\}$ is an exclusion system for $\mathcal{D} \subset \mathcal{L}$. In Section 3, we characterize the five π-versions of distributivity via exclusion systems; we show that $[M_3, \mathbf{3}^2)$, $\{L_{15}\}$, $\{L_{14}\}$, and $\{L_{13}\}$ are exclusion systems for $\mathcal{D}_{0\pi} \subset \mathcal{L}$, $\mathcal{D}_{m\pi} \subset \mathcal{D}_{0\pi}$, $\mathcal{D}_{\wedge\pi} \subset \mathcal{D}_{m\pi}$, and $\mathcal{D}_{\vee\pi} \subset \mathcal{D}_{m\pi}$, respectively, where $[M_3, \mathbf{3}^2)$ is defined in Section 3.

We study the notion of the order-skeleton $\widetilde{L}$ of a lattice L, which was first introduced in [8] and discussed in [9]. In Theorem 2.8, we prove that L satisfies $D_{0\pi}$ iff its order-skeleton $\widetilde{L}$ is distributive. Our main result, Corollary 3.6, is that L satisfies $D_{0\pi}$ iff $\widetilde{L}$ satisfies any of the five π-versions of distributivity presented. In particular, if a lattice L is isomorphic to its order-skeleton, then all the five π-versions of distributivity are equivalent to distributivity.

We conclude the paper by studying several other weakened distributivity conditions, giving the relation between them and our π-versions of distributivity.

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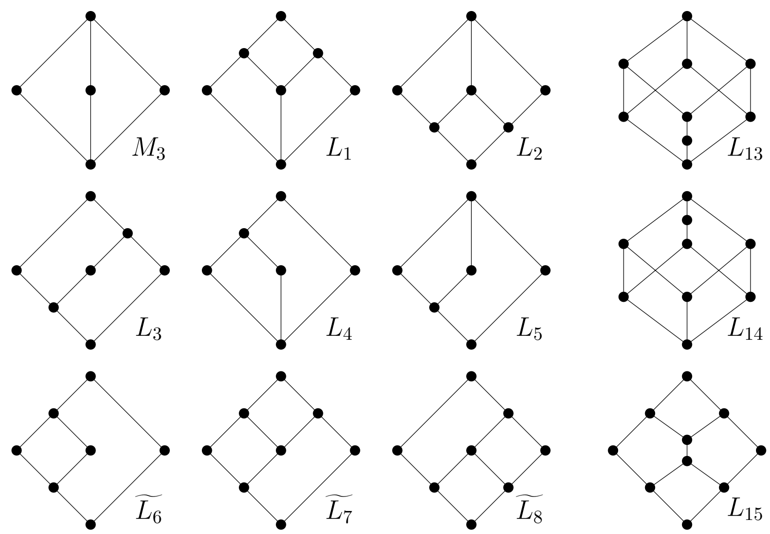
2. Order-skeletons

Let L be a lattice and $a, b \in L$. As usual, we write $[a,b] := \{x \in L \mid a \leqslant x \leqslant b\}$ and $[a,b) := \{x \in L \mid a \leqslant x < b\}$; we allow for the possibility that $a \nleqslant b$, in which case, of course, both sets are empty. Define $\pi(a) := \{b \in L \mid b \parallel a\}$. The following definition plays an important role throughout this paper:

$$a \sim b \text{ means } a \nparallel b \text{ and } \pi(a) = \pi(c) \text{ for all } c \in [a,b] \cup [b,a].$$



(a) A 9-element exclusion system for $\mathcal{D}_{0\pi} \subset \mathcal{L}$ (b)

FIGURE 1. A 12-element exclusion system for $\mathcal{D}_\pi \subset \mathcal{L}$

Following [16], a non-empty subset S of L is called *(order-)autonomous* in case, for all $p \notin S$, (1) if there is an $s \in S$ with $s \leqslant p$, then $x \leqslant p$ for all $x \in S$; and (2) if there is an $s \in S$ with $p \leqslant s$, then $p \leqslant x$ for all $x \in S$.

Lemma 2.1. *Let L be a lattice with $a, b \in L$. The following are equivalent.*

(i) $a \sim b$.
(ii) $[a, b]$ *or* $[b, a]$ *is a chain and* $\pi(a) = \pi(b)$.
(iii) $[a, b]$ *or* $[b, a]$ *is an autonomous chain.*

Proof. $(i) \Rightarrow (ii)$ We may assume that $a \sim b$ and $a \leqslant b$. We need only to show that $[a, b]$ is a chain. For any $c, d \in [a, b]$, we have $\pi(c) = \pi(a) = \pi(d)$, so that $c \nparallel d$. It follows that $[a, b]$ is a chain.
$(ii) \Rightarrow (iii)$ Since $\pi(a) = \pi(b)$, we have $a \nparallel b$. Assume $a \leqslant b$, so that $[a, b]$ is a chain. Let $c \in [a, b]$ and $p \notin [a, b]$. If $p \leqslant c$, then $p \leqslant b$. Since $\pi(a) = \pi(b)$, we have $p \nparallel a$. Since $p \notin [a, b]$, we have $p \leqslant a$. Thus, $p \leqslant x$ for all $x \in [a, b]$. Dually, if $c \leqslant p$, then $x \leqslant p$ for all $x \in [a, b]$. Therefore, $[a, b]$ is autonomous.
$(iii) \Rightarrow (i)$ We may assume that $[a, b]$ is an autonomous chain. Let $c \in [a, b]$ and $x \notin [a, c]$. Note that, $x \leqslant a$ iff $x \leqslant c$, and $a \leqslant x$ iff $c \leqslant x$. Thus, $x \parallel a$ iff $x \parallel c$. It follows that $\pi(a) = \pi(c)$, so that $a \sim b$. □

Recall that an equivalence relation θ on a lattice L is a congruence relation iff for any $a, b, c \in L$, $a \; \theta \; b$ implies that $(a \vee c) \; \theta \; (b \vee c)$ and $(a \wedge c) \; \theta \; (b \wedge c)$ (cf. [5]).

Lemma 2.2. *The relation $\sim$ defined on a lattice L is a congruence relation.*

Proof. First, we claim that $\sim$ is an equivalence relation. The reflexivity and symmetry follow directly from the definition. The transitivity follows from the fact that the subsets of autonomous chains are autonomous chains and the union of two autonomous chains having a non-empty intersection is an autonomous chain.

Now, we show that $\sim$ is a congruence relation. Let $a, b, c \in L$ and suppose that $a \sim b$. Since $a \nparallel b$, we may assume that $a \leqslant b$. We shall argue that $a \vee c \sim b \vee c$ by the following two cases.

Case 1. Suppose that $a \nparallel c$. Since $\pi(a) = \pi(b)$, we have $b \nparallel c$. Thus, $\{a, b, c\}$ is a chain. If $c \leqslant a \leqslant b$, then $a \vee c = a \sim b = b \vee c$. If $a \leqslant c \leqslant b$, then $a \vee c = c \sim b = b \vee c$. If $a \leqslant b \leqslant c$, then $a \vee c = c \sim c = b \vee c$. Therefore, in all cases, $a \vee c \sim b \vee c$.

Case 2. Suppose that $a \parallel c$. Since $a \leqslant b$, we have $a \vee c \leqslant b \vee c$. By Lemma 2.1, $[a, b]$ is an autonomous chain. Since $a \leqslant a \vee c$, we have $b \leqslant a \vee c$, so that $b \vee c \leqslant a \vee c$. Thus, $a \vee c = b \vee c$, and therefore, $a \vee c \sim b \vee c$.

By a dual argument, we have $a \wedge c \sim b \wedge c$. Therefore, $\sim$ is a congruence relation. □

Define $[a] := \{b \mid a \sim b\}$ and $\widetilde{L} := L/{\sim} = \{[a] \mid a \in L\}$. We call the quotient lattice $\langle \widetilde{L}; \vee_{\widetilde{L}}, \wedge_{\widetilde{L}} \rangle$ the *order-skeleton* of L. (The order-skeleton of a poset is introduced in [8].)

