5-regular simple planar graphs and D-operations

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Our goal is to prove a generating theorem for the class $\mathcal{E}_5$ of all 5-regular simple planar graphs. This is a progress report. First, we will see the general information from Euler's formula and the Discharge Method. Second, the basic graph operation *D-operation* will be introduced. Third, there are two cases to be discussed separately. A graph either contains an edge that is a part of three distinct triangles or does not contain such an edge. Fourth, list irreducible subgraphs under D-operation for each case. Then, we investigate irreducible subgraphs under "multiple D-operations" for each case.

The minimal graph in $\mathcal{E}_5$ is unique and is isomorphic to the graph of Icosahedron I by Euler's formula. Let W be a graph obtained from the wheel W_5 by removing an edge from the rim. We can prove that each graph in $\mathcal{E}_5$ contains W as a subgraph by the Discharge Method.

Now, we will define a graph operation called *D-operation*, which is an H-type operation in Toida [1]. Let G be a graph in $\mathcal{E}_5$ in this article. A D-operation will be applied to adjacent vertices $\{x, y\}$ or to an edge xy in G. Applying a D-operation to adjacent vertices $\{x, y\}$ is to delete $\{x, y\}$ from G and then reconnect neighbors of $\{x, y\}$ so that a resulting graph is 5-regular. Note that applying *D-operation* to $\{x, y\}$ is not unique. If you can obtain a resulting graph in $\mathcal{E}_5$ again, then we say that G is *reducible* by $\{x, y\}$, or the edge $\{x, y\}$ is *reducible.* If any adjacent vertices in G is not reducible, then we say that G is *irreducible* (under D-operation).

If G contains an edge xy that is a part of three distinct triangles, then we call xy a 3Δ *edge* and we say that G *is in the* 3Δ *case.* Otherwise, we call G *non-*3Δ *case* or 2Δ *case* since G contains W.

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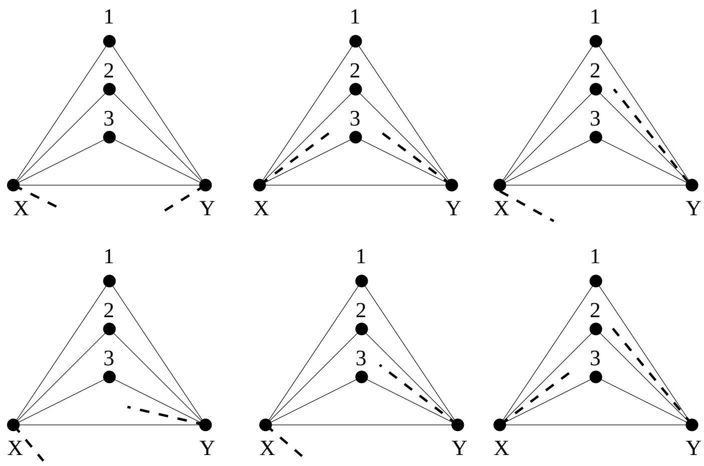
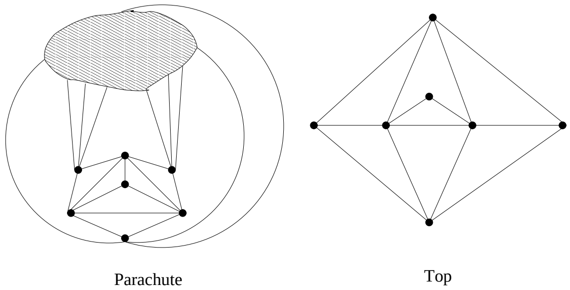
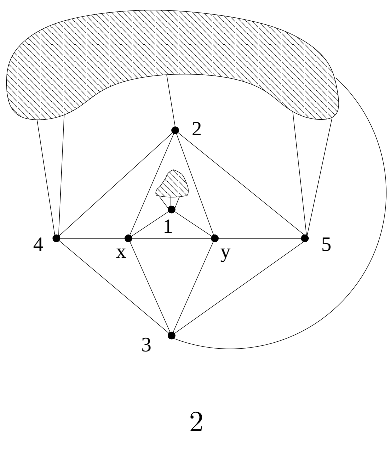


If G is in the 3Δ case, there are six different embeddings for a 3Δ edge xy and edges incident with x and y (see above). After a careful checking each of six cases, we conclude the following results where Oct is the graph of the octahedron.

