SPLITTER THEOREMS FOR 3- AND 4-REGULAR GRAPHS

A Dissertation

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Abstract

Let $\mathcal{G}$ be a class of graphs and $\leq$ be a graph containment relation. A splitter theorem for $\mathcal{G}$ under $\leq$ is a result that claims the existence of a set $\mathcal{O}$ of graph operations such that if G and H are in $\mathcal{G}$ and $H \leq G$ with $G \neq H$, then there is a decreasing sequence of graphs from G to H, say $G = G_0 \geq G_1 \geq G_2 \geq ... \geq G_t = H$, all intermediate graphs are in $\mathcal{G}$, and each G_i can be obtained from G_{i-1} by applying a single operation in $\mathcal{O}$.

The classes of graphs that we consider are either 3-regular or 4-regular that have various connectivity and girth constraints. The graph containment relation we are going to consider is the immersion relation. It is worth while to point out that, for 3-regular graphs, this relation is equivalent to the topological minor relation. We will also look for the minimal graphs in each family. By combining these results with the corresponding splitter theorems, we will have several generating theorems.

In Chapter 4, we investigate 4-regular planar graphs. We will see that planarity makes the problem more complicated than in the previous cases. In Section 4.5, we will prove that our results in Chapter 4 are the best possible if we only allow finitely many graph operations.

Chapter 1
Introduction

1.1 Introduction

We begin the dissertation by introducing some basic notations and results in graph theory. All concepts used but not defined in this dissertation can be found in D. West [25].

A *graph* G consists of a *vertex* set $V(G)$ and an *edge set* $E(G)$ where each edge is *incident* with two (possibly equal) vertices called *endpoints*. In particular, we allow the *empty graph*, which has no vertices. If $v, w \in V(G)$ are endpoints of an edge e, then we will write $e = vw$ and say that v and w are *adjacent*. If two edges e and f have a common endpoint, we say that e is *incident* with f or that e and f are *incident*. We allow that two or more than two edges have common endpoints. Then we call these edges, *multiple edges*. Also we allow edges with identical endpoints, which are called *loops*. If a graph G does not contain a loop, then we call G, *loopless*. We also call G *simple* if G is loopless and does not contain any multiple edges.

An *isomorphism* from G to H is a bijection $f : V(G) \to V(H)$ which preserves the adjacency of vertices. We say that "G is isomorphic to H," if there is an isomorphism from G to H. The proof of the following statement is easy and can be found in D. West [25].

Theorem 1.1.1. *Isomorphism is an equivalence relation.*

An *isomorphism class* of graphs is an equivalence class of graphs under the isomorphism relation. When discussing the structure of a graph G, we will only consider a fixed vertex set for G, but our comments apply to every graph isomorphic to G. When we define a graph by a picture, the picture is a representative of its isomorphism class. Also, when we know that two graphs are isomorphic, we often discuss them using the same name. For this reason, we write $G = H$ instead of writing "G is isomorphic to H."

The *degree* of a vertex v is the number of non-loop edges incident with v plus twice the number of loops incident with v. The minimum degree of a graph G is denoted by $\delta(G)$ and the maximum degree is $\Delta(G)$. A graph G is *regular* if $\delta(G)$ $=\Delta(G)$, and G is *r-regular* if $\delta(G) = \Delta(G) = r$. Other authers refer to 3-regular graphs as *cubic graphs*. We use $e(G)$ to denote the number of edges in G. Some authors call Theorem (1.1.2) *Handshaking Lemma*. It implies that there is no graph having an odd number of vertices of odd degrees. We can find a proof of (1.1.2) in D. West [25].

Theorem 1.1.2. *If G is a graph with vertex degrees $d_1, d_2, ..., d_n$, and $e(G)$ edges, then $\sum_{i=1}^{n} d_i = 2e(G)$.*

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When we say "H is a subgraph of G," we mean that H is isomorphic to a subgraph of G. If H is a subgraph of G and $G \neq H$, then we call H a *proper subgraph* of G. A subgraph H is called a *spanning subgraph* if $V(H) = V(G)$. An *induced subgraph* of G is a subgraph H such that every edge of G contained in $V(H)$ belongs to $E(H)$. If H is an induced subgraph of G with a vertex set X, then we write $H = G[X]$ and say that H is the subgraph of G "induced by X."

To *delete a vertex* $v \in V(G)$ from G, delete v together with the edges incident with v; we denote the resulting graph by $G-v$. When X is a subset of $V(G)$, *deleting vertex set* X from G, denoted by $G-X$, is defined by $G[\overline{X}]$, where $\overline{X} = V(G)-X$. Note that each induced subgraph H of G can be written as $G-(V(G)-V(H))$.

To *delete an edge* $e \in G$ from G, delete e from $E(G)$. Let T be an edge or a subset of $E(G)$. Then deleting T from G results in a graph obtained from G by eliminating T from $E(G)$, which is denoted by $G\backslash T$.

To *contract* an edge $e \in G$, replace both endpoints of e by a single vertex whose incident edges are all edges that were incident to the endpoints of e, except e itself. We denote the resulting graph by G/e. Visually, we think of contracting e as shrinking e to a single point. Contracting a set of edges $T \subseteq E(G)$ will be denoted by G/T.

A *complete graph* is a simple graph in which every pair of vertices forms an edge. We use K_n to denote a complete graph with n vertices, based on isomorphism classes (see (1.1.1)). In the following chapters, K_4 and K_5 play very important roles. Figure 1.1 shows two different drawings of K_4 and Figure 1.2 shows a drawing of K_5.

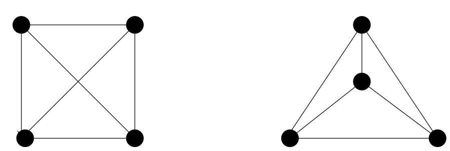

FIGURE 1.1. Two drawings of K_4.

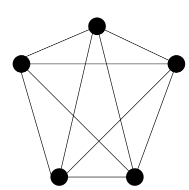

FIGURE 1.2. A drawing of K_5.

An *independent set* in a graph G is a vertex set $X \subseteq V(G)$ such that the induced subgraph $G[X]$ has no edges. A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. A *complete bipartite graph* is a bipartite graph in which the edge set consists of all pairs having a vertex from each of the two independent sets in the vertex set partition. We use $K_{l,m}$ to denote the complete bipartite graph with partite sets of sizes l and m. In Figure 1.3 there are two different drawings of $K_{3,3}$.

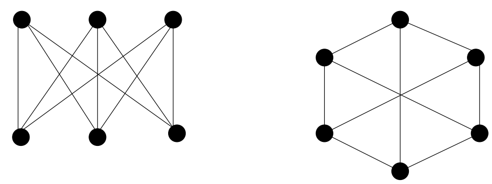

FIGURE 1.3. Two drawings of $K_{3,3}$.

A *walk of length k* is a sequence $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ is an edge for all i. A walk is *odd* or *even* if its length is odd or even, respectively. A *trail* is a walk with no repeated edge. A *path* is a walk with no repeated vertex. A *vw-walk* is a walk with first vertex v and last vertex w; these are its *endpoints*, and it is *closed* if $v = w$. A *cycle* is a closed trail of length at least one in which "first = last" is the only vertex repetition. We view closed walks and cycles as cyclic arrangements that can start at any vertex in the sequence. A loop is a cycle of length one. Every multiple edge is contained in a cycle of length two. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. A graph G is simple if and only if $g(G) \geq 3$.

The proofs of the following two statements, (1.1.3) and (1.1.4), are easy and can be found in D. West [25].

Theorem 1.1.3. *A graph is bipartite if and only if it has no odd cycles.*

Lemma 1.1.4. *Every closed odd walk contains an odd cycle.*

For vertex sets $X, Y \subseteq V(G)$, let $E(X, Y)$ be the set of all edges xy in G with $x \in X$ and $y \in Y$. We will use the following statement (1.1.5) in Section 4.5.

Lemma 1.1.5. *Let C be an odd cycle of graph G. If (X, Y) is a partition of $V(G)$, then $E(C)$ must contain odd number of edges in $E(G[X]) \cup E(G[Y])$.*

Proof. Note that we can partition $E(G)$ into three sets, $E(G[X])$, $E(G[Y])$, and $E(X, Y)$. There are two cases: $E(C)$ contains no edge of $E(X, Y)$ or $E(C)$ does. In the first case, (1.1.5) holds trivially. In the second case, $E(C)$ must contain an even number of edges in $E(X, Y)$ because C is closed. Then, (1.1.5) holds. □

In the rest of this section, we define four special graphs.

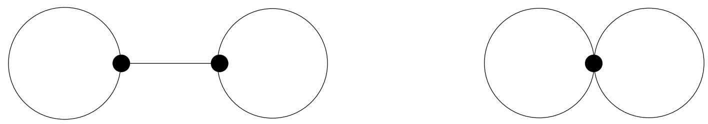

FIGURE 1.4. Drawings of K_2^L and $2L$.

The first two, K_2^L and $2L$, are shown in Figure 1.4. The other two graphs are $3K_2$ and $4K_2$ which consist of two vertices, say v and w, and of three and four multiple edges incident with v and w, respectively. Also, $3K_2$ and $4K_2$ are called 3-*linkage* and 4-*linkage* by some authors (see [24]). Moreover, $3K_2$ is called *theta graph* and denoted by θ in some papers (see [14]). However, note that the term *theta graph* and the symbol θ are used for a union of three internally disjoint paths with common ends (see [24]).

1.2 Connectivity and Girth

Connectivity is an important concept in graph theory and is strongly related to girth. We will focus on these properties in this dissertation. We will prove theorems for different connectivities and girths.

A graph G is *connected* if it has a vw-path for each pair $v, w \in V(G)$. Otherwise, we say that G is *disconnected*. The *components* of a graph G are its maximal connected subgraphs.

In this paper, we will consider two types of connectivity: vertex connectivity and edge-connectivity. Here, if we use just "connectivity", then we mean vertex connectivity.

There are well-known variants on the definitions of "k-connected" and "connectivity k" (see [23] and [26]). Roughly speaking, G is k-connected if G is connected and it can not be disconnected by deleting fewer than k vertices. This rough definition fails when G is a complete graph or can be obtained from a complete graph by adding edges because this case never produces a disconnected graph.

If G is connected and deleting a set of vertices from G results in a disconnected graph, then the set is called a *vertex cut*. If a vertex cut consists of one vertex v, then v is a *cut vertex*.

The *connectivity* of a graph G, denoted by $\kappa(G)$ is defined as follows.

(1) $\kappa(G) = 0$ if G is not connected.

(2) $\kappa(3K_2) = 2$.

(3) $\kappa(G) = |V(G)| - 1$ if $G \neq 3K_2$ and G contains a spanning complete graph.

(4) $\kappa(G) = j$ if G is connected, has a pair of non-adjacent vertices and j is the smallest integer such that G has a j-element vertex cut.

Note that connectivity is not affected by adding or deleting loops and multiple edges except $3K_2$, whenever it does not change the number of components. If k is a positive integer, then G is *k-connected* if $k \leq \kappa(G)$.

Compared to a definition of vertex connectivity, the definition of edge-connectivity is straightforward. For a positive integer k, a graph G is *k-edge connected* if it is connected and can not be disconnected by deleting fewer than k edges. We define G to be 0-edge connected if G is not connected. This definition is clear except for a graph having only one vertex. Since we study 3-regular and 4-regular graphs in this dissertation, it occurs only when the graph is $2L$.

An *edge-cut* is an edge set of the form $E(X, \overline{X})$, where X is a non-empty proper subset of $V(G)$ and $\overline{X} = V(G) - X$. We define an edge-cut to be the empty set if G is not connected. If the size of an edge-cut is $k(\geq 0)$, then we call the edge-cut a *k-edge-cut*. We also call a *cut edge* if $k = 1$. If every vertex of G has an even degree, then it follows from (1.1.2) that there is no odd edge-cut. This is a simple fact and very useful, especially when G is 4-regular in this paper. We state this as Lemma (1.2.1).

Lemma 1.2.1. *Every 4-regular graph does not contain an odd edge-cut.* □

Lemma (1.2.1) implies that every 1-edge connected 4-regular graph is 2-edge connected. Since $2L$ is connected (and hence 1-edge connected), it is natural for us to define it to be 2-edge connected.

The *edge-connectivity* of G, denoted by $\kappa'(G)$, is the maximum k such that G is k-edge connected. We define $\kappa'(2L) = 2$. Except in this special case, edge-connectivity is unaffected by adding or deleting loops, whenever it does not change the number of components. However, adding or deleting multiple edges does change edge-connectivity. The following inequality holds between connectivity and edge-connectivity, which was proved by H. Whitney in 1932.

Theorem 1.2.2. *If G is loopless, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.*

Note that both inequalities in (1.2.2) may be strict, but if G is a 3-regular simple graph, then the first inequality can be replaced by equality.

Lemma 1.2.3. *If G is a 3-regular simple graph, then $\kappa(G) = \kappa'(G)$.*

Proof. Since G is simple, we only need to show the lemma for the three cases, (1),(3) and (4), in the definition of connectivity above.

(1) If G is not connected, the lemma holds because $\kappa(G) = \kappa'(G) = 0$.

(3) Since G is simple and 3-regular, this case occurs only when $G = K_4$. From the definition, $\kappa(K_4) = 4-1 = 3$. By using (1.2.2), $\kappa'(K_4) = \kappa(K_4)$ because $\delta(K_4) = 3$.

(4) When G is connected and is not complete, let j be the smallest integer such that G has a j-element vertex cut, say Z. From the definition, $\kappa(G) = j$. Since each vertex in Z is of degree three, $G\backslash Z$ consists of either two or three components. By

(1.2.2), it is enough to show that G has a j-edge-cut. If $G\backslash Z$ has three components, say A, B and C, then each component must be connected with each vertex in Z because $\kappa(G) = j$. Thus $E(A,Z) = E(A, Z \cup B \cup C) = E(A, \overline{A})$ and $|E(A,Z)| = j$, which implies that G has a j-edge-cut.

Next, let $G\backslash Z$ consist of two components, A and B. For each vertex $z \in Z$, $|E(A,z)| = 1$ or 2. Let X be a union of $V(A)$ and all vertices of Z such that $|E(A,z)| = 2$. Therefore, if $z \notin X$, then $|E(z,X)| = 1$. Also, if $z \in X$, then $|E(z,\overline{X})| = 1$. Moreover, each edge of $E(X,\overline{X})$ is incident with a vertex of Z, which implies $|E(X,\overline{X})| = j$. □

The following lemmas will be very useful in this paper.

Lemma 1.2.4. *If $\kappa'(G) = t$ and T is a t-edge-cut with $t > 0$, then $G\backslash T$ consists of two components.*

Proof. If $G\backslash T$ consists of more than two components, then G has an edge cut consisting of fewer than t edges. Then G is not t-edge connected, which contradicts $\kappa'(G) = t$. □

Lemma 1.2.5. *If a connected 4-regular graph G contains a cut vertex v, then $G-v$ consists of two components.*

Proof. Suppose $G-v$ consists of more than two components. Since v is of degree four, it implies G contains a 1-edge-cut, which contradicts (1.2.1). □

The following theorem relates edge-connectivity and edge-disjoint paths, which is proved by K. Menger in 1927.

Theorem 1.2.6. *A graph G is k-edge connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.*

This theorem implies that contractions do not decrease edge connectivity because contractions only make paths shorter, while contractions decrease girth.

We study 3-regular and 4-regular graphs under different connectivity and edge connectivity, respectively. Note that if G is 3-regular, then $\kappa(G) \leq g(G)$. Hence 3-regular 3-connected graphs are simple. Also if G is 4-regular, then $\kappa'(G) \leq 2g(G)$ holds.

1.3 Topological Minor and Immersion

In this section, we define several graph containment relations, including the minor, the topological minor and the immersion relation.

A graph M is called a *minor* of a graph G if M can be obtained from G by a finite sequence of deletions and contractions.

Suppose H is a graph with $\delta(H) > 0$.

A graph H' is a *subdivision* of H if $V(H) \subseteq V(H')$ and there exists a family $(H'_e)_{e \in E(H)}$ of subgraphs of H' such that

(1) if $e \in E(H)$ joins two distinct vertices v, w,

then H'_e is a vw-path and $V(H'_e) \cap V(H) = \{v, w\}$,

(2) if $e \in E(H)$ is a loop incident with a vertex v,

then H'_e is a cycle and $V(H'_e) \cap V(H) = \{v\}$,

(3) for every pair e, f of distinct edges of H,

$V(H'_e) \cap V(H'_f) \subseteq V(H)$ and $E(H'_e) \cap E(H'_f) = \emptyset$,

(4) $H' = \bigcup_{e \in E(H)} H'_e$.

Also, a graph is a *pseudo-subdivision* of H if "$V(H'_e) \cap V(H'_f) \subseteq V(H)$ and" is omitted from condition (3) (see [15]). In this case, replace "vw-path" in (1) and "cycle" in (2) by "vw-trail" and "closed trail", respectively.

We say that a graph H is a *topological minor* of G if a subgraph of G is a subdivision of H, and that H is *immersed* in G if a subgraph of G is a pseudo-subdivision of H. Note that if H is 3-regular, then these containment relations are equal. In Chapter 2, H'_e is called an edge-path, denoted by P_e. In Chapter 3 and

Chapter 4, H'_e corresponds to an edge-trail or a redtrail, denoted by T_e or T_{vw} with $v, w \in V(H)$. We write $H \preceq G$ if H is a topological minor of G. We also write $H \prec G$ if $H \preceq G$ and $H \neq G$. We write $H \propto G$ if H is immersed in G.

Note that we have another equivalent definition for each concept. Suppose H is a topological minor of G. Then H can be obtained from G by a finite sequence of deletions and contractions of edges incident with vertices of degree two. When G and H are 3-regular, we can define a graph operation $\mathcal{R}$, which is a combination of a deletion and contractions of edges incident with vertices of degree two, and we can use it as an alternative definition of "topological minor." That is, Theorem (2.2.3) tells us that H is a topological minor of G if and only if H can be obtained from G by applying a sequence of $\mathcal{R}$. Also see Section 1.5.

To present an equivalent definition for immersion, we will define a new concept. Let (E_1, E_2) be a partition set of the edges incident with a vertex v. Then to *split* v (or to apply *vertex-splitting* to v), replace v by two new vertices v_1 and v_2 so that E_1 and E_2 are incident with v_1 and v_2, respectively. Visually, the vertex v was split to two vertices v_1 and v_2, and edges which were incident with v are incident with both of the new vertices. Suppose H is immersed in G. Then H can be obtained by applying a finite sequence of deletions, vertex-splittings, and contractions of edges incident with vertices of degree two. When G and H are 4-regular, we will introduce a graph operation Sp, which is a combination of vertex-splittings and contractions of edges incident with vertices of degree two, and we will show that we can use Sp to define immersion. Lemma (3.2.4) implies that H is immersed in G if and only if H can be obtained from G by applying a finite sequence of Sp.