Lemma 2.3. *Let L be a lattice with $a, b \in L$. Then*

(i) $[a] \leqslant_{\widetilde{L}} [b]$ *iff there exist* $a_1 \in [a]$ *and* $b_1 \in [b]$ *such that* $a_1 \leq b_1$*;*
(ii) $[a] <_{\widetilde{L}} [b]$ *iff* $a < b$ *and* $[a] \neq [b]$*;*
(iii) $[a] \vee_{\widetilde{L}} [b]$ *exists and equals* $[a \vee b]$*;*
(iv) $[a] \wedge_{\widetilde{L}} [b]$ *exists and equals* $[a \wedge b]$*;*
(v) $a \parallel b$ *iff* $[a] \parallel_{\widetilde{L}} [b]$*; and*
(vi) $\pi(a) = \pi(b)$ *iff* $\pi_{\widetilde{L}}([a]) = \pi_{\widetilde{L}}([b])$.

Proof. (i), (ii), (iii), and (iv) follow directly from the fact that $\sim$ is a congruence relation (cf. [5]). Also, (v) follows from (i) and the definition of $\sim$, and (vi) follows from (v). □

For a lattice L with $a \in L$, the element a is *join-reducible* if there exist $b, c < a$ such that $a = b \vee c$. A *meet-reducible* element is defined dually. An element a is *doubly-reducible* if it is both join-reducible and meet-reducible. Note that under this definition, 0 is not join-reducible and 1 is not meet-reducible.

For convenience of notation, we use a, b, c for elements in L and x, y, z for elements in $\widetilde{L}$. The following lemma ensures that there is at most one join-reducible (resp., meet-reducible) element of L in each $[a] \in \widetilde{L}$.

Lemma 2.4. *Let L be a lattice with $a \in L$ and $x \in \widetilde{L}$.*

(i) *If a is join-reducible in L, then $\bigwedge[a]$ exists and $\bigwedge[a] = a$.*
(ii) *If x is join-reducible in $\widetilde{L}$, then $\bigwedge x$ exists and $\bigwedge x \in x$.*

(*iii*) *If a is meet-reducible in L, then $\bigvee[a]$ exists and $\bigvee[a] = a$.*
(*iv*) *If x is meet-reducible in $\widetilde{L}$, then $\bigvee x$ exists and $\bigvee x \in x$.*

Proof. By duality, we need only prove (*i*) and (*ii*).
(*i*) Let $a \in L$ be join-reducible. We need only show that a is the lower bound of $[a]$. There exist $b, c \in L$ such that $b \parallel c$ and $a = b \vee c$. Since $b \parallel c$ and $a \nparallel c$, we have $[a] \neq [b]$. By Lemma 2.3 (*i*), we have $[b] \leqslant_{\widetilde{L}} [a]$, so that $[b] <_{\widetilde{L}} [a]$. Let $u \in [a]$. By Lemma 2.3 (*ii*), $b < u$. Similarly, $c < u$. Hence, $a = b \vee c \leqslant u$.
(*ii*) Since x is join-reducible in $\widetilde{L}$, there exist $b, c \in L$ such that $[b], [c] <_{\widetilde{L}} x$ and $x = [b] \vee_{\widetilde{L}} [c]$. By Lemma 2.3 (*ii*), for all $a \in x$, we have $b, c < a$, so that $b \vee c \leqslant a$. By Lemma 2.3 (*iii*), $x = [b \vee c]$, so $b \vee c \in x$. Therefore $b \vee c$ is the smallest element in x, i.e., $\bigwedge x = b \vee c \in x$. □

Lemma 2.5. *Let L be a lattice with $x \in \widetilde{L}$. Then*

(*i*) *x is a maximal autonomous chain in L;*
(*ii*) *$\widetilde{\widetilde{L}} := \widetilde{(\widetilde{L})} = \{\{x\} \mid x \in \widetilde{L}\}$, i.e., $\sim_{\widetilde{L}}$ is equality on $\widetilde{L}$.*
(*iii*) *$L \cong \widetilde{L}$ iff $\widetilde{L} = \{\{a\} \mid a \in L\}$.*

Proof. (*i*) Let $x \in \widetilde{L}$. For any $b, c \in x$, we have $b \sim c$, so that $b \nparallel c$; hence x is a chain. Let $p \notin x$ and $b, c \in x$. If $b \leqslant p$, then by Lemma 2.1, we have $[b, c]$ or $[c, b]$ is autonomous, so that $c \leqslant p$. Similarly, $p \leqslant b$ implies $p \leqslant c$. Hence, x is autonomous. We now show that x is maximal. Let $S \subseteq L$ be an autonomous chain containing x. For $a \in x$ and $s \in S$, we have that $[a, s]$ or $[s, a]$ is autonomous, so that by Lemma 2.1, $a \sim s$. Thus, $S \subseteq [a] = x$, i.e., x is a maximal autonomous chain.
(*ii*) Let $x, y \in \widetilde{L}$ and $x \sim_{\widetilde{L}} y$. Since $x \nparallel_{\widetilde{L}} y$, we may assume that $x \leqslant_{\widetilde{L}} y$, and hence, there exist $a \in x$ and $b \in y$ such that $a \leqslant b$. For any $c \in L$ with $a \leqslant c \leqslant b$, we have $x \leqslant_{\widetilde{L}} [c] \leqslant_{\widetilde{L}} y$. Thus, $\pi_{\widetilde{L}}(x) = \pi_{\widetilde{L}}([c])$, and hence, by Lemma 2.3 (*vi*), $\pi(a) = \pi(c)$. It follows that $a \sim b$, so that $x = y$. Therefore, $\widetilde{x} = \{x\}$.
(*iii*) If $\widetilde{L} = \{\{a\} \mid a \in L\}$, then the function $a \mapsto [a]$ is easily seen to be an isomorphism. For the converse, assume that $L \cong \widetilde{L}$ via the isomorphism $f \colon L \to \widetilde{L}$. Let $a, b \in L$ with $a \sim b$. We may assume that $a \leqslant b$. Since f is an isomorphism, we have $f(a) \leqslant_{\widetilde{L}} f(b)$. Let x be an arbitrary element in $[f(a), f(b)]$ and let $c := f^{-1}(x)$. We have $c \in [a, b]$, and thus, $\pi(c) = \pi(a)$ since $a \sim b$. Since f is an isomorphism, we have $\pi_{\widetilde{L}}(x) = \pi_{\widetilde{L}}(f(a))$. Therefore, by definition, $f(a) \sim_{\widetilde{L}} f(b)$. By (*ii*), we have $f(a) = f(b)$, so that $a = b$. Therefore, $\sim$ is equality on L, and $\widetilde{L} = \{\{a\} \mid a \in L\}$. □

Let L and K be two lattices. Recall that a *(lattice-)embedding* $f \colon L \hookrightarrow K$ is a one-to-one homomorphism from L to K. L is *(lattice-)embeddable* in K if there exists an embedding from L into K. The following lemma utilizes the Axiom of Choice.

Lemma 2.6. *Let L be a lattice. Consider the following conditions.*

(*i*) *Every doubly-reducible element in $\widetilde{L}$ is a singleton subset of L.*

(*ii*) *There exists an* embedding $\beta\colon \widetilde{L} \hookrightarrow L$ *such that* $\beta(x) \in x$ *for every* $x \in \widetilde{L}$.
(*iii*) $\widetilde{L}$ *is* embeddable *in* L.
(*iv*) *The cardinality of the set of doubly-reducible elements in* L *equals the cardinality of the set of doubly-reducible elements in* $\widetilde{L}$.

Then $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. *In particular, if* L *contains finitely many doubly-reducible elements, then the four conditions are equivalent to each other.*

Proof. $(i) \Rightarrow (ii)$ Assume that every doubly-reducible element in $\widetilde{L}$ is a singleton subset of L. Note that, by Lemma 2.4, if x is join-reducible (resp. meet-reducible) in $\widetilde{L}$, then $\bigwedge x$ (resp. $\bigvee x$) exists and $\bigwedge x \in x$ (resp. $\bigvee x \in x$). By assumption, if x is doubly-reducible in $\widetilde{L}$, then $\bigwedge x = \bigvee x$. Thus, there exists a selection function $\beta\colon \widetilde{L} \to L$ such that

(1) $\beta(x) \in x$,
(2) if x is join-reducible, then $\beta(x) = \bigwedge x$, and
(3) if x is meet-reducible, then $\beta(x) = \bigvee x$.