Theorem 0.1 *(J. Kanno and N. Richardson) If G is in the 3Δ case and is irrecucible under D-operation, then G contains a subgraph isomorphic to Oct, the Parachute, or the Top (see below).*

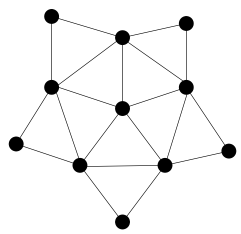


Theorem 0.2 *(N. Richardson) If G contains the Top with no edge between 1 and 2 (see below), and G is irreducible under D-operation, then $G \backslash S$ has exactly two components where $S = \{1, 2, 3, 4, 5, x, y\}$.*

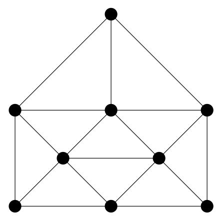


On the other hand, if G is in the 2Δ case, we have following results and conjectures. Let BD be the Birkhoff Diamond, which contains exactly two vertices of degree three, say v_1 and v_2. Then, let BD^* be the graphs obtained from BD by deleting v_1 or v_2.

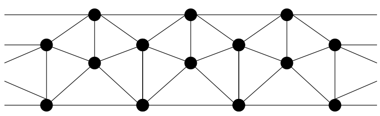
Theorem 0.3 *If G is in the 2Δ case, then G contains a subgraph isomorphic to the Lone Star, BD^*, a W_5-ladder, or an alternative W_5-ladder (see below).*



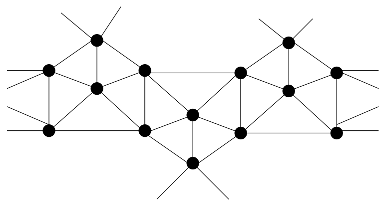
Lonestar



BD*

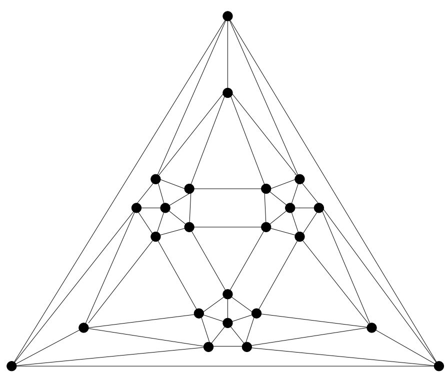


W_5 Ladder



Alternative W_5 Ladder

We define the cyclic n-W_5-ladder and the cyclic n-alternative-W_5-ladder for $n \geq 3$. Note that the cyclic 3-W_5-ladder is the graph of Icosahedron I and the cyclic 3-alternative-W_5-ladder in the next figure.



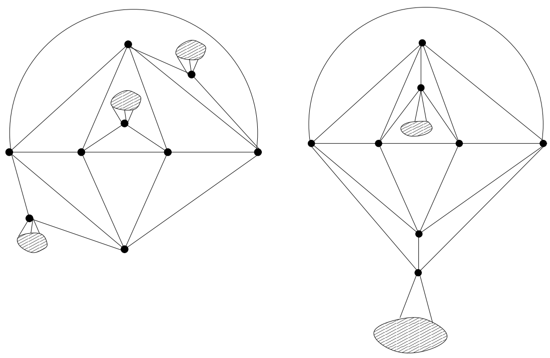
3−cyclic Alternative W_5 Ladder

Conjecture 0.1 *If G is in the 2Δ case and is irreducible under D-operation, then G is isomorphic to a cyclic n-W_5-ladder, or a cyclic n-alternative-W_5-ladder.*

It is worth knowing what would happen around a cut vertex. Notice that if an edge xy is a 1-edge-cut, then $\{x, y\}$ is reducible by D-operation. Since G is simple and every vertex is of degree 5, every cut vertex v in an irreducible graph is incident with a 2-edge-cut. Christopher Brandt proved the following about a cut vertex in G.

Theorem 0.4 *(C. Brandt) If G is irreducible under D-operation and G contains a cut vertex v, and $G\backslash v$ contains a component A connected to v by a 2-edge-cut such that A contains no 3Δ edge and has no cut vertices in A, then A is isomorphic to $I\backslash e$.*

After finding irreducible subgraphs under the D-operation, a natural question came up, what would happen if we apply D-operation more than once. That is, even if multiple edges appear after applying D-operation the first time, you will be allowed to apply D-operation repeatedly so that a final resulting graph is in $\mathcal{E}_5$. Let n be a positive integer and let D^n-*operation* be an operation where we allow D-operation n times at one time and consider the consecutive n operations as one operation. By using D^n-operation, in some special cases in the next figure, the graph *Oct* can be reduced as follows.



Theorem 0.5 *If G contains Oct and can be identified with the special cases in the figure above, then Oct is reducible by a D^3-operation or D^4-operation, respectively.*

Theorem 0.6 *(C. Brandt) If G is isomorphic to a cyclic n-W_5-ladder with $n > 3$ or a cyclic n-alternative- W_5-ladder with $n \geq 3$, then G is reducible by a D^2-operation.*

References

[1] Toida, S., Construction of quartic graphs, *J. Combin. Theory Ser. B* **16** (1974), 124-133.