Related topics and applications about immersion can be found in [5] and [6].

1.4 Planar Graphs

Here, we will investigate a little about topology. We will define the well-known concept of *drawing*. Then, we will define the concept of planarity and introduce some basic results used in Section 4.5.

A *polygonal curve* in the plane is a union of finitely many line segments. In a polygonal vw-curve, the beginning of first segment is v and the end of the last segment is w. An *open set* in the plane is a set $U \in \mathbb{R}^2$ such that for every $p \in U$, there is an ε-neighborhood of p belongs to U. A *region* is an open set U that contains a polygonal vw-curve for every pair $v, w \in U$. The *faces* of a plane graph are the maximal regions of the plane that are disjoint from the drawing. A curve in the plane is *closed* if its first and last points are the same, and it is a *simple curve* if it does not otherwise intersect itself. The following theorem (1.4.1) is famous in topology, called Restricted Jordan Curve Theorem.

Theorem 1.4.1. *A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.*

A *drawing* $D(G)$ of a graph G is the following realization of G in the plane: The vertices of G are different points in the plane, and edges between two vertices are polygonal curves between the corresponding points in such a way that two curves have at most one point in common, either an endpoint or a point of intersection, called *crossing* (see [8]). Note that any drawing of a graph does not contain any touching point.

A graph G is *planar* if it can be drawn in the plane without crossings. A *plane* graph is a particular drawing of a planar graph in the plane with no crossings. Here is an important characterization of planar graphs proved by K. Kuratowski in 1930. In Section 1.1, Figure 1.2 shows K_5 and Figure 1.3 shows $K_{3,3}$.

Theorem 1.4.2. *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

By Theorem (1.4.2), any drawing of K_5 has a crossing. Note that each drawing could contain a crossing but no touching. To obtain a *pinched graph* of a graph G, choose a drawing $D(G)$ of G and replace each crossing with a vertex. The resulting graph will be denoted by $\{D(G)\}^P$ or G^P. The replaced vertex will be called a *crossing vertex*. In other words, each vertex in $V(G^P) - V(G)$ is a crossing vertex because no touching in $D(G)$ or G^P. Note that pinched graphs depend on drawings and are planar by definition.

Notice that "pinching" is an inverse operation of "splitting" (see Section 1.3). Therefore, G is immersed in G^P because G can be obtained from G^P by splitting crossing vertices. For example, in Figure 4.4 (see Section 4.5), the graph G_n is a pinched graph of H_n and hence H_n is immersed in G_n.

The *length* of a face α in a plane graph G is the length of a minimum closed walk in G that bounds α. By the definition of region, a finite plane graph G has one unbounded face.

Lemma 1.4.3. *If every bounded face of G has even length, then the unbounded face of G has even length.*

Proof. If we sum the length of bounded faces, then we obtain an even number, because each bounded face length is even. This sum counts each edge of the unbounded face once. Each edge separating bounded faces gets counts twice, since each such edge is incident with two bounded faces. Hence, the unbounded face of G has even length. □

Lemma 1.4.4. *A plane graph G is bipartite if and only if every face of G has even length.*

Proof. Suppose G is bipartite and has a face α having odd length. Hence, the face α is bounded by a minimum closed walk C in G having odd length. By (1.1.4), the closed walk C must contain an odd cycle, which contradicts (1.1.3). Conversely, suppose that every face of G has even length and there is an odd cycle C in G. Since G has no crossings in the plane, C is laid out as a simple closed curve. Let F be the region enclosed by C. Delete all vertices except vertices in $F \cup C$ and call the resulting graph G'. Then, C bounds the unbounded face in G'. By (1.4.3), C can not be odd. □

1.5 Graph Operations

In section 1.3, we introduced some containment relations including topological minor and immersion. The role of graph operations in this dissertation is strongly related with these containment relations, almost they are equivalent. Most graph operations consist of deletions, contractions and their combinations.

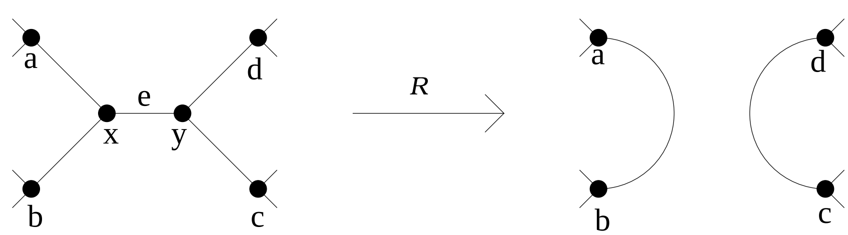

FIGURE 1.5. The operation $\mathcal{R}$.

The Figures 1.5 and 1.6 show a graph operation $\mathcal{R}$ for 3-regular graphs and another graph operation Sp for 4-regular graphs. We use the planar splitting PS for 4-regular plane graphs, which is the same as Sp except omit the "cross splitting."

The operation $\mathcal{R}$ will be applied to an edge, say e, of a 3-regular graph, and e and the edges incident with the endpoints of e will be changed. We may extend the definition of $\mathcal{R}$ to include $O_0(K_2^L)$, $O_0(3K_2)$, $O_1(L)$, and $O_2(3K_2)$ (see Section 2.2). Then we can conclude an important relation between topological minor and

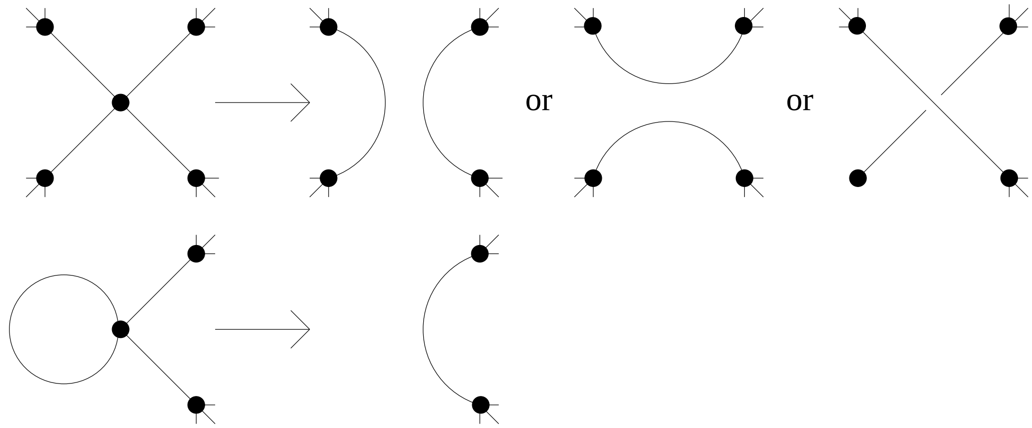

FIGURE 1.6. The operation Sp.

$\mathcal{R}$ by Theorem (2.2.3). For 3-regular graphs, H is a topological minor of G if and only if H can be obtained from G by applying a sequence of $\mathcal{R}$.

On the other hand, the operation Sp or PS will be applied to a vertex, say v, of a 4-regular or 4-regular planar graph unless $v \in 2L$, respectively. Note that after applying Sp or PS, the number of vertices and edges of G decreases by one and two, respectively. Applying Sp to v in a loop results in a unique graph, but in general, it is not unique. Observe that we can obtain the same resulting graph from $G - v$ by adding suitable one or two edges. Hence, there are at most three different resulting graphs because we can choose two pairs from four if v is not in a loop. We can deduce by Lemma (3.2.4) that for 4-regular graphs, H is immersed in G if and only if H can be obtained from G by applying a sequence of Sp.

We will use several (a finite number of) other graph operations. They are expressed by $O_i(K)$ with $i = 0, 1, 2, 3, 4$ and a special 3- or 4-regular graph K. The graph operations $O_i(K)$ will be applied to an induced subgraph S in G, which is a component of $G\backslash T$ for an i-edge-cut T. Thus if $i = 0$, then $S = K$.

Let us explain about $O_i(K)$ more detail for each $i = 1, 2, 3, 4$. If i is odd, then by (1.2.1), the graph operation $O_i(K)$ cannot be applied to 4-regular graphs. Let

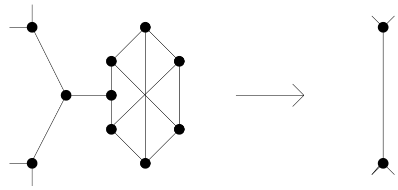

FIGURE 1.7. The operation $O_1(K_{3,3})$.

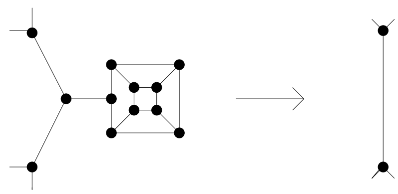

FIGURE 1.8. The operation $O_1(Q)$.

L be a loop. The operation $O_1(L)$ is defined by Figure 2.1 in Section 2.2. If $i = 1$ and $K \neq L$, an applied graph G has a cut edge e_1 such that $G \backslash e_1$ contains a component S, which is a subdivision of K. For example, see Figure 1.7 and 1.8, where $K = K_{3,3}$ and Q (the 3-cube), respectively.

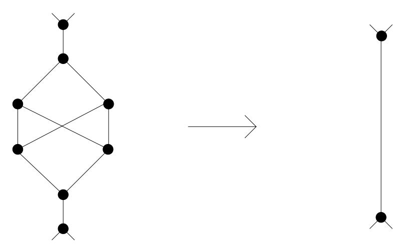

FIGURE 1.9. The operation $O_2(K_{3,3})$.

If we can apply $O_2(K)$ to an induced subgraph S in G, then $S = K \backslash e$ and applying $O_2(K)$ to S results in $G/E(S)/e_1$, where e_1 is in a 2-edge-cut T above. In Figure 1.9 and 1.10, $K = K_{3,3}$ and the 3-cube, respectively.

If we can apply $O_3(S)$ and $O_4(S)$ to an induced subgraph S in G, then $S = K - v$ and the resulting graph is $G/E(S)$. See Figure 1.11, 1.12, 1.13 and 1.15.

Note that we will see that if $O_i(K)$ is for 3-regular graphs, then it can be replaced by applying successive $\mathcal{R}$, and if it is for 4-regular or 4-regular planar graphs, then it can be replaced by applying a sequence of Sp or PS, respectively.

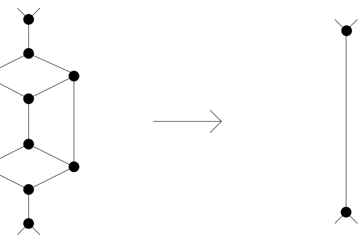

FIGURE 1.10. The operation $O_2(Q)$.

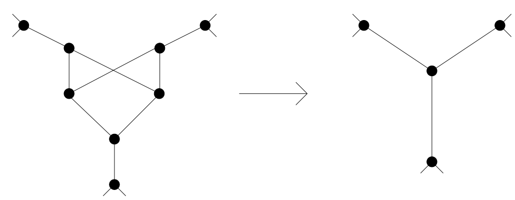

FIGURE 1.11. The operation $O_3(K_{3,3})$.

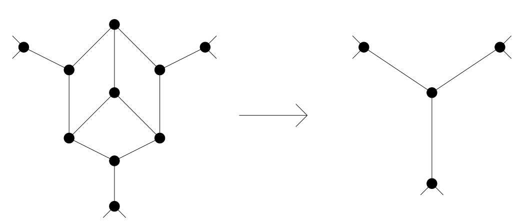

FIGURE 1.12. The operation $O_3(Q)$.

1.6 Splitter Theorems and Known Results

In this section, we explain the concept of a splitter theorem and mention some known splitter theorems. Also, we will discuss the obvious application called a generating theorem and introduce known results.

Suppose a graph G "contains" another graph H. Then how can G be built up from H in such a way that certain properties of G and H are preserved during the construction process? Probably the best-known result to answer this kind of question is the one by P. Seymour [19], for general matroids, and S. Negami [16], for graphs only, which explains the construction when the containment relation is the minor relation and the property to preserve is the 3-connectedness. These results are known as splitter theorems. Other splitter theorems can be found in [10], [11], [18] and [22]. In this dissertation, we will investigate splitter theorems for 3-regular graphs and 4-regular graphs under the immersion containment. Note that, for 3-regular graphs, topological minor and immersion are equivalent.

Let us clarify what is meant by "building G from H while maintaining certain properties." In fact, we will talk about *reducing* G to H, which will be equivalent to building G from H, yet it is much more convenient for stating our results. Suppose both G and H belong to a family $\mathcal{G}$ of graphs. Then we say that G can be *reduced to H within $\mathcal{G}$ by a set $\mathcal{O}$ of graph operations* if there is a sequence $G_0, G_1, ..., G_t$ of graphs in $\mathcal{G}$ such that $G_0 = G$, $G_t = H$, and each G_i is obtained from G_{i-1} by applying a single operation in $\mathcal{O}$. Moreover, in the sequence, $G_i \propto G_{i-1}$ holds for each i, and so $H \propto G_i \propto G$. Under this terminology, a *splitter theorem* is a result that claims the existence of $\mathcal{O}$ such that every $G \in \mathcal{G}$ can be reduced within $\mathcal{G}$ by $\mathcal{O}$ to any $H \in \mathcal{G}$ if $H \propto G$.

Let $\mathcal{G}(H)$ be the class of all graphs G in $\mathcal{G}$ with $H \propto G$. In the literature, a splitter theorem has the equivalent formulation, that there exists a set $\mathcal{O}$ of graph operations for which if both G and H are in a class $\mathcal{G}$ of graphs and $H \propto G$, then G can be *reduced within* $\mathcal{G}(H)$ by $\mathcal{O}$, that is, an operation in $\mathcal{O}$ can be applied to G such that the resulting graph belongs to $\mathcal{G}(H)$. It is clear that this formulation is implied by the first, while the first can also be proved by repeatedly using the second (as long as every operation in $\mathcal{O}$ always results in a graph of fewer edges). In this dissertation, all splitter theorems will be stated using the first formulation but be proved using the second.

A *generating theorem* for a certain family $\mathcal{G}$ of graphs tells us how to construct all the members of $\mathcal{G}$ from a set of graphs by using a set of graph operations. Ideally, the set of graphs and the set of graph operations are small. Suppose we have a splitter theorem for $\mathcal{G}$ under a containment relation. Then, if we can determine the minimal graphs in $\mathcal{G}$ with respect to the containment relation, we have a generating theorem because every graph G in $\mathcal{G}$ contains a minimal graph M and G can be reduced to M by the splitter theorem. By tracking the opposite direction of reduction, we have a generating theorem for $\mathcal{G}$.

Corollary (2.3.14) is a generating theorem for 3-regular 3-connected graphs. This result was first proved by W. Tutte in [23] in 1961, and by E. Steinitz and H. Rademacher for planar graphs (see [7] and [20]) in 1934. N. Wormald [26] first proved generating theorems for 3-regular connected simple graphs and for 3-regular 2-connected simple graphs in 1979. We will prove these as corollaries of the splitter theorems (2.3.10) and (2.3.11), respectively. It is also well known that E. Johnson gave another construction for 3-regular connected simple graphs by using a different operation called H-reduction in [9] and [17].

In 1974, S. Toida [21] showed that all 4-regular connected simple graphs can be generated from K_5 by H-type and V-type expansions, where H-type expansion is generalized from E. Johnson's work in [17].

In 1981, F. Bories, J-L. Jolivet, and J-L. Fouquet [2] showed that all 4-regular connected simple graphs can be generated by three extensions from K_5. We will obtain the same result as a corollary of Theorem (3.4.6).

Comparing these two results by S. Toida and by F. Bories and others, we can say the second one has a nice property, which is because each of the three operations in the second can keep a certain containment relation during the construction process, but H-type expansions in the first can not.

In 1979, P. Manca [13] began to show how to generate all 4-regular simple planar graphs from the octahedron by using some graph operations, and J. Lehel [12] completed this work in 1981. This is a consequence of Corollary (4.4.14). Also, in 1993, H. Broersma, A. Duijvestijn, and F. Göbel [3] showed how to generate all 4-regular 3-connected simple planar graphs from the octahedron by using some graph operations in such a way that all intermediate graphs are 4-regular 3-connected simple planar graphs. They used different operations from ours. This relates Corollary (4.4.16) because 4-regular 3-connected graphs are 4-edge connected by (1.2.1) and (1.2.2).

1.7 Main Results

Finally, here are the main results of this dissertation. We divide our results in three groups, and three chapters.

First, in Chapter 2, let $\Gamma_{k,g}$ be the family of 3-regular k-connected graphs with girth at least g. We prove splitter theorems for $\Gamma_{k,g}$, for $k = 0, 1, 2, 3$ and $g = 1, 2, 3, 4$. We show all the proofs except the splitter theorem for $\Gamma_{3,3}$, which is a consequence of a theorem in A. Kelmans [10]. In addition, we will also determine the $\preceq$-minimal graphs in each $\Gamma_{k,g}$. Then, combining with the corresponding splitter theorems, we will obtain results on how to generate all graphs in each family $\Gamma_{k,g}$ from each set of minimal graphs.

The following statements are the most difficult to prove in Chapter 2 and will have many applications. Let K be either $K_{3,3}$ or 3-cube Q in the following statements. The operations $O_i(K)$ with $i = 1, 2, 3$ are in Figure 1.7, 1.8, 1.9, 1.10, 1.11 and 1.12.

Theorem 2.4.17. *If G and H are in $\Gamma_{k,4}$ and $H \preceq G$, then G can be reduced to H within $\Gamma_{k,4}$ by applying a sequence of $\mathcal{R}$ and $O_i(K)$, where $i = 0, 2, 3$ for $k = 0$, and $i = 1, 2, 3$ for $k = 1$, and $i = 2, 3$ for $k = 2$, and $i = 3$ for $k = 3$.*

Let $\mathcal{O}_k$ be the set of operations used for each $k = 0, 1, 2, 3$ in Theorem (2.4.17).

Corollary 2.4.18. *Every 3-regular graph in $\Gamma_{k,4}$ can be reduced to $K_{3,3}$ or Q within $\Gamma_{k,4}$ by applying $\mathcal{O}_k$.*

Next, we consider 4-regular graphs and 4-regular planar graphs in Chapter 3 and Chapter 4, respectively. Let $\Phi_{k,g}$ and $P\Phi_{4,3}$ be the family of 4-regular k-edge connected graphs with girth at least g and the family of planar graphs in $\Phi_{k,g}$, respectively.