It is easy to see that $\beta\colon \widetilde{L} \to L$ is one-to-one. We now show that β is a homomorphism. Let $[a], [b] \in \widetilde{L}$. If $[a] \nparallel_{\widetilde{L}} [b]$, then $\beta([a] \vee_{\widetilde{L}} [b]) = \beta([a]) \vee \beta([b])$. If $[a] \parallel_{\widetilde{L}} [b]$, then both $[a] \vee_{\widetilde{L}} [b]$ and $a \vee b$ are join-reducible, so that by Lemma 2.3 (*iii*) and Lemma 2.4, $\beta([a] \vee_{\widetilde{L}} [b]) = \beta([a \vee b]) = \bigwedge[a \vee b] = \beta([a]) \vee \beta([b])$. Dually, we have $\beta([a] \wedge_{\widetilde{L}} [b]) = \beta([a]) \wedge \beta([b])$. Therefore, β is an embedding.

$(ii) \Rightarrow (i)$ Let $\beta\colon \widetilde{L} \hookrightarrow L$ be an embedding such that $\beta(x) \in x$ for every $x \in \widetilde{L}$. Let $x \in \widetilde{L}$ be a doubly-reducible element. Since x is join-reducible in $\widetilde{L}$, there exist $y, z \in \widetilde{L}$ such that $y \parallel_{\widetilde{L}} z$ and $x = y \vee_{\widetilde{L}} z$. By Lemma 2.3 (*v*), $\beta(y) \parallel \beta(z)$, so that $\beta(x) = \beta(y) \vee \beta(z)$ is join-reducible in L. By Lemma 2.4, $\beta(x) = \bigwedge x$. Dually, $\beta(x) = \bigvee x$, so that $\bigwedge x = \bigvee x$, i.e., x is a singleton subset of L.

$(ii) \Rightarrow (iii)$ This is trivial.

$(iii) \Rightarrow (iv)$ Let $f\colon \widetilde{L} \hookrightarrow L$ be an embedding. Let c_L and $c_{\widetilde{L}}$ be the cardinality of the doubly-reducible elements in L and $\widetilde{L}$ respectively. Note that f maps the doubly-reducible elements in $\widetilde{L}$ to the doubly-reducible elements in L, so that $c_{\widetilde{L}} \leqslant c_L$. Also note that every doubly-reducible element $a \in L$ corresponds to a doubly-reducible element $[a] \in \widetilde{L}$, so that $c_L \leqslant c_{\widetilde{L}}$. Therefore, $c_L = c_{\widetilde{L}}$.

Now assume that L contains finitely many doubly-reducible elements.

$(iv) \Rightarrow (i)$ Suppose that x is doubly-reducible in $\widetilde{L}$ but not a singleton subset of L. Since every doubly-reducible element $a \in L$ corresponds to a doubly-reducible element $[a] \in \widetilde{L}$, we have $c_L \leqslant c_{\widetilde{L}}$. Also note that there is no doubly-reducible element in L that corresponds to x. It follows that $c_L < c_{\widetilde{L}}$ which contradicts (*iv*). □

Note that (*i*), (*iii*), and (*iv*) are not in general equivalent to each other. For example, let L be the horizontal sum of L_{15} with countably many copies of the $\mathbf{3}^2$. Then $\widetilde{L}$ is the horizontal sum of countably many copies of the $\mathbf{3}^2$. Therefore $\widetilde{L}$ is embeddable in L, but L contains a horizontal summand isomorphic to L_{15} so that $\widetilde{L}$ contains a doubly-reducible element which is not a singleton subset of L, i.e., $(iii) \not\Rightarrow (i)$. Now let M be the vertical sum of two $\mathbf{2}^2$ so that there is a bi-reducible element in M. Let Q be the horizontal sum of L_{15} with countably many copies of M. In this example, both Q and $\widetilde{Q}$ have countably many bi-reducible elements, but $\widetilde{Q}$ is not embeddable in Q, i.e., $(iv) \not\Rightarrow (iii)$.

Define a *pentagon* in a lattice L to be a quintuple $\langle a,b,c,u,v\rangle$ such that $a,b,c,u,v \in L$ and

$$v < b < a < u,\ c \wedge a = v,\ c \vee b = u.$$

Lemma 2.7. *Let L be a lattice. If $\widetilde{L}$ is non-modular, then $\widetilde{L}$ contains a pentagon $\langle x,y,z,u,v\rangle$ and there exists an element $w \in \widetilde{L}$ such that either*
(i) $x = y \vee_{\widetilde{L}} w$ *and* $\{w,y,z\} \in \pi^3_{\widetilde{L}}$*; or* (ii) $y = x \wedge_{\widetilde{L}} w$ *and* $\{w,x,z\} \in \pi^3_{\widetilde{L}}$.

Proof. Suppose that $\widetilde{L}$ contains a pentagon $\langle x_1,y,z,u,v\rangle$. Since $x_1 \not\sim_{\widetilde{L}} y$, there exists $x_2 \in \widetilde{L}$ such that $y <_{\widetilde{L}} x_2 \leqslant_{\widetilde{L}} x_1$ and $\pi(y) \neq \pi(x_2)$. Note that, $\langle x_2,y,z,u,v\rangle$ is a pentagon. Since $\pi(y) \neq \pi(x_2)$, we have either (i^*) $\exists\, w \in \widetilde{L}$ with $w \parallel_{\widetilde{L}} y$ and $w \nparallel_{\widetilde{L}} x_2$, or (ii^*) $\exists\, w \in \widetilde{L}$ with $w \parallel_{\widetilde{L}} x_2$ and $w \nparallel_{\widetilde{L}} y$. By duality, we may assume that (i^*) holds. Since $y \leqslant_{\widetilde{L}} x_2$ and $y \nleqslant_{\widetilde{L}} w$, we have $x_2 \nleqslant_{\widetilde{L}} w$. Since $x_2 \nleqslant_{\widetilde{L}} w$ and $w \nparallel_{\widetilde{L}} x_2$, we have $w \leqslant_{\widetilde{L}} x_2$. Let $x = y \vee_{\widetilde{L}} w$. We have $y <_{\widetilde{L}} x \leqslant_{\widetilde{L}} x_2$, and thus, $\langle x,y,z,u,v\rangle$ is a pentagon. Since $w \nleqslant_{\widetilde{L}} y$ and $x \wedge_{\widetilde{L}} z = v \leqslant_{\widetilde{L}} y$, we have $w \nleqslant_{\widetilde{L}} x \wedge_{\widetilde{L}} z$. Since $w \nleqslant_{\widetilde{L}} x \wedge_{\widetilde{L}} z$ and $w \leqslant_{\widetilde{L}} x$, we have $w \nleqslant_{\widetilde{L}} z$. Since $w \leqslant_{\widetilde{L}} x$ and $z \nleqslant_{\widetilde{L}} x$, we have $z \nleqslant_{\widetilde{L}} w$. Thus, $w \parallel_{\widetilde{L}} z$, and $\{w,y,z\} \in \pi^3_{\widetilde{L}}$. Therefore, (i) holds. In the dual case (ii^*), (ii) holds. □

Theorem 2.8. *Let L be a lattice. The following are equivalent.*

(i) *L satisfies $D_{0\pi}$.*
(ii) *$\widetilde{L}$ satisfies $D_{0\pi}$.*
(iii) *$\widetilde{L}$ is distributive.*

Proof. First, we prove that (i) and (ii) are equivalent.
$(i) \Leftrightarrow (ii)$ If L satisfies $D_{0\pi}$, then $\{[a],[b],[c]\} \in \pi^3_{\widetilde{L}}$ implies $\{a,b,c\} \in \pi^3_L$ and $[a]\wedge_{\widetilde{L}}([b]\vee_{\widetilde{L}}[c]) = [a\wedge(b\vee c)] \leqslant_{\widetilde{L}} [(a\wedge b)\vee c] = ([a]\wedge_{\widetilde{L}}[b])\vee_{\widetilde{L}}[c]$, so that $\widetilde{L}$ satisfies $D_{0\pi}$. Now we assume that L does not satisfy $D_{0\pi}$. Then $\exists\{a,b,c\} \in \pi^3_L$ such that $a \wedge (b \vee c) \nleqslant (a \wedge b) \vee c$. Let $d = a \wedge (b \vee c)$ and $e = (a \wedge b) \vee c$, so that $d \nleqslant e$. Since $\{a,b,c\} \in \pi^3_L$, we have $\{[a],[b],[c]\} \in \pi^3_{\widetilde{L}}$. Since $d \leqslant a$, $c \leqslant e$, and $c \nleqslant a$, we have $e \nleqslant d$. Thus, $d \parallel e$. It follows, $[d] \parallel_{\widetilde{L}} [e]$. Since $[a] \wedge_{\widetilde{L}} ([b] \vee_{\widetilde{L}} [c]) = [d] \nleqslant_{\widetilde{L}} [e] = ([a] \wedge_{\widetilde{L}} [b]) \vee_{\widetilde{L}} [c]$, we have $D_0([a],[b],[c])$ does not hold. Thus, $\widetilde{L}$ does not satisfy $D_{0\pi}$.