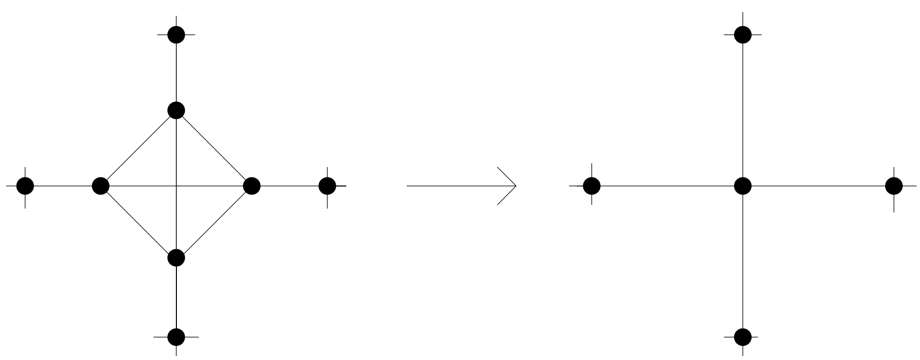

FIGURE 1.13. The operation $O_4(K_5)$.

Theorem 3.4.8. *If G and H are in $\Phi_{4,3}$, and $H \propto G$, then G can be reduced to H within $\Phi_{4,3}$ by applying a sequence of Sp and $O_4(K_5)$.*

We will see that K_5 is the unique $\propto$-minimal graph in $\Phi_{4,3}$. Then, by (3.4.8), the following corollary holds.

Corollary 3.4.9. *Every 4-regular 4-edge connected simple graph can be reduced to K_5 within $\Phi_{4,3}$ by applying a sequence of Sp and $O_4(K_5)$.*

Finally, we will study 4-regular planar graphs in Chapter 4, and note that an immersed graph H in a plane graph G is not necessarily planar. Recall that a pinched graph H^P of H is obtained from a drawing of H by replacing each crossing with a vertex and is planar. Splitter theorems for $g = 1, 2$ can be proved by the same arguments in Chapter 3, but for simple graphs, we need more preparations to prove the splitter theorems. We will introduce a splitter theorem for $P\Phi_{4,3}$ and in the following we will describe a family of minimal graphs in $P\Phi_{4,3}$. Figure 1.14 shows us two of them and we will denote the octahedron by *Oct*. Also see Figure 1.15 for the operation $O_4(Oct)$.

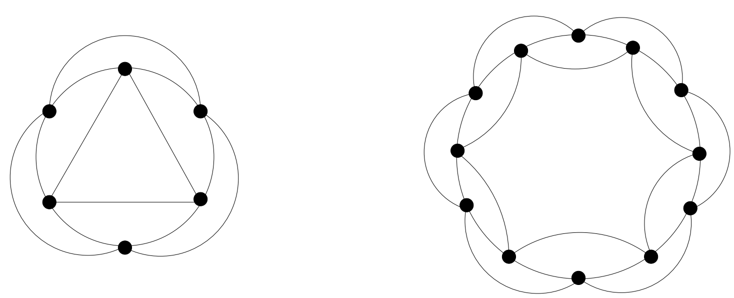

FIGURE 1.14. The octahedron and C_{12}^2.

Let $n \geq 3$. The *square* of an even cycle C_{2n}, denoted by C_{2n}^2 is the graph obtained from C_{2n} by connecting every two vertices two apart. The smallest graph this kind is the octahedron. We call C_{2n}^2 a *cyclic ladder* for each $n \geq 3$. Figure 1.14 shows the octahedron, C_6^2, and C_{12}^2. Figure 1.15 shows the graph operation called $O_4(Oct)$.

Theorem 4.4.15. *If $G \in P\Phi_{4,3}$, $H \in \Phi_{4,3}$, and $H \propto G$, then G can be reduced to H^P within $P\Phi_{4,3}$ by applying a sequence of PS and $O_4(Oct)$ without increasing the number of crossings, unless G is isomorphic to a cyclic ladder.*

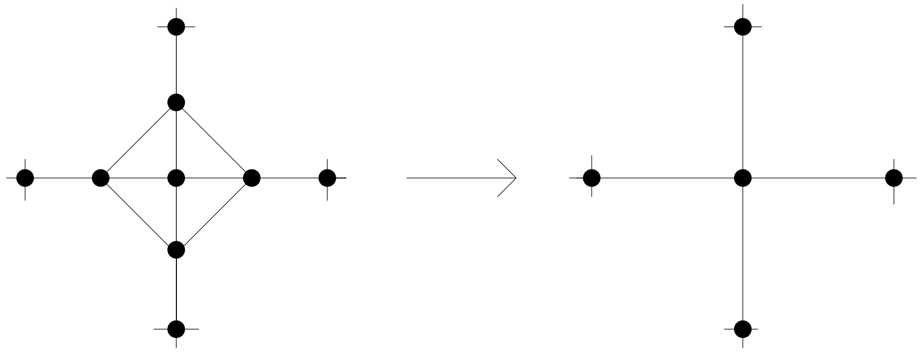

FIGURE 1.15. The operation $O_4(Oct)$.

Notice that if H is also a plane graph in Theorem (4.4.15), we can replace H^P by H because (4.4.15) guarantees we can reduce without increasing the number of crossings. This is true for all theorems and corollaries in Chapter 4.

Corollary 4.4.16. *Every 4-regular 4-edge connected simple plane graph can be reduced to a cyclic ladder within $P\Phi_{4,3}$ by applying a sequence of PS and $O_4(Oct)$.*

In Section 4.5, we will prove that our splitter theorems for 4-regular planar graphs can not be simplified if we allow only a finite number of graph operations. To prove this, we will show the existence of infinitely many pairs of (G, H) such that a 4-regular graph H is immersed in a 4-regular plane graph G and that there is no planar graph between G and H. By this, we mean that there is no 4-regular planar graph immersed in G and contains H as an immersion.

Chapter 2
Splitter Theorems for 3-regular Graphs

2.1 Introduction

In this chapter, we will investigate splitter theorems (see Section 1.6) for 3-regular graphs. The graph properties that we try to maintain are connectivity and girth. Let $\mathit{\Gamma}_{k,g}$ be the family of k-connected 3-regular graphs of girth at least g. Since only 3-regular graphs are considered, it is natural for us to assume that $k \leq 3$. It is also natural to assume $g > 0$ since every 3-regular graph has a cycle. In addition, notice that $\mathit{\Gamma}_{k,g} = \mathit{\Gamma}_{k,k}$ for all $g < k$, thus we also assume $g \geq k$.

In the following three sections, we prove the splitter theorems for $\mathit{\Gamma}_{k,g}$, for $g = 1, 2$, $g = 3$, and $g = 4$, respectively. In addition, we will also determine the $\preceq$-minimal graphs in each $\mathit{\Gamma}_{k,g}$. Then, combining with the corresponding splitter theorems, we will obtain results on how to generate all graphs in each $\mathit{\Gamma}_{k,g}$ from its minimal graphs. Table 2.1 lists the numbers of splitter theorems and generating theorems which will be proved in this chapter, and the names of authors who proved corresponding results.

2.2 On Non-simple Graphs

In this section, we consider the cases when $g = 1, 2$. It is easy to see that there are five classes, $\mathit{\Gamma}_{0,1}$, $\mathit{\Gamma}_{0,2}$, $\mathit{\Gamma}_{1,1}$, $\mathit{\Gamma}_{1,2}$, and $\mathit{\Gamma}_{2,2}$. Most proofs in this section are straightforward. We include them for the purpose of completeness.

Let us consider the graph operations $\mathcal{R}$, $O_1(L)$ and $O_2(3K_2)$ in Figure 2.1 (see Section 1.5). These three operations have a common character. Let e be the non-loop edge applied by $\mathcal{R}$ or $O_1(L)$, or a multiple edge applied by $O_2(3K_2)$, then in each operation, e and the loop are deleted and an incident non-loop edge with each

TABLE 2.1. Splitter theorems and generating theorems for 3-regular k-connected graphs with girth at least g.

	$g = 1$	$g = 2$	$g = 3$	$g = 4$
k=0	Thm 2.2.3	Thm 2.2.4	Thm 2.3.9	Thm 2.4.17
	Lem 2.2.2	Cor 2.2.6	Cor 2.3.12(a)	Cor 2.4.18
k=1	Thm 2.2.8	Thm 2.2.10	Thm 2.3.10	Thm 2.4.17
	Cor 2.2.9	Cor 2.2.11	Johnson, Wormald	Cor 2.4.18
k=2		Thm 2.2.14	Thm 2.3.11	Thm 2.4.17
		Cor 2.2.15	Wormald	Cor 2.4.18
k=3			Kelmans	Thm 2.4.17
			Tutte	Cor 2.4.18

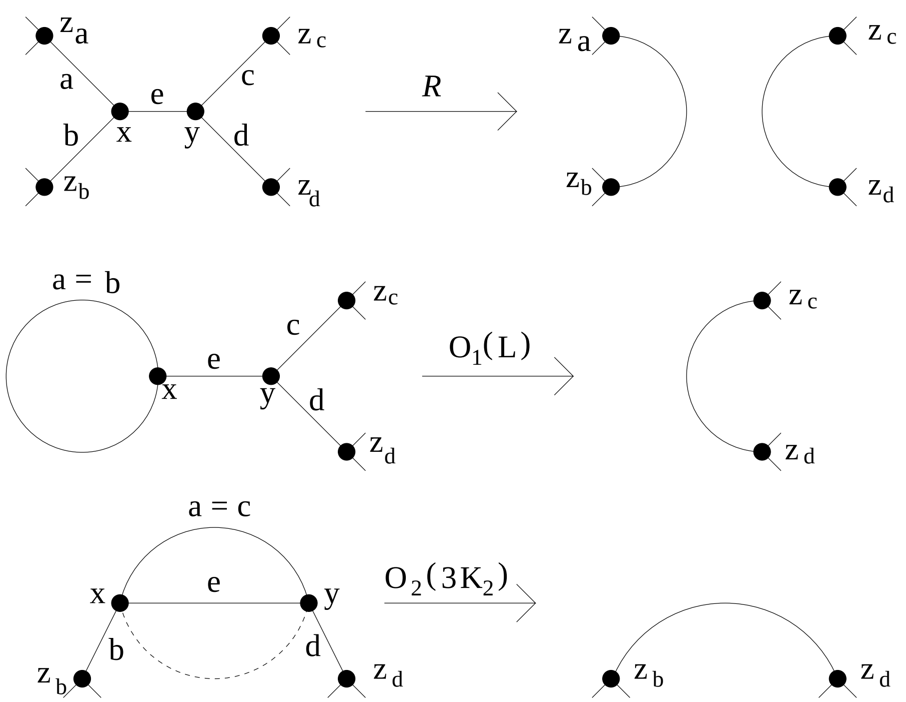

FIGURE 2.1. The operations $\mathcal{R}$, $O_1(L)$, and $O_2(3K_2)$.

endpoint of e is contracted. From these observations, we also denote the result of applying $\mathcal{R}$ or $O_1(L)$ or $O_2(3K_2)$ to e by $(G\backslash e)\breve{}$. The following are two observations on $\mathcal{R}$, $O_1(L)$ and $O_2(3K_2)$ that we will use frequently. We omit their proofs since they can be verified directly.

Lemma 2.2.1. (a) $(G\backslash e)\breve{}$ *is* 3*-regular;*

(b) *If* $G\backslash e$ *has a subgraph* H' *which is a subdivision of a* 3*-regular graph* H*, then* $H \preceq (G\backslash e)\breve{}$*.*

If $\mathcal{R}$ or $O_1(L)$ or $O_2(3K_2)$ cannot be applied to a non-loop edge e, then the component containing e can have only two vertices. By repeatedly applying $\mathcal{R}$, $O_1(L)$ or $O_2(3K_2)$ whenever it is possible, we conclude the following from (2.2.1a), which implies that $3K_2$ and K_2^L (see Section 1.1) are actually the only $\preceq$-minimal 3-regular graphs.

Lemma 2.2.2. *Every* 3*-regular graph can be reduced within* $\Gamma_{0,1}$ *by applying a sequence of* $\mathcal{R}$*,* $O_1(L)$ *and* $O_2(3K_2)$ *to a graph for which every component is either* $3K_2$ *or* K_2^L*.*

Theorem 2.2.3. *If* G *and* H *are* 3*-regular and* $H \preceq G$*, then* G *can be reduced to* H *within* $\Gamma_{0,1}$ *by applying a sequence of* $\mathcal{R}$*,* $O_1(L)$*,* $O_2(3K_2)$*,* $O_0(3K_2)$*, and* $O_0(K_2^L)$*.*

Proof. Let H' be a proper subgraph of G which is a subdivision of H. It follows that either $V(G)$ has a vertex x not in H' or H' has a vertex x of degree two. In both cases, it is easy to see that x is incident with a non-loop edge e of $E(G)\backslash E(H')$. If e is in a component S with only two vertices, then $V(S)$ is disjoint from $V(H')$ and thus the theorem follows since $O_0(S)$ reduces G to a smaller 3-regular graph without touching H'. If e is in a component with more than two vertices, then $\mathcal{R}$, $O_1(L)$ or $O_2(3K_2)$ can be applied to e and so the result follows from (2.2.1). □

We may extend the definition of $\mathcal{R}$ to include $O_1(L)$, $O_2(3K_2)$, $O_0(3K_2)$ and $O_0(K_2^L)$ so that only $\mathcal{R}$ is needed in (2.2.3). We choose to formulate the theorem in the current form just to make it more consistent with other theorems in this dissertation.

Theorem 2.2.4. *If G and H are loop-less 3-regular graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{0,2}$ by applying a sequence of $\mathcal{R}$, $O_2(3K_2)$, and $O_0(3K_2)$.*

Proof. Again, let a subdivision H' of H be a proper subgraph of G and let F denote $E(G)\backslash E(H')$. We first prove the following proposition.

Proposition 2.2.5. *Suppose some edge in F is incident with a multiple edge e. Then either F has a multiple edge that is not in a 3-cycle, or F has five distinct edges xy, xy, xz, yz, zv such that no multiple edge is incident with v.*

Proof. It follows from the assumption of (2.2.5) that the 2-cycle containing e is not a subgraph of H', as H is loop-less. Thus we may assume $e \in F$. Let us also assume that every multiple edge of F is in a 3-cycle. In particular, let $xy \in F$ be a multiple edge and let xy, xz, yz form a 3-cycle. Let zv be the other edge incident with z. Clearly, all these edges belong to F, since H is loop-less. Now let $f \neq zv$ be an edge incident with v. Suppose f is a multiple edge. Then $f \notin F$ since the only cycle containing f is a 2-cycle. However, $f \notin E(H')$ since H is loop-less. This contradiction proves that v is not incident with any multiple edge and thus the proof of (2.2.5) is complete. □

Now the proof of Theorem (2.2.4) is straightforward. Let us assume that no edge in F is contained in $3K_2$. By (2.2.1), we need only find an edge $g \in F$ such that $(G\backslash g)\breve{}$ is loop-less. If F has an edge g that is not incident with any multiple edge, then it is easy to see that $(G\backslash g)\breve{}$ is loop-less. By (2.2.5), F must have a multiple edge g such that it is not contained in a 3-cycle. Again, it is easy to see in this case that $(G\backslash g)\breve{}$ is loop-less, and thus (2.2.4) is proved. □

The following Corollary (2.2.4) is an analog of (2.2.2) for loop-less graphs.

Corollary 2.2.6. *Every loop-less* 3*-regular graph* G *can be reduced to a graph for which every component is* $3K_2$ *within* $\Gamma_{0,2}$ *by applying a sequence of* $\mathcal{R}$*,* $O_2(3K_2)$ *and* $O_1(3K_2)$*.*

Proof. By (2.2.4), we only need to show that $3K_2 \preceq G$. Let us apply $\mathcal{R}$ and $O_2(3K_2)$ to G repeatedly, as long as no loops are created. Then the resulting graph G' must have a 2-cycle C. In addition, if $e \in E(C)$, then e is contained in a 3-cycle D. Clearly, a union of C and D is a subdivision of $3K_2$. Thus $3K_2 \preceq G' \preceq G$, as required. □

The following Lemma (2.2.7) is useful. A graph containing no cycles is called *a tree*. A tree with a non-empty edge set must contain a vertex of degree one called *a pendant vertex.*

Lemma 2.2.7. *Let* G *be a connected graph with* $\delta(G) > 1$*. If* G *has a connected proper subgraph* A*, then* $G \backslash e$ *is connected for some* $e \in E(G) \backslash E(A)$*.*

Proof. If A is a spanning subgraph, then every edge in $E(G) \backslash E(A)$ has the required property. If A is not a spanning subgraph, then a spanning tree of G must have a pendant vertex v not in A because A is a connected proper subgraph. Choose an edge $e \notin A$ incident with v such that e is a loop if it is possible. Then e has the required property. □

Theorem 2.2.8. *If* G *and* H *are connected* 3*-regular graphs and* $H \preceq G$*, then* G *can be reduced to* H *within* $\Gamma_{1,1}$ *by applying a sequence of* $\mathcal{R}$*,* $O_1(L)$ *and* $O_2(3K_2)$*.*

Proof. As before, let a proper subgraph of G be a subdivision H' of H. By (2.2.7), $G \backslash e$ is connected for some $e \in E(G) \backslash E(H')$. If e is not a loop, then, by (2.2.1), we have $(G \backslash e)\breve{} \in \Gamma_{1,1}(H)$ and thus we are done. If e is a loop, then the unique edge f that is incident with e cannot be contained in $E(H')$. Clearly, $(G \backslash f)\breve{}$ is connected. Then, by (2.2.1) again, we have $(G \backslash f)\breve{} \in \Gamma_{1,1}(H)$. □

By combining (2.2.2) and (2.2.8) we conclude the following immediately.

Corollary 2.2.9. *Every connected 3-regular graph can be reduced to* $3K_2$ *or* K_2^L *within* $\Gamma_{1,1}$ *by applying a sequence of* $\mathcal{R}$, $O_1(L)$ *and* $O_2(3K_2)$.

For our following splitter theorem, we need $O_1(3K_2)$. This operation is illustrated in Figure 2.2.

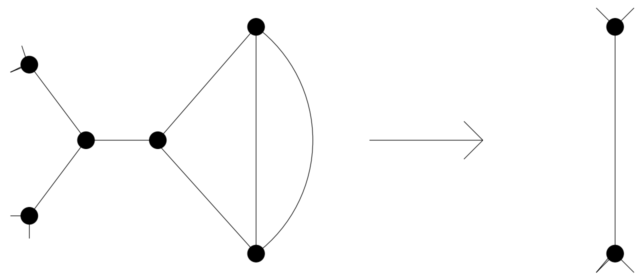

FIGURE 2.2. The operation $O_1(3K_2)$.

Theorem 2.2.10. *If* G *and* H *are connected loop-less 3-regular graphs and* $H \preceq G$, *then* G *can be reduced to* H *within* $\Gamma_{1,2}$ *by applying a sequence of* $\mathcal{R}$, $O_2(3K_2)$ *and* $O_1(3K_2)$.

Proof. Let a subdivision H' of H be a proper subgraph of G and let $F = E(G)\backslash E(H')$. By (2.2.7), $G\backslash e_0$ is connected for some $e_0 \in F$. We may assume that e_0 is incident with a multiple edge, for otherwise $(G\backslash e_0)\breve{}$ is connected and loop-less, and then (2.2.10) follows from (2.2.1). Similarly, we assume every multiple edge f of F is contained in a 3-cycle, for otherwise $(G\backslash f)\breve{}$ is connected and loop-less, and so, (2.2.10) follows from (2.2.1). Now by (2.2.5), it is clear that applying $O_1(3K_2)$ to zv results in a graph in $\Gamma_{1,2}(H)$. □

By (2.2.6) and (2.2.10) we deduce the following corollary immediately.

Corollary 2.2.11. *Every connected loop-less 3-regular graph can be reduced to* $3K_2$ *within* $\Gamma_{1,2}$ *by applying a sequence of* $\mathcal{R}$, $O_2(3K_2)$ *and* $O_1(3K_2)$.

The last splitter theorem in this section is for 2-connected 3-regular graphs. The following Lemma (2.2.12) is a lemma we will use in our proof and is the case $k = 2$ of Theorem 2.2 in [14]. Here is a direct proof.

Lemma 2.2.12. *If C is a cycle of 2-connected 3-regular graph G, then $G\backslash e$ is 2-connected for some $e \in E(C)$.*

Proof. If $G\backslash e$ is not 2-connected, then $G\backslash e$ has a cut edge $f \in E(C)$. Choose $e_0 \in E(C)$ so that $G\backslash e_0\backslash f_0$ contains a smallest component, say A, for some $f_0 \in E(C)$. Since e_0 and f_0 are not incident, A contains an edge $e_1 \in E(C)$. If e_1 is contained in a cycle in A, then $G\backslash e_1$ is 2-connected. Otherwise, $\{e_0, e_1\}$ is a 2-edge-cut in G because every cycle contains e_1 contains e_0. Then $G\backslash e_0\backslash e_1$ produces a smaller component than A. □

The following Lemma (2.2.13) is an analog of (2.2.7) for 2-connected graphs.

Lemma 2.2.13. *If a 2-connected 3-regular graph G has a subgraph G' which is also 2-connected, then $G\backslash e$ is 2-connected for some $e \in E(G)\backslash E(G')$.*

Proof. Let $e \in E(G)\backslash E(G')$ and suppose that $G\backslash e$ is not 2-connected. Then $G\backslash e$ has a cut edge f. Since G' is 2-connected, f cannot be in G'. It follows that G' is a subgraph of a component, say A, of $G\backslash e\backslash f$. Note that by (1.2.2), $\kappa'(G) = 2$, and hence by (1.2.4), $G\backslash e\backslash f$ consists of two components. Let B be the other component of $G\backslash e\backslash f$. Observe that e and f are non-incident, if G is 2-connected. Thus $\delta(B) > 1$ and so, B has a cycle. Now it is clear that the result follows from (2.2.12). □

Theorem 2.2.14. *If G and H are 2-connected 3-regular graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{2,2}$ by applying a sequence of $\mathcal{R}$ and $O_2(3K_2)$.*

Proof. Let a subdivision H' of H be a proper subgraph of G. Clearly, H' is 2-connected, if H is 2-connected. By (2.2.13), $E(G)\backslash E(H')$ has an edge e such that $G\backslash e$ is 2-connected. Then it is not difficult to see that $(G\backslash e)\breve{}$ is 2-connected. Now the theorem follows from (2.2.1). □

Since 2-connected 3-regular graphs are loop-less, the following Corollary (2.2.15) follows from (2.2.6) and (2.2.14).

Corollary 2.2.15. *Every* 2*-connected* 3*-regular graph can be reduced to* $3K_2$ *within* $\mathit{\Gamma}_{2,2}$ *by applying a sequence of* $\mathcal{R}$ *and* $O_2(3K_2)$*.* $\square$

2.3 On Simple Graphs

When considering graphs of girth at least three, we will need the following observation on connectivity.

Lemma 2.3.1. *Let G be k-connected and let $J \subseteq E(G)$. Suppose G/J is* 3*-regular and simple. Then G/J is k-connected.*

Proof. By (1.2.2), G is k-edge connected. It follows that G/J is also k-edge connected, since contracting edges does not decrease edge-connectivity. However, for 3-regular simple graphs, k-edge connected means k-connected from (1.2.3). Thus G/J is k-connected. $\square$

Let G and H be graphs in $\mathit{\Gamma}_{k,3}$. Let H' be a proper subgraph of G and be a subdivision of H. Let $F = E(G)\backslash E(H')$. In the following, we make a few observations on when G can be reduced within $\mathit{\Gamma}_{k,3}(H)$. The first is obvious since H is simple.

Lemma 2.3.2. *If $e \in F$ is in a component S with $|V(S)| = 4$, then applying $O_0(K_4)$ to S results in a graph in $\mathit{\Gamma}_{0,3}(H)$.* $\square$

A *tripod* T is a subgraph of G with distinct vertices t_1, t_2, ..., t_6 and edges t_1t_2, t_2t_3, t_3t_1, t_1t_4, t_2t_5, t_3t_6.

Lemma 2.3.3. *If T has an edge in F, then $(G\backslash e)\breve{} \in \mathit{\Gamma}_{k,3}(H)$ for some $e \in F$.*

Proof. Since H is simple, we may assume that the 3-cycle C of T has an edge e in F. Clearly, $(G\backslash e)\breve{}$ is simple. Then we deduce $(G\backslash e)\breve{} \in \mathit{\Gamma}_{0,3}(H)$ by (2.2.1). Notice that $(G\backslash e)\breve{}$ is actually isomorphic to $G/E(C)$, thus the result follows from (2.3.1). $\square$

A *necklace* N is a subgraph of G with vertices n_1, n_2, ..., n_6 and distinct edges n_1n_3, n_2n_4, n_3n_5, n_3n_6, n_4n_5, n_4n_6, n_5n_6. Vertices n_1 and n_2, which could be identical, are the *ends* of N.

Lemma 2.3.4. *Suppose an edge f of F is not contained in K_4 or any necklace. If $G\backslash f$ is k-connected, then $(G\backslash e)\breve{} \in \mathit{\Gamma}_{k,3}(H)$ for some $e \in F$.*

Proof. If $(G\backslash f)\breve{}$ is simple, then, since this graph can be considered as obtained from $G\backslash f$ by contracting edges, it follows by (2.2.1) and (2.3.1) that f can be chosen as e. Thus we may assume that $(G\backslash f)\breve{}$ is not simple. Next, let us verify that an end of f is contained in a 3-cycle C of G such f is not in C. Since G is simple, $(G\backslash f)\breve{}$ has no loops and so, must have 2-cycles. Since f is not in K_4 or any necklace, the two new edges of $(G\backslash f)\breve{}$ do not form a 2-cycle. It follows that each 2-cycle of $(G\backslash f)\breve{}$ consists of an old edge and a new edge. Clearly, such a 2-cycle corresponds to a 3-cycle C as claimed above. Now it is easy to see that the six edges incident with the three vertices of C form a tripod and thus the result follows by (2.3.3). □

A necklace N is *short* if its two ends are identical; it is *closed* if its two ends are adjacent; it is *open* if its two ends are not adjacent. Next we consider these three situations under the assumption that $E(N) \cap F \neq \emptyset$. But first, we have a simple observation which follows directly from the fact that H is simple.

Lemma 2.3.5. *$E(N) \cap E(H')$ is a (possibly empty) subpath of an edge-path P_e of H'.*

Lemma 2.3.6. *Suppose N is short. If e is the only edge between $V(N)$ and $V(G) - V(N)$, then $e \in F$. In addition, $(G\backslash e)\breve{}$ is simple, unless e is contained in a tripod or an open necklace.*

Proof. Since H is simple, it follows by (2.3.5) that $E(N) \cap E(H') = \emptyset$, and thus, $e \in F$. Let x be the end of e that is not in N. If x is not in a 3-cycle, then $(G\backslash e)\breve{}$ is simple. If x is in a 3-cycle, then e is contained in a tripod or an open necklace. □

Lemma 2.3.7. *Suppose N is closed. If e is the edge between the ends of N, then $(G\backslash e)\breve{} \in \mathit{\Gamma}_{k,3}(H)$.*

Proof. Let P be a path of N between its two ends. Since H is simple, it follows by (2.3.5) that, if $e \in E(H')$, then $E(N) \cap E(H') = \emptyset$. Therefore, by replacing e with P, if necessary, we may assume that $e \in F$. Clearly, $(G\backslash e)\breve{}$ is simple, and thus, by (2.2.1), we have $(G\backslash e)\breve{} \in \Gamma_{0,3}(H)$. Now it remains to show that $(G\backslash e)\breve{}$ is k-connected. This is trivial if $k = 0$. It is also obvious if $k = 1$, since adding e to P is a cycle containing e. If $k = 2$, then $G\backslash E(N)\backslash e$ must have a path Q between the ends of N. It follows that e is a chord of the cycle $P \cup Q$ and thus $G\backslash e$ is 2-connected. Therefore, by (2.3.1), $(G\backslash e)\breve{}$ is 2-connected. Finally, notice that the two ends of N form a cut of G, which means that G is not 3-connected, so the proof of (2.3.7) is complete. □

Let N be an open necklace and let e be one of the two edges of N that are incident with its ends. We define a new operation O_3 on N to be the contraction of $E(N)\backslash e$ in G. Notice that this is exactly $O_2(K_4)$. This operation is illustrated in Figure 2.3.

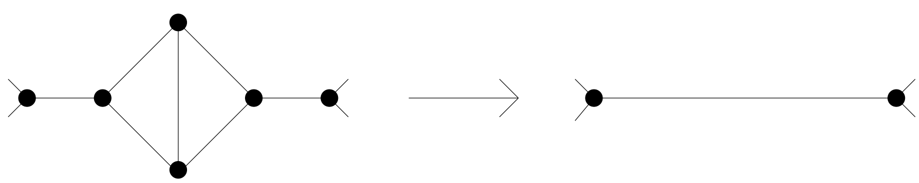

FIGURE 2.3. The operation $O_2(K_4)$.

The following observation on $O_2(K_4)$ is an analog of (2.2.1).

Lemma 2.3.8. *Let N be an open necklace and let G' be the result of applying $O_2(K_4)$ to N. Then*

(a) $G' \in \Gamma_{k,3}$;

(b) *If $F \cap E(N) \neq \emptyset$, then $H \preceq G'$.*

Proof. It is straightforward to verify that (a) follows from the definition of $O_2(K_4)$ and (2.3.1), and (b) follows from the definition of $O_2(K_4)$ and (2.3.5). □

Now we are ready to state and prove our splitter theorems.

Theorem 2.3.9. *If G and H are 3-regular simple graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{0,3}$ by applying a sequence of $\mathcal{R}$, $O_2(K_4)$ and $O_0(K_4)$.*

Proof. As before, let a proper subgraph H' of G be a subdivision of H and let F denote $E(G)\backslash E(H')$. By (2.3.2), (2.3.3), (2.3.7), and (2.3.8), we may assume that no edge of F is contained in a K_4, a tripod, a closed necklace, or an open necklace. If F has an edge which is contained in a short necklace, then the result follows from (2.3.6) and (2.2.1). Therefore, we can further assume that no edge of F is contained in any necklace. Now the result follows from (2.3.4). □

For our following splitter theorem, we need another operation, $O_1(K_4)$, which eliminate a short necklace. This operation is illustrated in Figure 2.4.

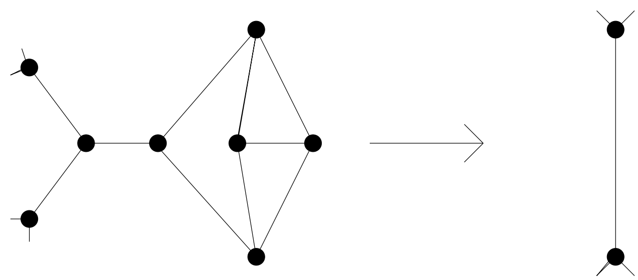

FIGURE 2.4. The operation $O_1(K_4)$.

Theorem 2.3.10. *If G and H are 3-regular simple connected graphs and $H \preceq G$, then G can be reduced to H within $\Gamma_{1,3}$ by applying a sequence of $\mathcal{R}$, $O_2(K_4)$ and $O_1(K_4)$.*

Proof. Let a proper subgraph H' of G be a subdivision of H and let $F = E(G)\backslash E(H')$. By (2.2.7), $G\backslash f$ is connected for some $f \in F$. Then we deduce from (2.3.4), (2.3.7), and (2.3.8) that f is contained in a short necklace N. By (2.3.6), either $O_1(K_4)$ can be applied to eliminate N, or some $e \in F$ is contained in a tripod or an open necklace. Thus (2.3.10) follows from (2.3.3) and (2.3.8). □

Theorem 2.3.11. *If G and H are 3-regular simple 2-connected graphs and $H \preceq G$, then G can be reduced to H within $\mathit{\Gamma}_{2,3}$ by applying a sequence of $\mathcal{R}$ and $O_2(K_4)$.*

Proof. Let a proper subgraph H' of G be a subdivision of H and let F denote $E(G)\backslash E(H')$. By (2.3.7) and (2.3.8), we may assume that no edge of F is contained in a necklace. Then (2.3.11) follows from (2.2.13) and (2.3.4). □

The following corollary follows immediately from the last three theorems and a well-known result of Dirac (see [4] or [17]) which says that every simple graph G with $\delta(G) \geq 3$ contains K_4 topologically. Note that the following (2.3.12b) and (2.3.12c) were first proved by Wormald [26]. It is also well known that Johnson gave another construction for $\mathit{\Gamma}_{1,3}$ from K_4 by using a different operation named H-reduction in [9] and [17].

Corollary 2.3.12. (a) *Every 3-regular simple graph can be reduced to a graph for which every component is K_4 within $\mathit{\Gamma}_{0,3}$ by applying a sequence of $\mathcal{R}$ and $O_2(K_4)$.*
(b) *Every 3-regular simple connected graph can be reduced to K_4 within $\mathit{\Gamma}_{1,3}$ by applying a sequence of $\mathcal{R}$, $O_2(K_4)$, and $O_1(K_4)$;*
(c) *Every 3-regular simple 2-connected graph can be reduced to K_4 within $\mathit{\Gamma}_{2,3}$ by applying a sequence of $\mathcal{R}$ and $O_2(K_4)$.*

The splitter theorem for $\mathit{\Gamma}_{3,3}$, which is stated below, is a consequence of a theorem in [10].

Theorem 2.3.13. *If G and H are 3-connected 3-regular graphs and $H \preceq G$, then G can be reduced to H within $\mathit{\Gamma}_{3,3}$ by applying a sequence of $\mathcal{R}$.*

Corollary 2.3.14. *Every 3-connected 3-regular graph can be reduced to K_4 within $\mathit{\Gamma}_{3,3}$ by applying a sequence of $\mathcal{R}$.*

This result of (2.3.14) was first proved by W. Tutte in [23], and by E. Steinitz and H. Rademacher for planar graphs (see [7] and [20]). In fact, W. Tutte reduced to $3K_2$, which is 3-connected according to his definition.

2.4 On Graphs with Girth Four

In this section, we discuss the cases when $g = 4$. First, we will state one of main theorems, Theorem (2.4.17), and a useful corollary. In the following, let K be either $K_{3,3}$ or the 3-cube Q and refer Section 1.5 for the graph operations $O_i(K)$ with $i = 0, 1, 2, 3$.

Theorem 2.4.17. *If G and H are in $\mathit{\Gamma}_{k,4}$ and $H \preceq G$, then G can be reduced to H within $\mathit{\Gamma}_{k,4}$ by applying a sequence of $\mathcal{R}$ and $O_i(K)$, where $i = 0, 2, 3$ for $k = 0$, and $i = 1, 2, 3$ for $k = 1$, and $i = 2, 3$ for $k = 2$, and $i = 3$ for $k = 3$.*

Let $\mathcal{O}_k$ be the set of operations used in Theorem (2.4.17).

Corollary 2.4.18. *Every 3-regular graph in $\mathit{\Gamma}_{k,4}$ can be reduced to $K_{3,3}$ or Q within $\mathit{\Gamma}_{k,4}$ by applying $\mathcal{O}_k$.*

In the rest of section, to prove the theorem above, we proceed by proving a sequence of lemmas. First, we will prove lemmas related with graph operations $O_i(K)$, and then will focus on $\mathcal{R}$ (or equally on $(G\backslash e)\tilde{}$) and connectivity. Note that we must keep three properties, containing H as a topological minor, connectivity and girth.

The following lemma is useful to maintain H as a topological minor. Let G and H be graphs in $\Gamma_{k,4}$ with $H \preceq G$, and H' be a subdivision of H. Suppose H' is a proper subgraph of G and let $F = E(G)\backslash E(H')$. Note that if applying $\mathcal{R}$ to an edge $f \in F$ whose endpoint is on an m- cycle C, not containing f, decreases the size of C by one (see Lemma 2.4.8). Thus we can apply $\mathcal{R}$ to f if $m \geq 5$.

Lemma 2.4.1. *Let T be a t-edge-cut in G, and S be a component of $G\backslash T$ with $|V(S)| \leq 8$. Assume that $e \in F$ is in $E(S)$, and at least one endpoint of each edge f of $E(S)$ is on a 4-cycle of $G\backslash f$. If $0 \leq t \leq 2$, then S has no vertex of degree three in H'. If $t = 3$, then S has at most one vertex of degree three in H'.*

Proof. Let X be a subset of $V(S)$ such that each member of X is of degree three in H'. Let $e = xy$ and x be in a 4-cycle C of $G\backslash e$, say $C = xuvw$. Because $g(G) \geq 4$, y is not in $V(C)$. Since $x \notin X$ and $g(G) \geq 4$, there is an edge f in $C \cap F$. By symmetry, $f = xu$ or $f = vw$. In both cases, $|V(S) - X| \geq 4$ and hence $|X| \leq 4$. If $|X| = 1, 2, 3, 4$, then $t \geq 3, t \geq 4, t \geq 5$ and $t \geq 4$, respectively. □

Let K be either $K_{3,3}$ or Q. By (2.4.1), the following holds.