Now we prove that (ii) and (iii) are equivalent.
$(ii) \Leftrightarrow (iii)$ Assume that $\widetilde{L}$ is distributive. For any $\{x,y,z\} \in \pi^3_{\widetilde{L}}$, $x\wedge_{\widetilde{L}}(y\vee_{\widetilde{L}}z) = (x\wedge_{\widetilde{L}}y)\vee_{\widetilde{L}}(x\wedge_{\widetilde{L}}z) \leqslant_{\widetilde{L}} (x\wedge_{\widetilde{L}}y)\vee_{\widetilde{L}}z$. Thus, $\widetilde{L}$ satisfies $D_{0\pi}$. Now we may assume that $\widetilde{L}$ is not distributive. Then $\widetilde{L}$ contains a sublattice isomorphic to M_3 or N_5. It is easy to verify that M_3 does not satisfy $D_{0\pi}$. We may assume that $\widetilde{L}$ contains a pentagon $\langle x,y,z,u,v\rangle$. By Lemma 2.7, we may assume that $\exists\, w \in \widetilde{L}$ such that $x = y \vee_{\widetilde{L}} w$ and $\{w,y,z\} \in \pi^3_{\widetilde{L}}$. Since $w \wedge_{\widetilde{L}} z \leqslant_{\widetilde{L}} x \wedge_{\widetilde{L}} z = v \leqslant_{\widetilde{L}} y$, we have $w \wedge_{\widetilde{L}} (z \vee_{\widetilde{L}} y) = w \nleqslant_{\widetilde{L}} y = (w \wedge_{\widetilde{L}} z) \vee_{\widetilde{L}} y$, i.e., $\widetilde{L}$ does not satisfy $D_{0\pi}$. □

We conclude this section by observing that a simple induction proves that a lattice L is π-meet-distributive iff $x \wedge (\bigvee(A - \{x\})) = \bigvee_{a \in A-\{x\}} (x \wedge a)$ for every finite π-set $A \subseteq L$ and $x \in A$. And the dual statement holds for π-join-distributivity. Note that this is not true when A is not finite. For example, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and consider the lattice $\langle \mathbb{N}_0; \mathrm{lcm}, \mathrm{gcd} \rangle$. Note that 0 is the top element and 1 is the bottom element. This is a distributive lattice, and thus, a π-meet-distributive lattice. Let O be the set of all odd integers, $A = O \cup \{2\}$, and $x = 2$. We have $2 \wedge (\bigvee O) = 2 \wedge 0 = 2$ but $\bigvee_{a \in O} (2 \wedge a) = \bigvee_{a \in O} 1 = 1$.

3. Exclusion systems

In this section, we characterize the five π-versions of distributivity introduced in Section 1 by the exclusion systems. We observe that they are not equivalent to each other. Note that in Figure 1, L_{13} is π-meet-distributive but not π-join-distributive, while L_{14} is π-join-distributive but not π-meet-distributive. L_{15} satisfies $D_{0\pi}$ but does not satisfy the π-median law. Also, both L_{13} and L_{14} satisfy $D_{0\pi}$ and the π-median law, but do not satisfy π-distributivity.

For a lattice L with $a, b, c \in L$, we use $\Gamma\{a, b, c\}$ to denote the sublattice of L generated by a, b, c. We write 1_K (resp. 0_K) for the top (resp. bottom) element of a sublattice K of L.

Lemma 3.1. *Let L be a lattice with $\{a, b, c\} \in \pi_L^3$. The following statements hold.*

(i) *If $a \wedge (b \vee c) \notin \{a \wedge b, a \wedge c\}$, then $\{a \wedge (b \vee c), b, c\} \in \pi_L^3$.*
(ii) *If $a \vee (b \wedge c) \notin \{a \vee b, a \vee c\}$, then $\{a \vee (b \wedge c), b, c\} \in \pi_L^3$.*
(iii) *If $a \vee b < 1_{\Gamma\{a,b,c\}}$ and $a \vee c < 1_{\Gamma\{a,b,c\}}$, then $\{(a \vee b) \wedge (a \vee c), b, c\} \in \pi_L^3$.*
(iv) *If $0_{\Gamma\{a,b,c\}} < a \wedge b$ and $0_{\Gamma\{a,b,c\}} < a \wedge c$, then $\{(a \wedge b) \vee (a \wedge c), b, c\} \in \pi_L^3$.*

Proof. By duality, we need only prove (i) and (iii).
(i) Let $a_1 := a \wedge (b \vee c)$ and assume that $a_1 \notin \{a \wedge b, a \wedge c\}$. Since $a_1 \neq a \wedge b$ and $a \wedge b \leqslant a_1$, we have $a \wedge b < a_1$. Since $a_1 \wedge b = a \wedge (b \vee c) \wedge b = a \wedge b < a_1$ and $a_1 \wedge b = a \wedge b < b$, we have $a_1 \parallel b$. By symmetry, $a_1 \parallel c$. It follows that $\{a_1, b, c\} \in \pi_L^3$.
(iii) Let $a_u := (a \vee b) \wedge (a \vee c)$. Since $a \leqslant a_u$ and $a \nleqslant b$, we also have $a_u \nleqslant b$. We have $b \nleqslant a_u$, otherwise, $b \leqslant a_u = (a \vee b) \wedge (a \vee c) \leqslant a \vee c$, which implies $1_{\Gamma\{a,b,c\}} = a \vee b \vee c = a \vee c$, contradicting $a \vee c < 1_{\Gamma\{a,b,c\}}$. Hence, $a_u \parallel b$. By symmetry, $a_u \parallel c$. Therefore, $\{a_u, b, c\} \in \pi_L^3$. □

Recall that $D_0(a, b, c)$ means $a \wedge (b \vee c) \leqslant (a \wedge b) \vee c$. Dually, we can define $D_0^*(a, b, c)$ to mean $(a \vee b) \wedge c \leqslant a \vee (b \wedge c)$. Note that $D_0^*(a, b, c) = D_0(c, b, a)$.

Lemma 3.2. *Let L be a lattice with $\{a, b, c\} \in \pi_L^3$. The following statements are equivalent.*

(*i*) $D_0(a,b,c)$ *holds.*
(*ii*) $c \vee (a \wedge (b \vee c)) = (a \wedge b) \vee c$.
(*iii*) $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b))$.
(*iv*) $\{a \wedge (b \vee c), b, (a \wedge b) \vee c\} \notin \pi_L^3$.