Lemma 2.4.2. *If $e \in F$ is in a component K of G, then applying $O_0(K)$ to K produces a graph in $\Gamma_{k,4}(H)$.* □

The following Lemma (2.4.3) is deduced from (2.3.1) and (2.4.1). Let K' be an induced subgraph of G isomorphic to $K - v$.

Lemma 2.4.3. *If $e \in F$ is in K', then applying $O_3(K)$ to K' produces a k-connected 3-regular graph which contains H topologically.* □

By (2.4.3), if applying $O_3(K)$ to K' does not give a graph in $\Gamma_{k,4}(H)$, then the only obstruction is girth.

Lemma 2.4.4. *If applying $O_3(K)$ to K' does not produce a graph in $\Gamma_{k,4}(H)$, then there is a path of length two or three in $G\backslash E(K')$ whose ends are in $V(K')$.* □

Let K'' be an induced subgraph of G isomorphic to $K\backslash e$. The following lemmas are analogs of (2.4.3) and (2.4.4), and are proved similarly.

Lemma 2.4.5. *If K'' has an edge $f \in F$, then applying $O_2(K)$ to K'' produces a k-connected 3-regular graph which contains H topologically.* □

Lemma 2.4.6. *If applying $O_2(K)$ to K'' does not produce a graph in $\Gamma_{k,4}(H)$, then there is a path of length two or three or four in $G\backslash E(K'')$ whose ends are in $V(K'')$.* □

Let M be a subgraph of G that is isomorphic to a graph obtained by replacing an edge of K with a path P_2 of length two (see Section 1.5). Let e be a cut edge incident with the vertex of degree two in P_2. By (2.3.1) and (2.4.1), if M has an edge $f \in F$, then applying $O_1(K)$ to M produces a connected 3-regular graph which contains H topologically. From this observation, the following holds.

Lemma 2.4.7. *If applying $O_1(K)$ to M does not produce a graph in $\mathit{\Gamma}_{1,4}(H)$, then at least one of endpoints of the cut edge e is in a 4-cycle in $G\backslash e$.* □

Now, we focus on $\mathcal{R}$ or equivalently on $(G\backslash e)\breve{}$. If $e \in F$, then endpoints of e are not from $V(H)$. Hence, the following lemma is clear.

Lemma 2.4.8. *If $(G\backslash e)\breve{}$ is not in $\mathit{\Gamma}_{k,4}(H)$, then $(G\backslash e)\breve{}$ is either not k-connected, or at least one of endpoints of e is in a 4-cycle of $G\backslash e$.* □

Let $\mathcal{O}_{k,4}$ be a union of all $\mathcal{O}_k$ from $k = 0$ to 3 and G be in $\mathit{\Gamma}_{k,4}(H)$. Then, we call G, *irreducible* in $\mathit{\Gamma}_{k,4}(H)$ if applying each operation in $\mathcal{O}_{k,4}$ to the all corresponding subgraphs of G produces a graph not in $\mathit{\Gamma}_{k,4}(H)$. If G is not irreducible, we say G is *reducible* in $\mathit{\Gamma}_{k,4}(H)$.

Next, concerning 4-cycles in $G \in \mathit{\Gamma}_{k,4}(H)$, we can observe the following. Recall that if a 4-cycle C is incident with $e \in F\backslash E(C)$, then C is not contained in H' completely. Take a contrapositive and combine it with (2.2.7), (2.2.13), and (2.3.13), the following is obvious by (2.4.8).

Lemma 2.4.9. *If H' is a proper subgraph of G in $\mathit{\Gamma}_{k,4}$, and every 4-cycle of G is contained in H' completely, then G is reducible in $\mathit{\Gamma}_{k,4}(H)$.* □

By (2.4.9), we may assume that there is a 4-cycle in G, not contained in H' completely. The following two observations (2.4.11) and (2.4.13) are very important to prove Theorem (2.4.17). To prove (2.4.11), we need the following Lemma (2.4.10).

Lemma 2.4.10. *Let $e = x_1y$ and $f = x_1x_2$ be two edges of G and let $C = x_1x_2x_3x_4$ be a 4-cycle of G that contains f but not e. If $e \in F$, then $H \preceq G\backslash\{e, f\}$.*

Proof. The result is clear if $f \in F$, so we assume $f \in E(H')$. Then, by $e \in F$, x_1x_4 must be in $E(H')$ because no vertex in H' is degree one. Since $g(H) \geq 4$, $F' = F \cap \{x_2x_3, x_3x_4\}$ is not empty. Let H'' be a graph obtained from H' by deleting x_1 and adding edges in F'. Clearly H'' contains neither e nor f. We will show that H'' is a subdivision of H, which proves the lemma.

If $|F'| = 2$, then $d_{H'}(x_3) \leq 1$ and thus x_3 is not in H'. It follows that H'' is the result of replacing the path $x_4x_1x_2$ of H' by another path $x_4x_3x_2$, so the lemma holds. If $|F'| = 1$, by symmetry, we may assume $F' = \{x_3x_4\}$, that is, $x_3x_4 \in F$ and $x_2x_3 \in E(H')$. Let j be the edge of G between x_2 and $V(G) - V(C)$. If $j \in F$, then H'' is the result of replacing the path $x_4x_1x_2x_3$ of H' by an edge x_4x_3, so the lemma also holds. If $j \in E(H')$, then $d_{H'}(x_2) = 3$ and $d_{H'}(x_i) = 2$ with $i = 1, 3, 4$. Let P be the edge-path of H' that contains x_1. Then H'' is the result of "shifting" one end of P from x_2 to x_3. Therefore, H'' is also a subdivision of H. □

Lemma 2.4.11. *Let $k = 0, 1, 2, 3$, and C be a 4-cycle in G which is not contained in H' completely. Let e be an edge in $E(C) \cap F$. Then $(G\backslash e)\breve{}$ is k-connected or G is reducible.*

Proof. For $k = 0$, the result holds immediately. For $k = 1$, the graph $G\backslash e$ is 1-connected because e is in a cycle, so $(G\backslash e)\breve{}$ is 1-connected. For $k = 2, 3$, we need more detail, and so let $e = v_1v_2$ and $C = v_1v_2v_3v_4$. For $k = 2$, suppose $G\backslash e$ is not 2-connected, then by (1.2.3), it is not 2-edge connected. So, $G\backslash e$ has a cut edge, say h. Thus $T = \{e, h\}$ is a 2-edge-cut of G, which implies that every cycle containing e must contain h, including C. Then, $h = v_3v_4$ because a 2-edge-cut are non-incident in a 2-connected graph. By (1.2.4), $G\backslash T$ consists of two components, say A and B. Since H is 2-connected and $e \in F$, without loss of generality, we may assume that H' is in A containing v_1. Note that, if $v_3 \in A$, T is not a 2-edge-cut, and hence $v_3 \notin A$. Then, $f = v_2v_3 \in F$.

Next, we will show $(G\backslash f)\breve{} \in \Gamma_{2,4}(H)$. Suppose $G\backslash f$ is not 2-connected. Then, by the same argument above for T, the edge set $\{f, h'\}$ with $h' = v_1v_4$ is a 2-edge-cut in G. It implies that G has a cut vertex, which is impossible. Therefore, $G\backslash f$ is 2-

connected, and so $(G\backslash f)\breve{}$ is 2-connected. Moreover, neither v_3 nor v_4 is in a 4-cycle in $G\backslash f$ because T is a 2-edge-cut. Thus, G is reducible.

For $k = 3$, we use the following proposition (see (11.1) in [19]), and the proof is similar.

Proposition 2.4.12. *If G is a 3-regular 3-connected simple graph with $|V(G)| \geq 5$ and $e \in E(G)$, then either G/e or $(G\backslash e)\breve{}$ is 3-connected.*

An edge-cut is *trivial* if it separates only one vertex from the rest of the graph. Suppose $(G\backslash e)\breve{}$ is not 3-connected, and we will show that G is reducible. Then, since the graph is 3-regular, $(G\backslash e)\breve{}$ has an edge-cut of size at most two. Consequently, $G\backslash e$ has a nontrivial edge-cut of size at most two, which in turn implies that, as G is 3-connected, G has a nontrivial edge-cut T' of size three with $e \in T'$. Since G is 3-connected, edges in T' are pairwise non-incident. Therefore, $h = v_3v_4$ must be in T' because $|E(C) \cap T'| \neq 1$.

By (1.2.4), $G\backslash T'$ consists of two components, say A' and B'. Without loss of generality, let A' contain v_1 and at least one degree-three vertex of H'. Since $e \in F$ and H is 3-connected, B' cannot contain any degree-three vertex of H'. Then $\{f\} = C \cap B'$ with $f = v_2v_3$ and we will prove $(G\backslash f)\breve{} \in \mathit{\Gamma}_{3,4}(H)$, which implies G is reducible.

Let vw be the third edge of T' with $w \in B'$. Observe that G/f is not 3-connected because w and the new vertex form a vertex cut of size two. By (2.4.12), $(G\backslash f)\breve{}$ is 3-connected. To prove $(G\backslash f)\breve{}$ has girth at least four, we only need to check that no 4-cycles of $G\backslash f$ contain either v_2 or v_3. If there is such a 4-cycle D, then D must contain $e = v_1v_2$ or $h = v_3v_4$. Since $e, h \in T'$, $G[T']$ cannot have a degree-three vertex, and edges in T' are pairwise non-incident, D does not exist. Let j be the third edge incident with v_3, different from f and h. Finally, we verify that $H \preceq (G\backslash f)\breve{}$. This is clear if $f \in F$. If $f \in E(H')$, since $e \in F$ and all degree-three vertices of H' are in A', we deduce that $H' \backslash E(A')$ is a path between v_4 and v. Notice that both $h = v_3v_4$ and $f = v_2v_3$ are in this path, therefore j is not and thus $j \in F$. Now we conclude by Lemma (2.4.10) that $H \preceq (G \backslash f)\breve{}$ and that completes the proof of the lemma. □

We use the following Lemma (2.4.13) to prove (2.4.14), a key theorem to prove our theorem.

Lemma 2.4.13. *Let G be a 2-connected 3-regular graph having a 2-edge or 3-edge-cut T including edges f and h. Let S be a 2-edge connected component of $G \backslash T$. If $e \in G \backslash E(S)$ is incident with f and h, then $G \backslash e$ is also 2-connected.*

Proof. Suppose that $G \backslash e$ is not 2-connected. Then $G \backslash e$ has a 1-edge-cut, say j. So, $\{e, j\}$ is a 2-edge-cut in G, which implies that every cycle containing e must contain j. Since S is 2-edge connected, by (1.2.6), there are at least two cycles, say C_1 and C_2, passing through $\{e\} \cup T \cup S$ such that $C_1 \cap S$ and $C_2 \cap S$ are edge disjoint. It implies that j must be in T and hence e and j are incident. Therefore, G has a cut vertex because a 2-edge-cut, e and j, are incident. This contradicts the fact that G is 2-connected. □

The following is a key theorem to prove our splitter theorem.

Theorem 2.4.14. *Suppose that G is irreducible in $\Gamma_{k,4}(H)$, and a 4-cycle C of G contains an edge e in F. Then, $k \neq 3$ and C is contained in a subgraph of G isomorphic to one of U_j with $j = 1, 2, 3, 4$, in Figure 2.5.*

Proof. We first show that C is contained in an subgraph S of G such that S is isomorphic to either $K_{3,3} - v$ or $Q - v$. Notice that S is an induced subgraph in G because G has girth at least 4. By (2.2.1) and (2.4.11), $(G \backslash e)\breve{} \in \Gamma_{k,1}(H)$. Since G is irreducible, by (2.4.8), at least one endpoint of e is in a 4-cycle C' in $G \backslash e$. Notice that $5 \leq |V(C) \cup V(C')| \leq 6$. If $V(C) \cup V(C')$ has five vertices, then the subgraph is $K_{2,3}$, and this is the subgraph S because $K_{2,3} = K_{3,3} - v$. If $V(C) \cup V(C')$ has

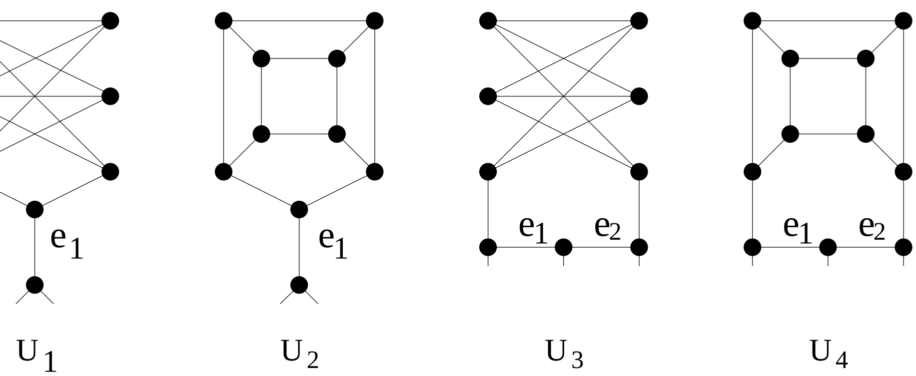

FIGURE 2.5. Subgraphs:U_1, U_2, U_3, and U_4.

six vertices, then $L = (V(C) \cup V(C'), E(C) \cup E(C'))$ consists of a 6-cycle and a chord f, which is the only common edge of C and C'. Clearly, e is one of the edges that are incident with f. By (2.4.10), we can assume that f is also in F. Then, by (2.2.1) and (2.4.11), $(G\backslash f)\breve{}$ is in $\Gamma_{k,1}(H)$. Since G is irreducible, at least one of endpoints of f is contained in a 4-cycle C'' of $G\backslash f$. Suppose C' was chosen with $V(C) \cup V(C')$ minimal. Then L must be an induced subgraph of G. Therefore, the subgraph induced by $V(L) \cup V(C'')$ is isomorphic to $Q - v$, which is S.

Let $K = K_{3,3}$ if $|V(S)| = 5$ and let $K = Q$ if $|V(S)| = 7$. Since we cannot apply $O_3(K)$ to S, by (2.4.4), the shortest path P of $G\backslash E(S)$ that is between two distinct vertices of S must have length either two or three. Let e_1, e_2, and e_3 be three edges incident with three degree-two vertices in S. Thus P is either P_2 or P_3.

We will show $P = P_2$. Suppose $P = P_3$. Without loss of generality, we can assume e_1, $e_2 \in E(P)$. Let e' be the middle edge of P, which is incident with both e_1 and e_2. Let $M = S \cup P$. Since every vertex of M is of degree three except three vertices, we can say M is a component of $G\backslash T$ where T is a subset of $E(G)\backslash E(M)$. We will show T is a 3-edge-cut. Since $e_3 \in T$, say $T = \{e_3, e_4, e_5\}$. If $e_3 = e_4$ or $e_3 = e_5$, then $P = P_2$. Hence, $e_3 \neq e_4$ and $e_3 \neq e_5$. Moreover, $e_4 \neq e_5$ because $e_4 = e_5$ implies that e' is a loop. Thus $|T| = 3$, and so T is a 3-edge-cut of G.

To prove $P \neq P_3$, we will show that G is reducible by applying $\mathcal{R}$ to e'. First, we must show $e' \in F$. Since $\{e_1, e_2, e_3\}$ is also a 3-edge-cut, by (2.4.1), S contains at most one degree-three vertex of H'. Hence, M contains at most three degree-three vertices. Observe that M cannot have two or three degree-three vertices of H' because of T. If M' has no degree-three vertices of H' and $e' \in E(H')$, then it is easy to move an edge-path containing e' to S. If a degree-three vertex v of H' is one of endpoints of e', then move v to S and shift an edge-path so that no edge-path contains e'. Thus we can assume $e' \in F$. Second, no endpoints of e' are in a 4-cycle of $G \backslash e'$ because the shortest path P has length three. Finally, by the fact that e' is in a cycle for $k = 1$, by (2.4.13) for $k = 2$, and by (2.4.12) for $k = 3$, applying $\mathcal{R}$ to e' results in $\mathit{\Gamma}_{k,4}(H)$. Thus, G is reducible if $P = P_3$. Hence, $P = P_2$.

First, we will prove that, if $P = P_2$, then G is not 3-connected. Without loss of generality, we can assume $E(P) = \{e_1, e_2\}$. Let $N = S \cup P$ and observe that N is isomorphic to $K \backslash h$ with an edge h in $E(K)$. To prove that G has a vertex cut or an edge cut of size two, it is enough to show that N is an induced subgraph of G. Suppose N is not an induced subgraph, which implies that N is in a component K. By (2.4.2), G is reducible. Hence, N is an induced subgraph of G, and so G is not 3-connected.

Finally, we will deduce that the 4-cycle C must be contained in one of U_i with $i = 1, 2, 3, 4$ in Figure 2.5 if $P = P_2$. Since G is irreducible, by (2.4.6), there is another path P' of length two, three or four in $G \backslash E(N)$ with $N = S \cup P$. Note that, if $P' = P_2$ or $P' = P_4$, then $N \cup P'$ is a subgraph of U_i with $i = 1, 2$ or with $i = 3, 4$, respectively. If $P' = P_3$, by (2.2.1),(2.4.1) and (2.4.13), applying $\mathcal{R}$ to the middle edge of P' results in $\mathit{\Gamma}_{k,4}(H)$. Thus, G is reducible if $P' = P_3$. $\square$

The following two lemmas tell us more detail about the four pictures of Figure 2.5 in (2.4.14).