Proof. It is easy to see that (*i*), (*ii*), and (*iii*) are equivalent. It is trivial that (*i*) implies (*iv*). We need only to show that (*iv*) implies (*i*).
Let $a_1 := a \wedge (b \vee c)$ and $c_1 := (a \wedge b) \vee c$. Note that $a_1 \leqslant a$ and $c \leqslant c_1$. Suppose that $\{a_1, b, c_1\} \notin \pi_L^3$. We have $a_1 \not\parallel b$, $b \not\parallel c_1$, or $a_1 \not\parallel c_1$. Note that, since $b \nleqslant a$ and $a_1 \leqslant a$, we have $b \nleqslant a_1$. Therefore, if $a_1 \not\parallel b$, then since $b \nleqslant a_1$, we have $a_1 < b$, and thus, $a_1 \leqslant a \wedge b \leqslant (a \wedge b) \vee c = c_1$. Similarly, since $c \nleqslant b$ and $c \leqslant c_1$, we have $c_1 \nleqslant b$. Therefore, if $b \not\parallel c_1$, then since $c_1 \nleqslant b$, we have $b < c_1$, and thus, $a_1 = a \wedge (b \vee c) \leqslant b \vee c \leqslant c_1$. Thus, we may assume that $a_1 \not\parallel c_1$. Since $c \leqslant c_1$, $a_1 \leqslant a$, and $c \nleqslant a$, we have $c_1 \nleqslant a_1$. Thus, $a_1 \leqslant c_1$. □

Lemma 3.3. *Let L be a lattice. The following conditions are equivalent.*

(*i*) *L does not satisfy $D_{0\pi}$.*
(*ii*) *$\exists\{a,b,c\} \in \pi_L^3$ such that $a \leqslant b \vee c$ and $a \wedge b \leqslant c$.*
(*iii*) *$\exists\{a,b,c\} \in \pi_L^3$ such that $a \wedge b = 0_{\Gamma\{a,b,c\}}$ and $b \vee c = 1_{\Gamma\{a,b,c\}}$.*

Proof. Let $a_1 := a \wedge (b \vee c)$ and $c_1 := (a \wedge b) \vee c$. Note that $a_1 \leqslant a$ and $c \leqslant c_1$.
(*i*) ⇒ (*ii*) Since L does not satisfy $D_{0\pi}$, $\exists\{a,b,c\} \in \pi_L^3$ such that $a_1 \nleqslant c_1$. By Lemma 3.2 (*iv*), $\{a_1, b, c_1\} \in \pi_L^3$. Thus, $a_1 \wedge b = a \wedge b \leqslant (a \wedge b) \vee c = c_1$. Similarly, $a_1 \leqslant b \vee c_1$.
(*ii*) ⇒ (*iii*) It is trivial.
(*iii*) ⇒ (*i*) $a \wedge (b \vee c) = a \nleqslant c = (a \wedge b) \vee c$, i.e., $D_0(a,b,c)$ does not hold. □

As mentioned in the introduction, distributivity implies $D_{0\pi}$, but $D_{0\pi}$ does not imply distributivity. Moreover, for $\{a,b,c\} \in \pi_L^3$, if either $D(a,b,c)$ or $D^*(c,b,a)$ holds, then $D_0(a,b,c)$ holds. But the converse is not true. Figure 2 is an example of a lattice satisfying $D_{0\pi}$, but not $D(a,b,c)$ or $D^*(c,b,a)$ (this can be verified by applying the above lemma).

The following lemma is well known.

Lemma 3.4. $\{M_3\}$ *is an exclusion system for $\mathcal{D} \subset \mathcal{M}$.*

In [6], the following lemma is proved.

Lemma 3.5. *Let L be a lattice containing a pentagon $\langle a,b,c,u,v\rangle$ and an element d such that $a = b \vee d$ and $\{b,c,d\} \in \pi_L^3$. Then L contains a sublattice isomorphic to L_1, L_3, L_4, $\widetilde{L_6}$, $\widetilde{L_7}$, or $\widetilde{L_8}$.*

We follow the notation from [14, 11, 12]. Note that M_3, L_1, L_2, L_3, L_4, L_5, L_{13}, L_{14} and L_{15} are subdirectly irreducible lattices and each $\widetilde{L_i}$ is the order-skeleton of the corresponding lattice L_i for $i = 6, 7, 8$ as found in these references.

Note that L_1 and L_2 are dual; L_4 and L_5 are dual. By Lemma 2.7, Lemma 3.5 and its dual, and the fact that the eight lattices $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, and $\widetilde{L_8}$ are not modular, we have the following corollary.

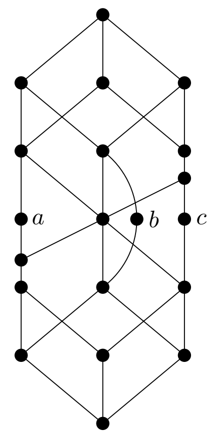


FIGURE 2. A lattice L satisfying $D_{0\pi}$ but satisfying neither $D(a,b,c)$ nor $D^*(c,b,a)$.

Corollary 3.6. *Let L be a lattice. The order-skeleton $\widetilde{L}$ is modular iff $\widetilde{L}$ contains no sublattice isomorphic to $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, or $\widetilde{L_8}$.*

Note that the lattices $M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, and $\widetilde{L_8}$ satisfy the condition (i) in Lemma 2.6. Therefore, we have the following corollary.

Corollary 3.7. *For $F \in \{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}$, if $\widetilde{L}$ contains a sublattice isomorphic to F, then L contains a sublattice isomorphic to F.*

Let $\langle \mathcal{L}_F; \leqslant \rangle$ be the poset of all finite lattices with the ordering defined by *order-embedding*, i.e., $L_1 \leqslant L_2$ iff there exists a one-to-one mapping $f\colon L_1 \hookrightarrow L_2$ such that $x \leqslant y$ iff $f(x) \leqslant f(y)$ for all $x, y \in L_1$. We define the half open interval in $\mathcal{L}_F$ by $[L_1, L_2) := \{L \in \mathcal{L}_F : L_1 \leqslant L < L_2\}$. One can verify that

$$[M_3, \mathbf{3}^2) = \{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}.$$

We now characterize the condition $D_{0\pi}$.

Theorem 3.8. *$[M_3, \mathbf{3}^2)$ is an exclusion system for $\mathcal{D}_{0\pi} \subset \mathcal{L}$.*

Proof. It is easy to verify that the nine lattices in $[M_3, \mathbf{3}^2)$ do not satisfy $D_{0\pi}$. We argue that, if L does not satisfy $D_{0\pi}$, then L contains a sublattice isomorphic to a lattice in $[M_3, \mathbf{3}^2)$.

Suppose that L does not satisfy $D_{0\pi}$. By Theorem 2.8, $\widetilde{L}$ is not distributive. If $\widetilde{L}$ is modular, then by Lemma 3.4, it contains a sublattice isomorphic to M_3. If $\widetilde{L}$ is non-modular, then by Corollary 3.6, it contains a sublattice isomorphic to one of $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}$. Hence, by Corollary 3.7, L contains a sublattice isomorphic to one of $M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}$. □

Recall that a lattice L satisfies the π-*median law* iff $D_m(a,b,c)$ holds for all $\{a,b,c\} \in \pi_L^3$. For convenience of notation, for $a,b,c \in L$, define $a_u := (a \vee b) \wedge (a \vee c)$, $b_u := (a \vee b) \wedge (b \vee c)$, and $c_u := (a \vee c) \wedge (b \vee c)$. Dually, define

$a_l := (a \wedge b) \vee (a \wedge c)$, $b_l := (a \wedge b) \vee (b \wedge c)$, and $c_l := (a \wedge c) \vee (b \wedge c)$. Also, define $m_u := (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ and $m_l := (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$.

It is easy to verify that the nine lattices in $[M_3, \mathbf{3}^2)$ do not satisfy the π-median law, so that $\mathcal{D}_{m\pi} \subset \mathcal{D}_{0\pi}$.

Lemma 3.9. *Let L be a lattice and $\widetilde{L} \cong \mathbf{3}^2$. If L does not satisfy the π-median law, then L_{15} is embeddable in L.*

Proof. Note that $\widetilde{L} \cong \mathbf{3}^2$ is generated by its 3-element antichain. Since L does not satisfy the π-median law, there exists $\{a, b, c\} \in \pi^3_L$ such that $D_m(a, b, c)$ does not hold. Since $\{[a], [b], [c]\} \in \pi^3_{\widetilde{L}}$, we have $\widetilde{L} = \Gamma_{\widetilde{L}}\{[a], [b], [c]\}$. Without loss of generality, we may assume that $[a]$ and $[c]$ are complements in $\widetilde{L}$. Since $D_m(a, b, c)$ does not hold, we have $m_l = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) < (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = m_u$. Since $m_l = b_l \leqslant b \leqslant b_u = m_u$ and $m_l \sim m_u$, we have $[m_l] = [b] = [m_u]$. Consider the inverse image of $\sim: L \to \widetilde{L}$ in L, one can verify that $\{0, 1, a, b_u, b_l, c, a \wedge b, b \wedge c, a \vee b, b \vee c\} \cong L_{15}$. □

Lemma 3.10. *$\{L_{15}\}$ is an exclusion system for $\mathcal{D}_{m\pi} \subset \mathcal{D}_{0\pi}$.*

Proof. It is easy to verify that L_{15} does not satisfy the π-median law. Now assume that L satisfies $D_{0\pi}$, but does not satisfy the π-median law. There exists $\{a, b, c\} \in \pi^3_L$ such that $D_m(a, b, c)$ does not hold, so that $m_l < m_u$.

We claim that at least two of a, b, and c are incomparable with m_u. Otherwise, we may assume that $m_u \not\parallel a$ and $m_u \not\parallel b$. If $m_u \leqslant a$ and $m_u \leqslant b$, then $m_u \leqslant a \wedge b \leqslant m_l$, contradicting $m_l < m_u$. If $a < m_u$ and $b < m_u$, then $a \leqslant m_l$ and $b \leqslant m_l$, so that $m_u \leqslant a \vee b \leqslant m_l$, contradicting $m_l < m_u$. If $a \leqslant m_u$ and $m_u \leqslant b$, then $a \leqslant b$, contradicting $a \parallel b$. If $b \leqslant m_u$ and $m_u \leqslant a$, then $b \leqslant a$, contradicting $a \parallel b$. Therefore, two of a, b, and c are incomparable with m_u.

We may assume that $m_u \parallel a$ and $m_u \parallel b$, so that $\{a, b, m_u\} \in \pi^3_L$. By Theorem 2.8, $\widetilde{L}$ is distributive, so that $m_l \sim m_u$. Observe that $a \wedge b \leqslant m_u \leqslant a \vee b$, i.e., a and b are complement elements in $\Gamma\{a, b, m_u\}$. One can verify that $\Gamma_{\widetilde{L}}\{[a], [b], [m_u]\} \cong \mathbf{3}^2$, so that by Lemma 3.9, L_{15} is a sublattice of L. □

By Theorem 3.8 and Lemma 3.10, we have the following theorem.

Theorem 3.11. *$[M_3, \mathbf{3}^2) \cup \{L_{15}\}$ is an exclusion system for $\mathcal{D}_{m\pi} \subset \mathcal{L}$.*

It is easy to verify that the ten lattices in $[M_3, \mathbf{3}^2) \cup \{L_{15}\}$ are not π-meet-distributive, so that $\mathcal{D}_{\wedge\pi} \subset \mathcal{D}_{m\pi}$.

Lemma 3.12. *$\{L_{14}\}$ is an exclusion system for $\mathcal{D}_{\wedge\pi} \subset \mathcal{D}_{m\pi}$.*

Proof. It is easy to verify that L_{14} is not π-meet-distributive. Now assume that L satisfies the π-median law, but does not satisfy the π-meet-distributivity. There exists $\{a_1, b, c\} \in \pi^3_L$ such that $D(a_1, b, c)$ does not hold. Let $a := a_1 \wedge (b \vee c)$. Since $a_1 \wedge b \leqslant (a_1 \wedge b) \vee (a_1 \wedge c) < a_1 \wedge (b \vee c) = a$, we have $a \neq a_1 \wedge b$. By symmetry, $a \neq a_1 \wedge c$. By Lemma 3.1 (*i*), $\{a, b, c\} \in \pi^3_L$. We have $(a \wedge b) \vee (a \wedge c) = (a_1 \wedge b) \vee (a_1 \wedge c) < a_1 \wedge (b \vee c) = a \wedge (b \vee c)$, i.e., $D(a, b, c)$ does not hold. Let $F = \Gamma\{a, b, c\}$. We have $b \vee c = a \vee b \vee c = 1_F$.

Since L satisfies the π-median law, we have $m_l = m_u$. Since $b \vee c = 1_F$, $m_u = (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \vee b) \wedge (a \vee c) = a_u$. Since $a_l = (a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c) = a \wedge 1_F = a \leqslant a_u$ and $a_l \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = m_l = m_u = a_u$, we have $0_F < b \wedge c$. Since $a \leqslant a_u$ and $a \nleqslant b$, we have $a_u \nleqslant b$. Since $a_u \nleqslant b$ and $b_l \leqslant b$, we have $a_u \neq b_l$. Since $a_u \neq b_l$ and $b_l \vee (a \wedge c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = m_l = m_u = a_u$, we have $0_F < a \wedge c$. By symmetry, $0_F < a \wedge b$. By Lemma 3.1 (iv), $\{a_l, b, c\} \in \pi_L^3$, $\{a, b_l, c\} \in \pi_L^3$, and $\{a, b, c_l\} \in \pi_L^3$. Since $b_l \nleqslant c$ and $c_l \leqslant c$, we have $b_l \nleqslant c_l$. By symmetry, $c_l \nleqslant b_l$. Thus, $b_l \parallel c_l$. It follows, $\{a, b_l, c_l\} \in \pi_L^3$. Since $a \leqslant a_u = m_u = m_l = b_l \vee c_l$, $(a \wedge b_l) \vee (a \wedge c_l) \leqslant (a \wedge b) \vee (a \wedge c) < a = a \wedge (b_l \vee c_l)$, i.e., $D(a, b_l, c_l)$ does not hold. Since $a \wedge b \leqslant a_l \leqslant a$ and $a \wedge b \leqslant b_l \leqslant b$, we have $a \wedge b \leqslant a_l \wedge b_l \leqslant a \wedge b$, so that $a_l \wedge b_l = a \wedge b$. Similarly, $b_l \wedge c_l = b \wedge c$ and $a_l \wedge c_l = a \wedge c$. Observe that $a_l \vee b_l = b_l \vee c_l = a_l \vee c_l = m_l$, so that $\Gamma\{a_l, b_l, c_l\} \cong \mathbf{2}^3$. Since $a_l \leqslant a \leqslant m_l$, we have $\Gamma\{a, b_l, c_l\} \cong L_{14}$. □

By Theorem 3.11, Lemma 3.12, and their dual statements, we have the following theorem.

Theorem 3.13. (i) $[M_3, \mathbf{3}^2) \cup \{L_{14}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\wedge\pi} \subset \mathcal{L}$.
(ii) $[M_3, \mathbf{3}^2) \cup \{L_{13}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\vee\pi} \subset \mathcal{L}$.
(iii) $[M_3, \mathbf{3}^2) \cup \{L_{13}, L_{14}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\pi} \subset \mathcal{L}$.

Remark 3.14. By definition, π-distributivity implies both π-meet- and π-join-distributivity. By comparing the exclusion systems of the π-versions of distributivity, one can also see that either one of π-meet-distributivity and π-join-distributivity implies the π-median law, and the π-median law implies $D_{0\pi}$. In particular, π-distributivity implies $D_{0\pi}$.

By using the exclusion systems with Theorem 2.8, we have the following corollary.

Corollary 3.15. *Let L be a lattice. The following statements are equivalent.*

(i) *L satisfies* $D_{0\pi}$.
(ii) *L contains no sublattice isomorphic to a lattice in* $[M_3, \mathbf{3}^2)$.

(iii) *$\widetilde{L}$ is distributive.*
(iv) *$\widetilde{L}$ is π-distributive.*
(v) *$\widetilde{L}$ is π-meet-distributive.*
(vi) *$\widetilde{L}$ is π-join-distributive.*
(vii) *$\widetilde{L}$ satisfies the π-median law.*
$(viii)$ *$\widetilde{L}$ satisfies* $D_{0\pi}$.

This corollary tells us that, if a lattice L is isomorphic to its own order-skeleton, then all these π-properties are equivalent to distributivity. We conclude this section by observing that no two of the properties (iii) - $(viii)$ are equivalent for general lattices.