Lemma 2.4.15. *We can assume the edge e_1 of U_i with $i = 1, 2, 3, 4$ in Figure 2.5 is in F.*

Proof. Let K be either $K_{3,3}$ or Q, and S be $K\backslash f$ with an edge f in K. Since each U_i contains S as an induced subgraph, by (2.4.1), the subgraph S of G has no degree-three vertex of H'. For U_1 and U_2, suppose e_1 is in $E(H')$, and for U_3 and U_4, suppose both e_1 and e_2 are in $E(H')$. Then, H has a loop or a 3-cycle in U_i with $i = 1, 2$ or with $i = 3, 4$, respectively. Hence, for U_1 and U_2, immediately, and for U_3 and U_4, by symmetry, we can assume that e_1 is in F. □

Lemma 2.4.16. *Let G be k-connected with $k = 0, 1, 2$. If G contains U_3 or U_4 of Figure 2.5 as a subgraph, then $G\backslash e_1$ is k-connected.*

Proof. For $k = 0$, it is trivial, and for $k = 1$, the result holds as e_1 is in a cycle. So, we only need to show that $G\backslash e_1$ is 2-connected when G is 2-connected. Suppose that $G\backslash e_1$ is not 2-connected. Then $G\backslash e_1$ has a cut edge, say h. Here, note that $\{e_1, h\}$ is a 2-edge-cut of G. Let $T = \{a, b\}$ be the 2-edge-cut of G contained in U_3 or U_4 and let a be incident with e_1. Let g_1, g_2 and g_3 be the three edges of U_3 or U_4 located in the bottom of Figure 2.5, and let e_1 be incident with g_1 and g_2, and let e_2 be incident with g_2 and g_3. Since $\{e_1, h\}$ is a 2-edge-cut of G, every cycle containing e_1 must contain h. Then, because there is a cycle passing through e_1 and a and b, and $T = \{a, b\}$ is a 2-edge-cut of G, the edge h is in T. If $h = a$, then g_1 is a cut edge of G because the 2-edge-cut $\{e_1, a\}$ has a common endpoint. So, we may assume that $h = b$. Then there is no cycle containing e_1 and g_1 and g_2, and no cycle containing e_1 and g_1 and g_3. It implies that g_1 is a cut edge of G, which contradicts the fact G is 2-connected (and hence 2-edge connected). Therefore, if G is 2-connected, then $G\backslash e_1$ is 2-connected. □

In the following theorem, we allow the empty graph for $\Gamma_{k,4}$ with $k = 0, 1, 2, 3$ because it is convenient to prove that $K_{3,3}$ and the 3-cube Q are the only $\preceq$-minimal 3-regular graphs in $\Gamma_{k,4}$ for $k = 0, 1, 2, 3$.

Theorem 2.4.17. *If G and H are in $\Gamma_{k,4}$ and $H \preceq G$, then G can be reduced to H within $\Gamma_{k,4}$ by applying a sequence of $\mathcal{R}$ and $O_i(K)$ with $K = K_{3,3}$ or Q, where $i = 0, 2, 3$, for $k = 0$, and $i = 0, 1, 2, 3$, for $k = 1$, and $i = 0, 2, 3$, for $k = 2$, and $i = 0, 3$, for $k = 3$.*

Proof. Let H' be a proper subgraph of G which is a subdivision of H. Let $F = E(G)\backslash E(H')$. From (2.4.9), we may assume that there is a 4-cycle C which contains an edge in F. Then, by (2.4.11), for every edge $e \in E(C) \cap F$, the graph $(G\backslash e)\breve{}$ is k-connected or G is reducible by (2.4.11). Our goal is to show that G is always reducible in $\Gamma_{k,4}$. By Lemma (2.4.14), for $k = 3$, the result holds. So, let $k \neq 3$ and suppose G is irreducible. Then by (2.4.14), C can be extended to one of U_i with $i = 1, 2, 3, 4$. We know each e_1 of U_i is in F by (2.4.15). By (2.4.7), (2.4.8), and (2.4.16), we can assume that at least one of endpoints of e_1 is in a 4-cycle C_1 in $G\backslash e_1$. Since the edge $e_1 \in F$ is incident with a vertex of $V(C_1)$, the 4-cycle C_1 is not contained in H' completely. By using (2.4.14) again, the 4-cycle C_1 must be extended to one of U_i in Figure 2.5 above. However, by checking a cut edge or a 2-edge-cut in U_i, we can find that U_i can not provide to C_1 the same neighborhood as before. Therefore, there is no 4-cycle not contained in H' completely, or G is not irreducible. In both cases, we can conclude that G is reducible in $\Gamma_{k,4}$. □

In the last theorem, we can allow the empty graph for $\Gamma_{k,4}$ with $k = 0, 1, 2, 3$. Let H be the empty graph in the theorem. Then $H \preceq G$ holds for every 3-regular graph $G \in \Gamma_{k,4}$ with $k = 0, 1, 2, 3$. So, by the theorem, every 3-regular graph $G \in \Gamma_{k,4}$ with $k = 0, 1, 2, 3$, can be reduced to the empty graph. Consider the last graph operation in this process. The only possible operations are $O_0(K_{3,3})$ and $O_0(Q)$.

This implies that $K_{3,3}$ and Q are the only $\preceq$-minimal 3-regular graphs in $\mathit{\Gamma}_{k,4}$ for $k = 0, 1, 2, 3$. From this observation we have the following as a corollary of (2.4.17).

Let $\mathcal{O}_k$ be the set of operations used in Theorem (2.4.17).

Corollary 2.4.18. *Every* 3*-regular graph in* $\mathit{\Gamma}_{k,4}$ *can be reduced to* $K_{3,3}$ *or* Q *within* $\mathit{\Gamma}_{k,4}$ *by applying* $\mathcal{O}_k$.

Chapter 3
Splitter Theorems for 4-regular Graphs

3.1 Introduction

We will use the "immersion" containment relation (see Section 1.3) in the rest of this paper and prove several splitter theorems and generating theorems (see Section 1.6) for 4-regular graphs. Lemma (3.2.4) tells us that a 4-regular graph H is immersed in another 4-regular graph G if and only if H can be obtained from G by applying a sequence of splitting operations Sp (see Section 1.5 and Section 3.2). Thus, we can use this as an alternative definition of immersion.

The graph properties that we try to maintain are edge-connectivity and girth. Let $\Phi_{k,g}$ be the family of k-edge connected 4-regular graphs of girth at least g. Here, note that $k \leq 2g$ and, by (1.2.1), $\Phi_{2k-1,g} = \Phi_{2k,g}$. Since only 4-regular graphs are considered, it is natural for us to assume that $k \leq 4$. It is also natural to assume $g > 0$ since every 4-regular graph has a cycle. We define $\kappa'(2L)$ to be two (see Section 1.1).

In the following three sections, we will prove the splitter theorems for $\Phi_{k,g}$, for $g = 1$, $g = 2$, and $g = 3$, respectively. Table 3.1 shows the numbers of splitter theorems and generating theorems that will be proved in this chapter, and the names of authors who proved a corresponding result.

3.2 4-regular Graphs

Since we consider the cases of $g = 1$ in this section, we study all 4-regular graphs, including graphs having loops. Note that we have only two classes, $\Phi_{0,1}$ and $\Phi_{2,1}$ because $k \leq 2g$ and $\Phi_{1,1} = \Phi_{2,1}$. Most proofs in this section are straightforward. We include them for the purpose of completeness.

TABLE 3.1. Splitter theorems and generating theorems for 4-regular k-edge connected graphs with girth at least g

	$g=1$	$g=2$	$g=3$
k=0	Thm 3.2.7	Thm 3.3.4	Thm 3.4.5
	Cor 3.2.8	Cor 3.3.7	Cor 3.4.7
k=2	Thm 3.2.7	Thm 3.3.5	Thm 3.4.6
	Cor 3.2.8	Cor 3.3.7	Toida, Bories etc.
k=4		Thm 3.3.8	Thm 3.4.8
		Cor 3.3.9	Cor 3.4.9

The Figure 1.6 (see Section 1.5) shows us the operation Sp unless the applied vertex x is in $2L$. Note that applying Sp to a vertex x where a loop is incident with x results in a unique graph, but in general, it is not unique. There are at most three different resulting graphs because we can choose two pairs from four. To maintain edge-connectivity, the following Lemma (3.2.1) is useful.

Lemma 3.2.1. *If G is connected and applying one type of Sp to x produces a disconnected graph, then there is another type of Sp which can apply to x so that the resulting graph is connected.*

Proof. Clearly, x is not a vertex of a loop. Since the resulting graph can be obtained from $G-x$ by adding edges (see Section 1.5), $G-x$ is also disconnected. It follows that x is a cut vertex of G. So, no loop is incident with x. By (1.2.5), $G-x$ has just two components, say A and B. Let y be the new vertex of $G/E(A)$. Let z be a vertex of B. Since $G/E(A)$ is connected, there is a path connecting y and z. The path must pass x, so it must pass two edges incident with x, say e_i and e_j. For a desired splitting, choose e_i and e_j as a pair, which is a different pair from the original pair. The other pair can be chosen the remaining edges. When the new Sp is applied to x, the pair of e_i and e_j will be replaced by an edge, say e. Then the resulting graph is connected because both A and B are connected and are connected by e. □

There are several important lemmas about Sp in the following. Let both G and H be 4-regular graphs and H is immersed in G, denoted by $H \propto G$. The proof for the first is straightforward.

Lemma 3.2.2. *Applying Sp to a vertex of a 4-regular graph results in a 4-regular graph.*

Now, we want to find graph operations in which the resulting graphs maintain H as an immersion. We assume that we will not apply any operations to a vertex from $V(H)$. We denote all 4-regular graphs containing H as an immersion by $\Phi(H)$ or $\Phi_{k,g}(H)$. Let H' be a pseudo-subdivision of H and recall $H' = \bigcup_{vw \in E(H)} T_{vw}$ where T_{vw} is a vw-trail satisfying the conditions (see Section 1.3).

Call a trail T of G, a *redtrail* if there is an edge $vw \in E(H)$ such that T is a vw-trail in H'. We call an edge e of G, *red* if $e \in E(H')$, and *white* otherwise. A vertex x of G is *white* if all edges incident with x are white. Note that if x is from $V(H)$, then all edges incident with x must be red, but not vice verse.

The followings, (3.2.3) and (3.2.4), show us that if a vertex x of G is not from $V(H)$, we can find a type of Sp such that applying the type of Sp to x results in $\Phi(H)$. Suppose x is neither white nor a vertex of a loop. Then there are two or four red edges incident with x. Possibly, a redtrail T passes through x twice. Let $e_i = xx_i$ with $i = 1, 2, 3, 4$ be edges incident with x and let e_1 and e_2 be edges used in a redtrail in the first time. We call a type of applying Sp, *faithful* if the type of Sp pairs e_1 and e_2 as a pair, and e_3 and e_4 as another pair.

Lemma 3.2.3. *If a vertex x is not white, not a vertex of a loop and not from $V(H)$, then applying a faithful splitting of Sp to x results in a graph of $\Phi(H)$.*

Proof. By (3.2.2), the resulting graph is 4-regular. So, we only need to show that the resulting graph contains a pseudo-subdivision of H. After applying a faithful splitting, the resulting graph can by obtained by replacing $x_1e_1xe_2x_2$ and $x_3e_3xe_4x_4$ with x_1x_2 and x_3x_4, respectively. Let H'' be a graph obtained from H' by replacing paths $x_1e_1xe_2x_2$ and $x_3e_3xe_4x_4$ with edges x_1x_2 and x_3x_4, respectively. A faithful splitting makes (at most) two redtrails in H' shorter, but a vw-trail is still a vw-trail for every $v, w \in V(H)$ in H''. Thus H'' is also a pseudo-subdivision of H. □

Lemma 3.2.4. *For each x not from $V(H)$, there exists a type of splitting Sp so that applying the type of Sp to x results in a graph of $\Phi(H)$.*

Proof By (3.2.2), every resulting graph after applying Sp to a 4-regular graph is 4-regular. So, we only need to show that there is a type of Sp such that the resulting graph contains H as a immersion. There are three cases we have to cover. A loop is incident with x, x is a white vertex, and x is neither a vertex of a loop nor white. The first and second case are clear. The last case follows by (3.2.3). □

If we use (3.2.4) to all vertices not from $V(H)$ repeatedly, then we will obtain H. So, if $H \propto G$, then there is a sequence of Sp in G to obtain H. Conversely, if there is such a sequence, clearly $H \propto G$. Therefore, we may also use this equivalent condition as an alternative definition of immersion.

The following Lemma (3.2.5) will be used frequently in Chapter 3 and Chapter 4.

Lemma 3.2.5. *If a white edge is incident with a vertex x, then applying all possible types of Sp to x always results in $\Phi(H)$.*

Proof. If x is a white vertex, then the result holds. If a loop is incident with x, the result holds by (3.2.4). So, we may assume that no loop is incident with x and that a redtrail T passes through x containing $x_1e_1xe_2x_2$ as a subtrail. Then, e_3

and e_4 are white by the condition. By (3.2.3), we only need to find other pseudo-subdivisions of H which contains e_1, e_3 or e_1, e_4 in a redtrail, respectively. As every vertex of H' is of degree two or four, no or two or four white edges are incident with every vertex of G. In other words, each component of H' is Eulerian, and white edges form a union of Eulerian graphs. So, we can find a white cycle C, $xe_3P(x_3, x_4)e_4x$ where $P(x_3, x_4)$ is a x_3x_4-path.

Let T' be a trail obtained from T by replacing $x_1e_1xe_2x_2$ with $x_1e_1xe_3P(x_3, x_4)e_4xe_2x_2$ and let H'' be a graph obtained from H' by replacing T with T'. Then T' contains e_1, e_3 and H'' is also a pseudo-subdivision of H because $V(T) \cap V(H) = V(T') \cap V(H)$. Similarly, let T'' be a trail obtained from T by replacing $x_1e_1xe_2x_2$ with $x_1e_1xe_4P(x_3, x_4)e_3xe_2x_2$. Then T'' contains e_1, e_4 and the graph obtained from H' by replacing T with T'' is also a pseudo-subdivision of H. □

To prove a splitter theorem for $\Phi_{2,1}$, we only need to prove the following lemma.

Lemma 3.2.6. *Let G be in $\Phi_{2,1}(H)$. After applying a type of Sp to x, if the resulting graph is disconnected, then there is another type of Sp at x such that the resulting graph is in $\Phi_{2,1}(H)$.*

Proof. By (3.2.1) and (3.2.5), we only need to show that a white edge is incident with x. Let G' be the disconnected graph. By (1.2.4), G' has just two components, say A and B. Then G' can be obtained from $G - x$ by adding two new edges, say f_1 and f_2. Here, f_1 and f_2 belong to different components of G' because, otherwise, G is disconnected. Since $H \propto G'$ and H is connected, all vertices from $V(H)$ belong to a component, say A. Then all edges of B are white. Therefore, f_1 or f_2 is white. It follows that a white edge is incident with x in G. □

Note that the process of (3.2.6) does not decrease girth because the new edges are contained in an edge cut set in the resulting graph. By (3.2.2), (3.2.4) and

(3.2.6), the following theorem holds. Also note that, since we cannot use Sp to a vertex in $2L$, we need $O_0(2L)$ for the case of connectivity zero in the following.

Theorem 3.2.7. *Let $k = 0, 1, 2$. If G and H are in $\Phi_{k,1}$, and $H \propto G$, then G can be reduced to H within $\Phi_{k,1}$ by applying a sequence of Sp and $O_0(2L)$.* □

Notice that every 4-regular graph G contains $2L$ as an immersion because each component of G is Eulerian. By (3.2.7), the following corollary holds.

Corollary 3.2.8. *Every connected 4-regular graph can be reduced to $2L$ within $\Phi_{2,1}$ by a sequence of Sp.* □

3.3 4-regular Loopless Graphs

In this section, we consider the cases when $g = 2$. So, multiple edges are allowed, but no loops. Let G and H be 4-regular loopless graphs and $H \propto G$. To prove the splitter theorems in this section, we must not only make each resulting graph be in $\Phi(H)$, but all of the resulting graphs must remain loopless. There are three distinct classes: $\Phi_{0,2}$, $\Phi_{2,2}$, and $\Phi_{4,2}$ because $k \leq 2g$, and $\Phi_{1,2}{=}\Phi_{2,2}$ and $\Phi_{3,2}{=}\Phi_{4,2}$.

Let x be a vertex of G not from $V(H)$. If applying a type of Sp to x produces a loop, then there are two multiple edges ($2K_2$), say e_1 and e_2, such that the trail ye_1xe_2y was replaced by an edge, which forms a new loop incident with y.

Note that, if three or four multiple edges are incident with x, then the resulting graph after applying Sp is unique and it has a loop. So, we have no type to avoid loops if we use only Sp as a graph operation. It implies that we need new graph operations if we do not want to create a new loop. We will use two operations, $O_2(4K_2)$ is for three multiple edges and $O_0(4K_2)$ is for four multiple edges.

Let us investigate the operations used here, Sp, $O_0(4K_2)$ and $O_2(4K_2)$. We need to maintain the two properties for each resulting graph: being in $\Phi(H)$ and looplessness. Clearly, applying each of them to a 4-regular graph results in a 4-regular graph.

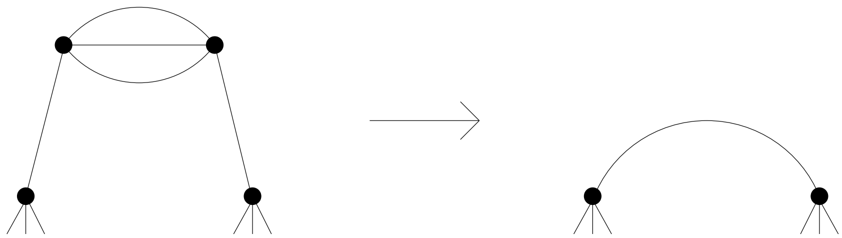

FIGURE 3.1. The operation $O_2(4K_2)$.

To check that every resulting graph in $\Phi(H)$, the following lemmas are useful. The first result holds because H has no loop.

Lemma 3.3.1. *Let H have no loop and $H \propto G$. If one of the endpoints of three or four multiple edges is not from $V(H)$, then the other endpoint is also not from $V(H)$.* □

Lemma 3.3.2. *Let S be a subgraph to which we can apply one of Sp, $O_0(4K_2)$ and $O_2(4K_2)$. Suppose at least one vertex of $V(S)$ is not from $V(H)$. Then, each resulting graph is in $\Phi(H)$.*

Proof. We prove one by one. By (3.2.4), the lemma holds for Sp. By (3.3.1), applying $O_0(4K_2)$ to a component $S = 4K_2$ results in a graph in $\Phi(H)$. To prove the lemma for the last operation $O_2(4K_2)$, recall that for this graph operation, S consists of three multiple edges because $S = 4K_2 \backslash e$ (see Section 1.5). Let x be an endpoint of S not from $V(H)$. we only need to show two cases; x is a white vertex or not. If x is a white vertex, then the other endpoint is also a white vertex because H' has no vertex of degree 1. In this case, clearly the resulting graph is in $\Phi(H)$. If x is not a white vertex, then there is a redtrail passing through x. By (3.3.1), we can change the redtrail so that only one multiple edge is red. □

Next, we need to take care of looplessness together with the condition that every resulting graph must be in $\Phi(H)$. For $O_0(4K_2)$, we do not have to anything because removing a component does not create any new loops and the resulting graph contains H as an immersion by (3.3.2). For the other operations, we have to show that, if a resulting graph has a loop, then there is another type satisfying both conditions above. Instead of showing this for Sp and $O_2(4K_2)$, we will prove it for a generalized case.

Lemma 3.3.3. *Let $G, H \in \Phi_{0,2}$ with $H \propto G$. Suppose applying some graph operations to G results in a graph $G' \in \Phi(H)$ and G' has a loop incident with a vertex x. Then, applying any type of Sp at x in G results in a graph in $\Phi(H)$.*

Proof. Since H has no loop, x is not from $V(H)$. By (3.2.5), it is enough to show that we can assume a white edge is incident with x in G. If x is in $2L$ in G', then the loop is white because x is not from $V(H)$. If $x \notin 2L$, then two non-loop edges are incident with x in G'. To prove the result, suppose all edges incident with x are red in G. Then, since x is not from $V(H)$, the red loop in G' is waste. Thus we can assume the loop is white in G', which follows that a white edge is incident with x in G. □

By (3.3.2) and (3.3.3), the following theorem holds.

Theorem 3.3.4. *If G and H are in $\Phi_{0,2}$, and $H \propto G$, then G can be reduced to H within $\Phi_{0,2}$ by applying a sequence of Sp, $O_0(4K_2)$ and $O_2(4K_2)$.*

Proof. If the resulting graph does not contain any loop, then the result holds by (3.3.2). If it contains a loop incident with x after applying a graph operation O, by (3.3.3), we can replace O by a type of Sp. By (3.3.3), we only need to show that the resulting graph is loopless. If applying a type of Sp produces a loop, try another type of Sp to x. Note that there exists at least one type of Sp at x such that the resulting graph does not produce a loop unless $x \in 3K_2$. In this case, $x \notin 3K_2$ because a loop was incident with x. □

Now, to maintain connectivity is not difficult.

Theorem 3.3.5. *If G and H are connected loopless 4-regular graphs with $H \propto G$, then G can be reduced to H within $\Phi_{2,2}$ by applying a sequence of Sp and $O_2(4K_2)$.*

Proof. We only need to show that there is a graph operation such that the resulting graph is in $\Phi_{2,2}(H)$. By (3.3.4), we can use Sp and $O_2(4K_2)$ unless the resulting graph is disconnected. Since $O_2(4K_2)$ consists of contractions, applying $O_2(4K_2)$ to a connected graph results in a connected graph. So, we only need to check Sp. By (3.2.6) and the note after (3.2.6), the result holds. □

For a generating theorem for $\Phi_{2,2}$, the following lemma (3.3.6) is useful.

Lemma 3.3.6. *Every connected loopless 4-regular graph G contains $4K_2$ as an immersion.*

Proof. Note that, by (1.2.1), $\kappa'(G) \geq 2$ and $\kappa'(G) > 2$ implies $\kappa'(G) \geq 4$. Hence, if $\kappa'(G) > 2$, the lemma holds by (1.2.6). Thus, we can assume $\kappa'(G) = 2$. Then, there is a 2-edge-cut T of G such that $G \backslash T$ has a component having no 2-edge-cut, say A. Thus, A is 3-edge connected (A is not 4-regular). Here A contains more than one vertex because G has no loop. Let v and w be distinct vertices in A. By (1.2.6), at least three pairwise edge-disjoint paths connect v and w in A. By using $G \backslash E(A)$, we can find another path connecting v and w, which is certainly edge-disjoint from paths in A. □

By (3.3.5) and (3.3.6), the following corollary holds.

Corollary 3.3.7. *Every connected loopless 4-regular graph can be reduced to $4K_2$ within $\Phi_{2,2}$ by applying a sequence of Sp and $O_2(4K_2)$.*

Next, since every 4-edge connected graph has no 2-edge cut, every graph of $\Phi_{4,2}$ has no three multiple edges. So, we do not have to use $O_2(4K_2)$. The following is a splitter theorem for $\Phi_{4,2}$.

Theorem 3.3.8. *If G and H are in $\Phi_{4,2}$ with $H \propto G$, then G can be reduced to H within $\Phi_{4,2}$ by a sequence of Sp.*

Proof. By (3.3.5), applying Sp to a vertex x which is not from $V(H)$, results in a graph in $\Phi_{2,2}(H)$. We only need to show that, if applying one of three types of Sp at x results in a d graph G' with $\kappa'(G') < 4$, then there is another type of Sp at x such that the resulting graph is in $\Phi_{4,2}$. By (1.2.1), G' has a 2-edge-cut, say $T = \{t_1, t_2\}$. Then, by (1.2.4), $G' \backslash T$ has just two components, say A and B. Since H is 4-edge connected, all vertices of H belong to a component, say A. Note that G' can be obtained from $G - x$ by adding two new edges, say f_1 and f_2. Let f_1 belong to A. If one of f_i is white, then a white edge is incident with x. Then (3.2.5) implies the result because in (3.2.5), new edges in the resulting graph are in a subset of a edge cut and hence new cycles do not decrease girth.

So, suppose both f_i are red. Then t_1 and t_2 must be red and must belong to a redtrail T together with f_2 because no vertex of B is from $V(H)$. We only need to show that T can be replaced by a shorter trail which does not contain f_2. If t_1 and t_2 are incident with a vertex v in $V(B)$, then T contains a waste close trail passing through f_2. In this case, omit the waste. So, we can assume t_1 and t_2 are not incident in B. Hence, t_1 and t_2 have distinct endpoints in B, say v_1 and v_2. Since B is connected, there is a path P in B connecting v_1 and v_2. Since $f \notin E(B)$, P does not contain f. Replace a subtrail of T connecting v_1 and v_2 by P. Thus we can assume f_2 is white. □

By (1.2.6), every 4-edge connected graph contains $4K_2$ as an immersion. Thus, the following generating theorem holds by (3.3.8).

Corollary 3.3.9. *Every 4-edge connected 4-regular graph can be reduced to $4K_2$ within $\Phi_{4,2}$ by applying a sequence of Sp.* □

3.4 4-regular Simple Graphs

In this section, we consider the cases when $g = 3$. So, every graph must contain no loops and no multiple edges here. We have three distinct classes, $\Phi_{0,3}$, $\Phi_{2,3}$, and $\Phi_{4,3}$

because $k \leq 4$, and $\Phi_{1,3}{=}\Phi_{2,3}$, and $\Phi_{3,3}{=}\Phi_{4,3}$. However, the splitter theorem for $\Phi_{0,3}$ is crucial. To maintain connectivity is not difficult comparing to keep simplicity. Let us introduce very useful two lemmas related to simplicity. Let $H \in \Phi_{0,3}$ and G is a 4-regular graph with $H \propto G$. The proof of the first lemma is very elementary, so it is eliminated.

Lemma 3.4.1. *Suppose H is a 4-regular simple graph and $G \in \Phi_{0,1}(H)$. Let T be an t-edge cut of G and let S be a component of $G \backslash T$.*

(a). If t =2, and $V(S)$ has vertices from $V(H)$ less than five, then $V(S)$ has no vertices from $V(H)$.

(b). If t = 4, then $V(S)$ cannot have exactly two or three vertices from $V(H)$.

□

The resulting graph has no loop after applying Sp because G is simple. Hence, only obstruction about applying Sp is to produce of a multiple edge. Note that if we apply Sp to a vertex x, and one of two pairs of four incident edges with x on a cycle, then the size of the cycle decreases by 1 after applying Sp. So, a 2-cycle or a multiple edge appears only when we choose two edges on a 3-cycle as one of two pairs for a splitting Sp. The following saves this situation.

Here we say that a type of Sp *releases a multiple edge* if the splitting at one of the endpoints separates the multiple edge.

Lemma 3.4.2. *Let H be a 4-regular simple graph and $G \in \Phi_{0,1}(H)$. If G has a multiple edge, say xy, then applying two splittings at x that releases the multiple edge, results in a graph in $\Phi(H)$.*

Proof. Since H is simple, both x and y can not be from $V(H)$. There are two cases; one case is that either x or y is from $V(H)$, and the other is that none of them is from $V(H)$. First, let y be from $V(H)$. Then x must have a faithful splitting releasing the multiple edge because H is simple. The result holds because two splittings releasing the multiple edge have the same resulting graph. If x is from $V(H)$, then y is not. In this case, y has a faithful splitting that releases the multiple edge and preserves H because H is simple. Move a vertex of $V(H)$ from x to y. Then the result holds because H is simple and xy is a multiple edge. In the second case, neither x nor y is from $V(H)$. Assume that none of them has the releasing splitting. Then the multiple edge xy is white because a redtrail needs a vertex from $V(H)$, but none of the endpoints is from $V(H)$. Hence, any splitting at x preserves H by (3.2.5). □

In the following, let both H and G be in $\Phi_{0,3}$ and $H \propto G$. Since a 4-regular simple graph needs at least five vertices, the complete graph K_5 is the smallest graph in $\Phi_{k,3}$ with $0 \leq k \leq 4$.

Let us study three operations $O_i(K_5)$ with $i = 0, 2, 4$ (see Section 1.5). Let S be an induced subgraph in G which will be applied $O_i(K_5)$. Recall $S = K_5, K_5\backslash e$ and $K_5 - v$ for $i = 0, 2$ and 4, respectively. Since K_5 is the smallest graph here, the fact that one of $V(K_5)$ is not from $V(H)$ implies that none of $V(K_5)$ is from $V(H)$. Thus, applying $O_0(K_5)$ to a component S results in a graph $\Phi(H)$ unless all vertices of the K_5 are from $V(H)$. By (3.4.1a), applying $O_2(K_5)$ to S results in a graph in $\Phi_{0,1}(H)$ unless all five vertices of S are from H. However, note that operations $O_0(K_5)$ and $O_2(K_5)$ are equivalent to five consecutive Sp to five points of S.

Finally, see Figure 1.13 for $O_4(K_5)$. Let T be the 4-edge-cut of G in $G\backslash E(S)$ incident with $V(S)$. Clearly, applying $O_4(K_5)$ to $S = K_5 - v = K_4$ results in a 4-regular graph. By (4.1b), the resulting graph in $\Phi(H)$ unless all four vertices of S are from $V(H)$.

We can see that applying $O_4(K_5)$ results in a simple graph unless some of T are incident in G. Notice that if just three edges of T have a common vertex, then G

has a subgraph isomorphic to $K_5\backslash e$. If all four edges have a common vertex, then G has a component isomorphic to K_5. The following lemma is very important. Let both G and H be in a class $\Phi_{0,3}$ and $H \propto G$.

Lemma 3.4.3. *If G contains a subgraph S isomorphic to $K_5\backslash e$, then G is reducible in $\Phi_{0,3}(H)$ unless all vertices of S are from $V(H)$.*

Proof. Suppose G is irreducible in $\Phi_{0,3}(H)$. We will prove S is an induced subgraph in G. If S is not an induced graph in G, then G has a component isomorphic to K_5. Since at least one vertex of K is not from $V(H)$, we can use the operation $O_0(K_5)$ and resulting graph is in $\Phi_{0,3}(H)$. Hence, S must be an induced graph of G. Then G has a 2-edge-cut, say $\{f, g\}$, such that $G\backslash\{f, g\}$ has a component S. Hence, by (3.4.1a), none of $V(S)$ is from $V(H)$. So, applying $O_2(K_5)$ to S results in a graph G' in $\Phi_{0,1}(H)$. Therefore, from our assumption, G' is not simple.

There are two cases that G' contains a loop or a multiple edge. Note that f and g are incident or have a common incident edge, respectively. In the first, suppose G' has a loop. Let x be the vertex incident with the loop. Note that, in G, four edges are incident with x including f and g. Let f' and g' be the other edges incident with x. Then, by (3.3.3), applying any types of Sp to x results in $\Phi(H)$. So, instead of applying $O_2(K_5)$ to S, apply the following type of Sp to x. Choose $\{f, f'\}$ and $\{g, g'\}$ as two pairs of the splitting. The only chance to have a non-simple graph after a splitting is to have a multiple edge. However, there is no multiple edge because each pair is not on a 3-cycle, as f and g are a 2-edge cut.

In the second, suppose G' has a multiple edge, say xy. Since G is simple, xy consists of two edges, a new edge and an old edge from G, say e. By (3.4.2), applying two types of Sp to x which releases the multiple edge, results in $\Phi(H)$. Because of our assumption, after applying (3.4.2), the resulting graph must be non-simple. It implies that in G, there must exist two triangles containing the common edge e, say xyz and xyw. Now, applying each of the two types of Sp produces another multiple edge, yz or yw. Then we can apply (3.4.2) to y according to yz or yw. Either case tells us that G must have an edge zw because of our assumption. It follows that the four vertices, $\{x, y, z, w\}$, span a K_4 in G. Note that G has two 2-edge-cuts, $\{f, g\}$ and, say $\{a, b\}$, such that $G\backslash f\backslash g\backslash a\backslash b$ has the component K_4 containing $\{x, y, z, w\}$. Since each of them is a 2-edge-cut of G, $\{f, g\} \cap \{a, b\} = \emptyset$. Moreover, $\{f, g\}$ has no common vertex in this case. Therefore, applying $O_4(K_5)$ has a problem only when a and b have a common vertex, say v. In that case, the resulting graph has a multiple edge again. Applying (3.4.2) to v, we have a simple graph because $\{a, b\}$ is a 2-edge cut in G. That is, G is reducible in $\Phi_{0,3}(H)$. □

Now we can prove the following key lemma to prove a splitter theorem for $\Phi_{0,3}$.

Lemma 3.4.4. *If G contains a K_4, then G is reducible in $\Phi_{0,3}(H)$ unless all vertices of the K_4 are from $V(H)$.*

Proof. By (3.4.1b), we can assume at most one vertex of a K_4 is from $V(H)$. Let $T = \{a, b, c, d\}$ be the 4-edge-cut of G such that $G\backslash T$ contain the K_4. By (3.4.3), we can assume no three of T have a common vertex. Also, if four of S have a common vertex, then we can use the graph operation $O_0(K_5)$. Moreover, if none of T have a common vertex, then we can apply $O_4(K_5)$ because the resulting graph is simple and is in $\Phi_{0,3}(H)$. Hence, we only need to show the case that two of T have a common vertex, say x. In this case, the resulting graph after applying $O_4(K_5)$ has a multiple edge containing x, say xy. By (3.4.2), applying two splittings at x releasing the multiple edge, results in $\Phi(H)$. If both of them result in a non-simple graph, then three of T have a common vertex. □

After the key lemma above, it is not difficult to prove the following splitter theorem for $\Phi_{0,3}$.

Theorem 3.4.5. *If G and H are in $\Phi_{0,3}$, and $H \propto G$, then G can be reduced to H within $\Phi_{0,3}$ by applying a sequence of Sp, $O_4(K_5)$, $O_2(K_5)$ and $O_0(K_5)$.*

Proof. If $G = H$, then the result holds. So, we can assume that there is a vertex, say v, not from $V(H)$. Then, by (3.2.4), there is a type of Sp at v such that the resulting graph, say G', is in $\Phi(H)$. Suppose G is not reducible in $\Phi_{0,3}(H)$. Hence, G' is not simple and since G is simple, G' has a multiple edge, say xy. Note that xyv spans a triangle. By (3.4.2), applying two splittings at x releasing the multiple edge, results in $\Phi(H)$. Since G is irreducible and we can assume no K_4 in G by (3.4.4), xy or xv is contained in three different triangles. Say, xv is the common edge of three different triangles. By symmetry, we can apply (3.4.2) to y according to the same multiple edge above. Then from the same argument above, yx or yv is contained in three different triangles. This means G contains a K_4. Thus, the result holds. □

The following are splitter theorems and generating theorems for G having a higher connectivity. Here, if we allow H to be the empty graph, then we need to add $O_0(K_5)$ in the list of operations.

Theorem 3.4.6. *If G and H are in $\Phi_{2,3}$ with $H \propto G$, then G can be reduced to H within $\Phi_{2,3}$ by applying a sequence of Sp, $O_4(K_5)$ and $O_2(K_5)$.*

Proof. By (3.4.5), we only need to take care of connectivity. Since both $O_4(K_5)$ and $O_2(K_5)$ consist of contractions, they do not decrease the connectivity of the resulting graph after applying them. So, we only need to show the result for Sp. The same argument as in (3.2.6) works because of the note after (3.2.6). □

In the following, we can obtain a generating theorem as a corollary of (3.4.6), which is the same result of F. Bories, J-L. Jolivet, and J-L. Fouquet [2]. To prove the result, we only need to find $\propto$-minimal graphs in $\Phi_{2,3}$. In (3.4.6), note that if we allow H to be the empty graph, then we only need to add the operation

$O_0(K_5)$ to have the result hold. Since any graph G contains the empty graph as an immersion, so does G in $\Phi_{2,3}$. In each sequence of the reducing process from G of $\Phi_{2,3}$ to the empty graph, there is a K_5 before the empty graph. Hence, K_5 is a $\propto$-minimal graph in $\Phi_{2,3}$. Moreover, this is only one $\propto$-minimal graph because the only operation to reach the empty graph is $O_0(K_5)$. So, the following corollary holds by (3.4.6).

Corollary 3.4.7. *Every connected* 4*-regular simple graph can be reduced to* K_5 *within* $\Phi_{2,3}$ *by applying a sequence of Sp,* $O_4(K_5)$ *and* $O_2(K_5)$*.*

The following is a splitter theorem for 4-edge connected 4-regular simple graphs.

Theorem 3.4.8. *If* G *and* H *are in* $\Phi_{4,3}$*, and* $H \propto G$*, then* G *can be reduced to* H *within* $\Phi_{4,3}$ *by applying a sequence of Sp and* $O_4(K_5)$*.*

Proof. By (3.4.6), we only need to keep that applying Sp to a vertex of G results in a 4-edge connected graph. We can use the exactly same argument as the one in (3.3.8). □

From the same observation before (3.4.7), K_5 is the unique $\propto$-minimal graph in $\Phi_{4,3}$. Hence, by (3.4.8), the following corollary holds.

Corollary 3.4.9. *Every* 4*-edge connected* 4*-regular simple graph can be reduced to* K_5 *within* $\Phi_{4,3}$ *by applying a sequence of Sp and* $O_4(K_5)$*.* □

Chapter 4

Splitter Theorems for 4-regular Planar Graphs

4.1 Introduction

We have proved splitter theorems for 4-regular graphs in Chapter 3. In this chapter, we will prove splitter theorems (see Section 1.6) for 4-regular planar graphs. We will assume that a 4-regular graph H is immersed in a 4-regular planar graph G, denoted by $H \propto G$. Since H could be non-planar, we will prove that G can be reduced to a pinched graph H^P of H, instead of reducing to H itself.

Recall that vertices in $V(H^P) - V(H)$ are called crossing vertices (see Section 1.4). A *crossing point* in G is a vertex $v \in V(G)$ where an edge-trail intersects edges incident with v. If H is immersed in G, then a crossing vertex is a crossing point, but not vice versa.