4. Motivation and related results

In another paper ([9]), we study the residuated approximation to order-preserving maps on complete lattices. A mapping between two posets is *residuated* if the inverse image of every principle ideal in Q is a principle ideal in L (see [2]). If the posets are complete lattices, then the mapping is residuated iff it preserves all joins. Any function $f\colon L \to Q$ between two complete lattices dominates a largest residuated mapping, called ρ_f, the *residuated approximation* of f. In [1], Andréka, Greechie, and Strecker introduced the *shadow* σ_f of f in order to efficiently approximate ρ_f by iterating σ_f where σ_f is defined by $\sigma_f(x) := \bigwedge\{q \in Q \mid x \leqslant \bigvee f^{-1}(\downarrow q)\}$. They proved the following theorem.

Theorem 4.1. *For a complete lattice Q, Q is completely distributive iff for every complete lattice L and every order-preserving mapping $f\colon L \to Q$, the shadow σ_f is residuated.*

Also, Greechie and Janowitz [10, 9, 8] proved a result implying the following corollary.

Corollary 4.2. *Let L and Q be two complete lattices and $f\colon L \to Q$ be an order-preserving mapping. If L is completely distributive, then σ_f is residuated.*

Notice that both the theorem and the corollary are concerned with distributivity conditions. But even if both L and Q are non-distributive, it is still possible that the shadow is residuated. In [9], we study the structure of L in order to make the shadow σ_f residuated and found the following theorem.

Theorem 4.3. *Let L be a lattice with no infinite chains. The following statements are equivalent.*

(*i*) *σ_f is residuated whenever Q is a complete lattice and $f\colon L \to Q$ is an order-preserving mapping.*
(*ii*) *L satisfies $D_{0\pi}$.*
(*iii*) *$\widetilde{L}$ is distributive.*

This theorem motivated us to study the π-versions of distributivity and the order-skeleton of a lattice.

Now we discuss the relation between the π-versions of distributivity in this paper to some weakened conditions found elsewhere [7, 13].

Note that both $\mathbf{2}$ and N_5 are π-distributive lattices, but $\mathbf{2} \times N_5$ does not satisfy $D_{0\pi}$. Thus, by Comment 3.14, both $\mathbf{2}$ and N_5 satisfy the five π-versions of distributive conditions, but $\mathbf{2} \times N_5$ does not satisfy any of these five conditions. Therefore, the five classes of lattices satisfying the various π-versions of distributivity described in this paper are not lattice varieties since they are not closed under products.

A lattice L is *semi-distributive* whenever, for every $a, b, c \in L$,

(SD1) $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$; and
(SD2) $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$.

It is easy to show that the π-version of semi-distributivity (again by applying the conditions only to $\{a, b, c\} \in \pi_L^3$) is equivalent to semi-distributivity. Davey, Poguntke, and Rival proved that $\{M_3, L_1, L_2, L_3, L_4, L_5\}$ is an exclusion system for $\mathcal{SD} \subset \mathcal{L}$ where $\mathcal{SD}$ is the class of all semi-distributive lattice [4].

A lattice L is *near-distributive* whenever, for every $a, b, c \in L$,

(ND1) $a \wedge (b \vee c) = a \wedge (b \vee (a \wedge (c \vee (a \wedge b))))$ and
(ND2) $a \vee (b \wedge c) = a \vee (b \wedge (a \vee (c \wedge (a \vee b))))$.

As with semi-distributivity, it is not difficult to show that the π-version of near-distributivity is equivalent to near-distributivity. It is also easy to show that near-distributivity implies semi-distributivity. We now show that $D_{0\pi}$ implies near-distributivity.

Lemma 4.4. *Let L be a lattice. If L satisfies $D_{0\pi}$, then L is near-distributive.*

Proof. Since near-distributivity and its π-version are equivalent, we need only to show that the π-version holds. Let $\{a, b, c\} \in \pi_L^3$. By Lemma 3.2 (*iii*), we know that $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b))$. Let $r := a \wedge (b \vee (a \wedge (c \vee (a \wedge b))))$ be the right hand side of (ND1). Since $a \wedge (c \vee (a \wedge b)) \leqslant a$ and $a \wedge (c \vee (a \wedge b)) \leqslant b \vee (a \wedge (c \vee (a \wedge b)))$, we have $a \wedge (c \vee (a \wedge b) \leqslant r$. Also, we have $r \leqslant a \wedge (b \vee (a \wedge (c \vee b))) \leqslant a \wedge (b \vee (c \vee b)) = a \wedge (b \vee c)$. Therefore, $a \wedge (b \vee c) = r = a \wedge (b \vee (a \wedge (c \vee (a \wedge b))))$, i.e., (ND1) holds. By duality, (ND2) holds. Therefore, L is near-distributive. □

A lattice L is *almost-distributive* if it is near-distributive and for every $x, y, z, u, v \in L$,

(AD1) $v \wedge (u \vee c) \leqslant u \vee (c \wedge (v \vee a))$; and
(AD2) $v \vee (u \wedge c') \geq u \wedge (c' \vee (v \wedge a'))$,

where $a = (x \wedge y) \vee (x \wedge z)$, $c = x \wedge (y \vee (x \wedge z))$, $a' = (x \vee y) \wedge (x \vee z)$, and $c' = x \vee (y \wedge (x \vee z))$. Note that a' is the dual of a, c' is the dual of c, and (AD2) is the dual of (AD1).

Lemma 4.5. *Let L be a lattice. If L satisfies $D_{0\pi}$, then L is almost distributive.*

Proof. Let L be a lattice satisfying $D_{0\pi}$. By Lemma 4.4, L is near-distributive. Thus, by duality, we need only to show that (AD1) holds.

Since L satisfies $D_{0\pi}$, by Theorem 2.8, $\widetilde{L}$ is distributive. Recall that $\sim$ is a congruence relation on L. In $\widetilde{L}$, $[a] = [(x \wedge y) \vee (x \wedge z)] = ([x] \wedge_{\widetilde{L}} [y]) \vee_{\widetilde{L}} ([x] \wedge_{\widetilde{L}} [z]) = [x] \wedge_{\widetilde{L}} ([y] \vee_{\widetilde{L}} [z]) = [x \wedge (y \vee z)] = [c]$. Thus, $a \sim c$ and clearly, $a \leqslant c$. If $a \parallel v$, then $c \parallel v$ and by Lemma 2.4, $v \vee a = \bigwedge [v \vee a] = \bigwedge [v \vee c] = v \vee c$, so that $v \wedge (u \vee c) \leqslant u \vee c = u \vee (c \wedge (v \vee c)) = u \vee (c \wedge (v \vee a))$. If $v \leqslant a$, then $v \wedge (u \vee c) \leqslant v \leqslant a \leqslant u \vee a = u \vee (c \wedge (v \vee a))$. If $c \leqslant v$, then $v \wedge (u \vee c) \leqslant u \vee c = u \vee (c \wedge (v \vee a))$. If $a \leqslant v \leqslant c$, then $v \wedge (u \vee c) = v \leqslant u \vee v = u \vee (c \wedge (v \vee a))$. Thus, (AD1) holds. □

Note that the converse of this lemma is not true. For example, $\widetilde{L_6}$ is almost distributive but does not satisfy $D_{0\pi}$.

In [15], Rose proved that for any subdirectly irreducible lattice L, L is almost distributive iff $L \cong D[d]$ for some distributive lattice D and $d \in D$ (see also [11, 13]). Here $D[d]$ is the "doubling" construction introduced by Day [3]. Note that $\widetilde{D[d]} \cong \widetilde{D}$. If $\widetilde{D} \cong D$, then the order-skeleton $\widetilde{D[d]}$ is isomorphic to D, and every block of the order-skeleton is a singleton subset except one block which is a doubleton subset.

Recall the definition of a pentagon $\langle a, b, c, u, v\rangle$ preceding Lemma 2.7. The following lemma follows from the definition of the "doubling" construction.

Lemma 4.6. *Let $L \cong D[d]$ for some distributive lattice D with $d \in D$. If L contains a pentagon $\langle a, b, c, u, v\rangle$ and θ is the smallest congruence relation that identifies a and b, then $L/\theta \cong D$ and θ is the congruence relation that identifies only a and b.*

Lemma 4.7. *Let L be a π-distributive lattice with $x, y, z \in L$ and $L \cong D[d]$ for some distributive lattice D with $d \in D$. If L contains a pentagon $\langle a, b, c, u, v\rangle$ and $x \vee v = y \vee v$, then $(x \wedge z) \vee v = (y \wedge z) \vee v$.*

Proof. Assume that $(x\wedge z)\vee v \neq (y\wedge z)\vee v$. Let θ be the smallest congruence relation that identifies a and b. By Lemma 4.6, L/θ is distributive, so that $[(x\wedge z)\vee v]_\theta = ([x]_\theta \wedge_\theta [z]_\theta)\vee_\theta [v]_\theta = ([x]_\theta \vee_\theta [v]_\theta)\wedge_\theta ([z]_\theta \vee_\theta [v]_\theta) = [(x\vee v)\wedge(z\vee v)]_\theta = [(y\vee v)\wedge(z\vee v)]_\theta = [(y\wedge z)\vee v]_\theta$. Since $(x\wedge z)\vee v\ \theta\ (y\wedge z)\vee v$, we have $\{(x\wedge z)\vee v, (y\wedge z)\vee v\} = \{a, b\}$ and we may assume that $(x\wedge z)\vee v = a$ and $(y\wedge z)\vee v = b$. Since $a \nleqslant b$, we have $x \nleqslant y$. Note that $x\vee a = x\vee v = y\vee v = y\vee b$. Since $a \leqslant x\vee a = y\vee b$ and $a \nleqslant b$, we have $y \nleqslant b$. Since $x \leqslant x\vee a = y\vee v$ and $x \nleqslant y$, we have $b \nleqslant y$. Thus, $y \parallel b$. Since $b \leqslant y\vee b = y\vee v \leqslant y\vee c$ and $b \nleqslant c$, we have $y \nleqslant c$. Since $y < y\vee b = y\vee v \leqslant y\vee c$, we have $c \nleqslant y$. Hence, $y \parallel c$, so that $\{y, b, c\} \in \pi_L^3$. Since $b\wedge y < b$ and $a\wedge y\ \theta\ b\wedge y$, we have $a\wedge y = b\wedge y$. Since $a \leqslant x\vee a = y\vee v \leqslant y\vee c$ and L is π-distributive, we have $a = a\wedge(y\vee c) = (a\wedge y)\vee(a\wedge c) = (b\wedge y)\vee(b\wedge c) \leqslant b$, contradicting $b < a$. Therefore, $(x\wedge z)\vee v = (y\wedge z)\vee v$. $\square$

Lemma 4.8. *Let L be a subdirectly irreducible π-distributive lattice. If L contains a pentagon $\langle a, b, c, u, v\rangle$, then $u = 1$ and $v = 0$.*

Proof. Let L be a subdirectly irreducible π-distributive lattice with a pentagon $\langle a, b, c, u, v\rangle$. Assume that $u \neq 1$ or $v \neq 0$. By duality, we may assume $v \neq 0$. Define a relation α on L by $x\ \alpha\ y$ iff $x\vee v = y\vee v$. It is easy to see that α is an equivalent relation such that for any $x, y, z \in L$ with $x\ \alpha\ y$, we have $x\vee z\ \alpha\ y\vee z$ and, by Lemma 4.7, $x\wedge z\ \alpha\ y\wedge z$, so that α is a congruence relation. Let θ be the smallest congruence relation that identifies a and b. By Lemma 4.6, θ identifies only a and b. Note that, since $a\vee v = a \neq b = b\vee v$, we have $a \not\alpha\ b$, so that $\theta \neq \alpha$. Let β be a congruence relation with $\beta \subseteq \alpha$ and $\beta \subseteq \theta$, and suppose that $c, d \in L$ with $c\ \beta\ d$. Since $c\ \theta\ d$, we have $c = d$ or

$\{c, d\} = \{a, b\}$. Since $c \;\alpha\; d$, we have $c = d$. Therefore, β is equality, which implies that L is not subdirectly irreducible, contradicting the assumption. □

Theorem 4.9. *Let L be a subdirectly irreducible lattice with $L \neq N_5$. Then L is distributive iff L is π-distributive.*

Proof. Let L be a subdirectly irreducible π-distributive lattice with $L \neq N_5$. By Comment 3.14, L satisfies $D_{0\pi}$, and by Lemma 4.5, L is almost distributive, so that $L \cong D[d]$ for some distributive lattice D and $d \in D$. Assume that L is not distributive, so that L contains a sublattice isomorphic to M_3 or N_5. Since M_3 does not satisfy $D_{0\pi}$, L contains a pentagon $\langle a, b, c, u, v \rangle$. By Lemma 4.8, $u = 1$ and $v = 0$. Since $L \neq N_5$, there exists $e \in L$ such that $e \notin \{a, b, c, 0, 1\}$. We have $a \not< e$, for if $a < e$, then $\langle a, b, c \wedge e, b \vee (c \wedge e), 0 \rangle$ is a pentagon, so that, by Lemma 4.8 again, $1 = b \vee (c \wedge e) \leqslant e$, contradicting $e \neq 1$. Similarly, $e \not< b$. Since L is subdirectly irreducible, $e \notin [b, a]$, so that $e \parallel a$ and $e \parallel b$. Since $\{a, b, e, a \vee e, a \wedge e\}$ is a pentagon, by Lemma 4.8, $a \vee e = b \vee e = 1$ and $a \wedge e = b \wedge e = 0$. Let θ be the smallest congruence relation that identifies a and b. By Lemma 4.6, θ identifies only a and b. Since $[c]_\theta = [c \wedge (a \vee e)]_\theta = [(c \wedge a) \vee (c \wedge e)]_\theta = [c \wedge e]_\theta$, we have $c \;\theta\; c \wedge e$, so that $c = c \wedge e$. Similarly, $c = c \vee e$, so that $c = e$, contradicting the assumption. □

Theorem 4.10. *Let L be a subdirectly irreducible lattice. Then L is almost distributive iff L satisfies $D_{0\pi}$.*

Proof. By Lemma 4.5, $D_{0\pi}$ implies almost distributivity. We now prove the sufficiency. Let L be a subdirectly irreducible almost distributive lattice. There exists a distributive lattice D and an element $d \in D$ such that $L \cong D[d]$. Notice that the order-skeleton $\widetilde{L} \cong \widetilde{D[d]} \cong \widetilde{D}$ is distributive, by Corollary 3.15, L satisfies $D_{0\pi}$. □

In [7], Erné introduced n-zipper-distributivity and the conditions of H_n where $n \geq 3$. It turns out that the π-version of n-zipper-distributivity is also equivalent to n-zipper-distributivity. In Figure 3, we present a diagram indicating the implications between the various weakened distributive conditions discussed above. We observe that nothing collapses except the five π-versions of distributivity discussed in Corollary 3.15 even if a lattice L is isomorphic to its own order-skeleton. Details can be found in [17].

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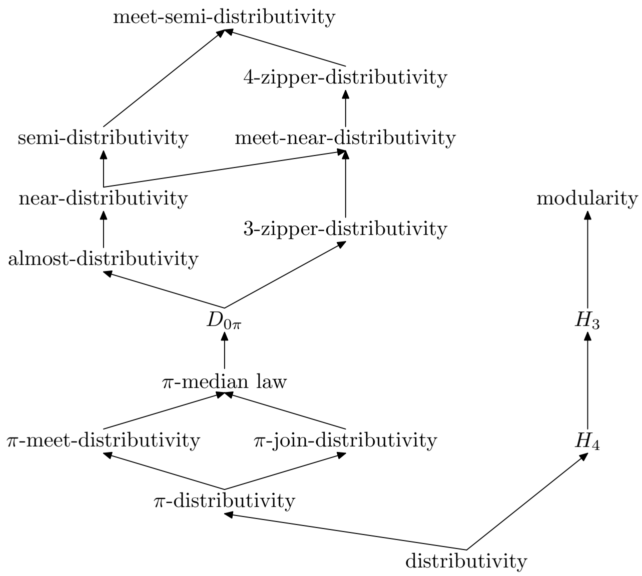

Figure 3. The implications between various weakened distributive conditions.

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