Let $\Phi_{k,g}$ be the family of k-edge connected 4-regular graphs of girth at least g, and let $P\Phi_{k,g}$ be all planar graphs in $\Phi_{k,g}$. We will prove that we can reduce G to H^P within $P\Phi_{k,g}$ without increasing the number of crossing points of G if $G, H \in \Phi_{k,g}$ with $H \propto G$ and G is a plane graph.

If a plane graph H is immersed in a plane graph G without any crossing points, then we can reduce from G to H itself. In Section 4.5, we will prove that we cannot replace H^P by H in the splitter theorems in Section 4.2 and Section 4.3 if we allow only a finite number of graph operations.

In the following three sections, we prove splitter theorems for $P\Phi_{k,g}$, for $g = 1$, $g = 2$, and $g = 3$, respectively. In addition, we will also determine $\propto$-minimal graphs in each $P\Phi_{k,g}$. Then, combining $\propto$-minimal graphs in $P\Phi_{k,g}$ with a corresponding splitter theorem, we will obtain a generating theorem (see Section 1.6) in $P\Phi_{k,g}$. Table 4.1 shows the numbers of splitter theorems and generating theorems that will be proved in this chapter, and the names of authors who proved a corresponding result.

TABLE 4.1. Splitter theorems and generating theorems for 4-regular k-edge connected planar graphs with girth at least g

	$g = 1$	$g = 2$	$g = 3$
k=0	Thm 4.2.6	Thm 4.3.4	Thm 4.4.12
	Cor 4.2.9	Cor 4.3.7	Cor 4.4.14
k=2	Thm 4.2.8	Thm 4.3.5	Thm 4.4.13
	Cor 4.2.9	Cor 4.3.7	Manca, Lehel
k=4		Thm 4.3.8	Thm 4.4.15
		Cor 4.3.9	Cor 4.4.16

4.2 4-regular Planar Graphs

In this section, we consider the cases when $g = 1$. So, we allow loops here. There are only two classes, $P\Phi_{0,1}$ and $P\Phi_{2,1}$ because $k \leq 2g$ and $P\Phi_{1,1} = P\Phi_{2,1}$.

We will use only one operation in this section, which is called *planar splitting*, denoted by PS (see Figure 4.1). Let H be a 4-regular graph and let G be a 4-regular planar graph with $H \propto G$. For convenience we fix a plane graph isomorphic to G and denote the plane graph by G.

There are several important notes about PS in the following. Let both G and H be 4-regular graphs and let G be a plane graph with $H \propto G$. Note that after applying each PS, the number of vertices of G decreases by one, and the number of edges of G decreases by two. The resulting graph is still a 4-regular graph. Moreover, no crossing points are produced by PS. Thus applying PS to a plane graph maintains planarity. Hence, the following note holds.

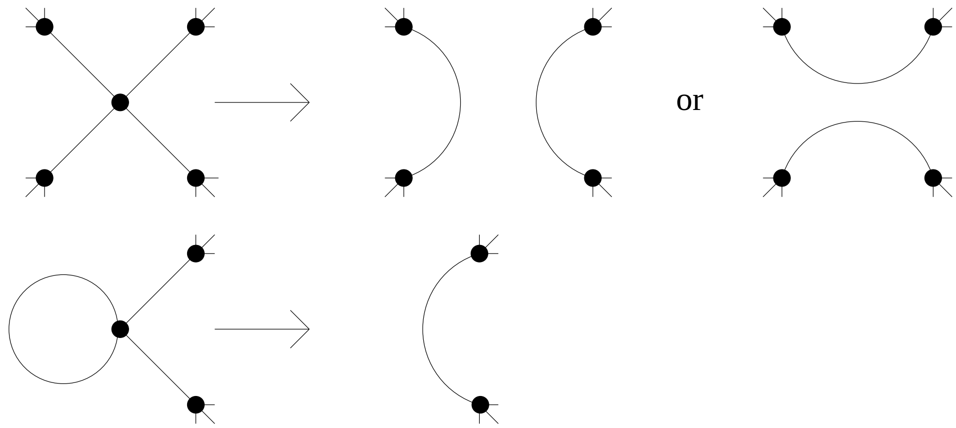

FIGURE 4.1. The operation PS.

Lemma 4.2.1. *Applying PS to a vertex of a 4-regular planar graph results in a 4-regular planar graph.*

To maintain edge-connectivity, the following Lemma (4.2.2) is useful. The proof is exactly the same as (3.2.1).

Lemma 4.2.2. *If G is a connected plane graph and applying PS to x results in a disconnected graph, then there is another planar splitting at x that produces a connected graph.* □

Now, we would like to pursue PS in such a type that the resulting graph maintains H as an immersion without increasing the number of crossing points in G. Let $\Phi(H)$ be a class consisting of all graphs G with $H \propto G$. The following lemma shows us that if x is not from $V(H)$, we can find a type of PS such that the resulting graph is in $\Phi(H)$. By (3.2.3) and (3.2.5), the following holds.

Lemma 4.2.3. *If x is not from $V(H)$ and no redtrail crosses with a red or white trail at x, then applying faithful PS to x results in $\Phi(H)$ without increasing the number of crossing points in G.* □

By (4.2.3), we can assume that there is no white vertex, and no redtrails are tangent with any redtrails. So, every vertex is a vertex from $V(H)$ or a crossing point of G. The following lemma tells us that we can reduce a crossing point x if a white edge is incident with x.

Lemma 4.2.4. *If a white edge is incident with a vertex x, then we can apply all possible types of PS at x so that the resulting graph is in $P\Phi_{0,1}(H)$ and the number of crossing points in G does not increase.*

Proof. We can deduce the lemma from (3.2.5) except the condition about the number of crosssings. We only need to show that the white cycle C in the proof of (3.2.5) does not increase the number of crossing points in G. Observe that if a vertex x of C is white, then we can choose edges incident with x so that we do not increase the number of crossing points in G. If a red edge incident with x and a redtrail crosses white edges, then we also do not increase the number of crossing points in G because that the crossing point was counted already. □

By (4.2.4), we can assume that we have no white edges in G. So, by (4.2.3), we may assume that each vertex of G is either from $V(H)$ or a crossing point of two redtrails.

Lemma 4.2.5. *If every vertex of G is from $V(H)$ or a crossing point of two redtrails, then G can be reduced to H^P without increasing the number of crossing points in G by modifying redtrails and using PS.*

Proof. Suppose that G is not a pinched graph of H. There are three types of obstruction. First, there is an edge trail crossing itself. Second, there are two trails crossing each other more than once. Third, there are two adjacent edge-trails crossing each other. For the first type, it is easy to change the trail from crossing itself to touching itself. Note that this modification decreases the number of crossing points in G. Then, we can use PS by (4.2.3). Thus, we can eliminate the first type. Similarly, for the second type, we can change the two trails from more than one crossing points to at most one crossing point because two crossing points in G can be changed to two tangent points. This modification also decreases the number of crossing points in G. Then, we can use (4.2.3). For the third type, we can use the same strategy as the other types. Since two redtrails are adjacent at a vertex $v \in V(H)$, they will be a vu-trail and a vw-trail with $u, w \in V(H)$. Then, change the crossing points to the tangent points, which decreases the number of crossing points in G. Also the resulting graph is in $P\Phi(H)$. By using (4.2.3) again, we can eliminate the third type of crossing points in G. □

By (4.2.1), (4.2.3), (4.2.4) and (4.2.5), the following theorem holds.

Theorem 4.2.6. *If H is a 4-regular graph and G is a 4-regular plane graph with $H \propto G$, then G can be reduced to H^P within $P\Phi_{0,1}$ by applying a sequence of PS without increasing the number of crossing points in G.* □

To prove a splitter theorem for $P\Phi_{2,1}$, we only need to prove the following lemma.

Lemma 4.2.7. *Let G be in $P\Phi_{2,1}(H)$. If applying PS to x results in a disconnected graph, then there is another type of PS at x such that the resulting graph is in $P\Phi_{2,1}(H)$ and the number of crossing points in G does not increase.*

Proof. By the same observation as in (3.2.6), we can conclude that a white edge is incident with x. Then, by (4.2.4) and (4.2.2), the lemma holds. □

By (4.2.6) and (4.2.7), the following theorem holds.

Theorem 4.2.8. *If H is in $\Phi_{2,1}$ and G is in $P\Phi_{2,1}$ with $H \propto G$, then G can be reduced to H^P within $P\Phi_{2,1}$ by applying a sequence of PS without increasing the number of crossing points in G.* □

Since every 4-regular plane graph contains $2L$ as an immersion, the following corollary holds by (4.2.8). Notice that we do not have to consider a pinched graph because we can assume that $2L$ is immersed in G without any crossing points.

Corollary 4.2.9. *Every connected 4-regular plane graph can be reduced to $2L$ within $P\Phi_{2,1}$ by applying a sequence of PS.* □

4.3 4-regular Loopless Planar Graphs

In this section, we consider the cases when $g = 2$. So, multiple edges are allowed, but no loops. Let G and H be 4-regular loopless graphs, and let G be a plane graph with $H \propto G$. Since no loop is incident with each vertex x of G, we can name four edges incident with x clockwise, say e_0, e_1, e_2, and e_3. There are three distinct classes: $P\Phi_{0,2}$, $P\Phi_{2,2}$, and $P\Phi_{4,2}$ because $k \leq 2g$, and $P\Phi_{1,2}$=$P\Phi_{2,2}$, and $P\Phi_{3,2} = P\Phi_{4,2}$.

Since all graphs in this section are 4-regular, we can apply the splitter theorems in the previous section here unless applying PS produces a loop. Also we can use lemmas in Section 3.2 except that we have to watch crossing points. Thus the following lemmas overlap some part, but we list these for the completeness without proofs in detail. The first lemma, which overlaps with (3.3.1), holds because H does not contain a loop.

Lemma 4.3.1. *Let H have no loop and $H \propto G$. If a vertex of $2K_2$ in G is not from $V(H)$, then the other vertex of the graph is also not from $V(H)$.* □

The following almost overlaps with (3.3.2). Let S be an induced subgraph of a plane graph G to which we can apply one of $O_0(4K_2)$ and $O_2(4K_2)$. The proof is the same as in (3.3.2) because (3.3.1)and (4.3.1) are equivalent and we can replace (3.2.4) by (4.2.3).

Lemma 4.3.2. *Suppose at least one of $V(S)$ is not from $V(H)$ and an applied vertex for PS is not a crossing point. Then, applying one of PS, $O_0(4K_2)$ and $O_2(4K_2)$ to S results in a in $P\Phi(H)$, and these operation does not increase the number of crossing points.* □

Lemma 4.3.3. *Let H have no loop and let G be a 4-regular planar graph with $H \propto G$. If G has a loop incident with a vertex x, then applying any type of Sp to*

x results in a graph in $P\Phi(H)$ without increasing the number of crossing points in G.

Proof. By the same argument in (3.3.3), we can assume a white edge is incident with x. By (4.2.4), the result holds. □

The following Theorem (4.3.4) holds because we can use the same argument in (3.3.4) except replacing (3.3.2) and (3.3.3) by (4.3.2) and (4.3.3), respectively.

Theorem 4.3.4. *If G and H are in $\Phi_{0,2}$, and G is planar with $H \propto G$, then G can be reduced to H^P within $P\Phi_{0,2}$ by applying a sequence of PS, $O_0(4K_2)$ and $O_2(4K_2)$ without increasing the number of crossing points.* □

Now, to maintain connectivity is not difficult.

Theorem 4.3.5. *If G and H are connected loopless 4-regular graphs, and G is a plane graph with $H \propto G$, then G can be reduced to H^P within $P\Phi_{2,2}$ by applying a sequence of PS and $O_2(4K_2)$ without increasing the number of crossing points.*

Proof. By (4.2.1) and (4.3.4), we only need to show that there is a graph operation such that the resulting graph is in a connected graph in $\Phi(H)$. Since $O_2(4K_2)$ consists of a contraction, applying $O_2(4K_2)$ to a connected graph results in a connected graph. So, we only need to check PS. By (4.2.2) and (4.2.6), the lemma follows from the exact same argument in (4.2.7). Note that changing a type of PS is unique because of planarity and the new edge does not decrease the girth since it belongs to a edge cut of the new resulting graph. □

For a generating theorem for $P\Phi_{2,2}$, the following lemma is useful and is the same as (3.3.6).

Lemma 4.3.6. *Every connected loopless 4-regular graph G contains $4K_2$ as an immersion.* □

By (4.3.5) and (4.3.6), the following Corollary (4.3.7) holds.

Corollary 4.3.7. *Every connected loopless* 4*-regular planar graph can be reduced to* $4K_2$ *within* $P\Phi_{2,2}$ *by applying a sequence of* PS *and* $O_2(4K_2)$ *without increasing the number of crossing points.* □

Since every 4-edge connected graph has no 2-edge cut, every graph of $P\Phi_{4,2}$ contains no $3K_2$ as an induced subgraph. So, we do not have to use $O_2(4K_2)$. The following is a splitter theorem for $P\Phi_{4,2}$.

Theorem 4.3.8. *If* G *and* H *are in* $\Phi_{4,2}$*, and* G *is a plane graph with* $H \propto G$*, then* G *can be reduced to* H^P *within* $P\Phi_{4,2}(H)$ *by applying a sequence of* PS *to* H^P *without increasing the number of crossing points.*

Proof. By (4.3.5), applying PS to a vertex x not from $V(H)$ results in a graph in $P\Phi_{2,2}(H)$. We only need to show that, if applying a planar splitting at x results in a not 4-edge connected graph, then there is another type of PS at x such that the resulting graph is in $P\Phi_{4,2}$. By the same argument in (3.3.8), a white edge is incident with x. By (4.2.4), the theorem holds. □

By (1.2.6), every 4-edge connected graph contains $4K_2$ as an immersion. Thus, the following generating theorem follows by (4.3.8). Since $4K_2$ can be immersed in G without any crossing points, we can reduce G to $4K_2$ itself, instead of reducing to a pinched graph of $4K_2$.

Corollary 4.3.9. *Every* 4*-edge connected* 4*-regular plane graph* G *can be reduced to* $4K_2$ *within* $P\Phi_{4,2}$ *by applying a sequence of* PS. □

4.4 4-regular Simple Planar Graphs

In this section, we consider the cases when $g = 3$. In other words, we will prohibit loops and multiple edges in any graph. We have three distinct classes, $P\Phi_{0,3}$, $P\Phi_{2,3}$, and $P\Phi_{4,3}$ because $k \leq 4$, and $P\Phi_{1,3}$=$P\Phi_{2,3}$, and $P\Phi_{3,3}$=$P\Phi_{4,3}$. However, the splitter theorem for $P\Phi_{0,3}$ is crucial. Let $G \in P\Phi_{0,3}$ and H is a 4-regular simple graph with $H \propto G$. The following Lemma (4.4.1) is contained in (3.4.1).

Lemma 4.4.1. *Suppose H is a 4-regular simple graph and $G \in P\Phi_{0,1}(H)$. Let T be an t-edge cut of G and let S be a component of $G \backslash T$.*

(a). If $t = 2$, and $V(S)$ has vertices from $V(H)$ less than five, then $V(S)$ has no vertices from $V(H)$.

(b). If $t = 4$, then $V(S)$ cannot have exactly two or three vertices from $V(H)$.

□

Let G be a 4-regular simple plane graph. To maintain girth, we will prove a key lemma, which is similar to (3.4.2) but we need more detail because of planarity. Let i be an integer modulo 4 and let e_i be the edges incident with a vertex x such that the incident edges are numbered clockwise around x in the plane graph G. Then, note that if we apply a type of PS to x, say e_i and e_{i+1} are paired, then an n-cycle containing e_i and e_{i+1}, will result in an $n-1$-cycle after applying PS. Thus, the resulting graph has no loop after applying PS because G has no 2-cycles. Hence, the only problem is when $n = 3$. We call this planar splitting a *triangle splitting* with the 3-cycle. The following Lemma (4.4.2) solves this problem.

Recall that we say that a type of *Sp releases a multiple edge* or a 2-cycle if applying the type of *Sp* to a vertex of the 2-cycle separates the multiple edges. Note that if a 2-cycle consists of e_i and e_{i+1}, then there is only one PS at x that releases the 2-cycle, but if a 2-cycle consists of e_i and e_{i+2}, then there are two types of PS at x that release the 2-cycle. Call the vertex in the last case, *u-vertex*. Let x, y be distinct vertices of a 2-cycle. Then observe that if x is a u-vertex, then y must be also a u-vertex by (1.2.1). So, if x is not a u-vertex, then neither is y.

Lemma 4.4.2. *Let H be a 4-regular simple graph and $G \in P\Phi_{0,1}(H)$. If G has a multiple edge, say xy, then applying the two types of PS at x (they are possibly isomorphic) that release the multiple edge results in $P\Phi_{0,1}(H)$ and they do not increase the number of crossing points.*

Proof. Since H is simple, at least one of x and y is not be from $V(H)$. So, there are two cases; only one vertex is from $V(H)$ or none of them are from $V(H)$.

First, if y is from $V(H)$, then x is not. Then two redtrails passing through x and they are adjacent because both of them have the endpoint y. Thus if the two redtrails cross at x, then we can change the trails from crossing at x to touching at x without losing redtrails and can reduce the number of crossing points in G. Hence, the lemma holds in this case. On the other hand, if x is from $V(H)$, then y is not. In this case, we can switch x and y without losing redtrails and the switching does not increase the number of crossing points in G.

In the second, neither x nor y is from $V(H)$. Suppose that there are no types of Sp at x and y such that the types can release the multiple edge. By (4.2.4), we only need to show that a white edge is incident with x. Suppose every multiple edge xy is red. Then a redtrail must be closed because there are no types of Sp at x and y that releases the multiple edge. Thus, we can change a multiple edge xy from red to white. So, we may assume that there is a type of Sp that releases the multiple edge at x or y. Suppose that there is no type of Sp that releases the multiple edge at x. Suppose all edges incident with x are red. By our assumption, there is a type of Sp at y that releases the multiple edge. Suppose x is a u-vertex. Then y is also a u-vertex and there is no crossing at y because, otherwise, there is no releasing planar splitting at y. Notice that in this case we can change a multiple edge xy from red to white because there is a waste closed sub-redtrail. So, we may assume that x is not a u-vertex. Since there is no type of Sp at x that releases the multiple edge, there is a waste red cycle that contains y. So, we can change a red multiple edge xy to white.

Therefore, we can assume that there is a type of Sp at x that releases multiple edge and all edges incident with x is red. If there is no crossing at x, then the lemma holds by (4.2.3). If x is a crossing point, then we can assume that y is not a crossing point because, otherwise, we can change these two crossing points to two touching points at a time. Observe that in this case, we can switch x and y so that there is a type of non-crossing Sp or PS at x, which releases the multiple edge, does not lose any redtrails and does not increase the number of crossing points. $\square$

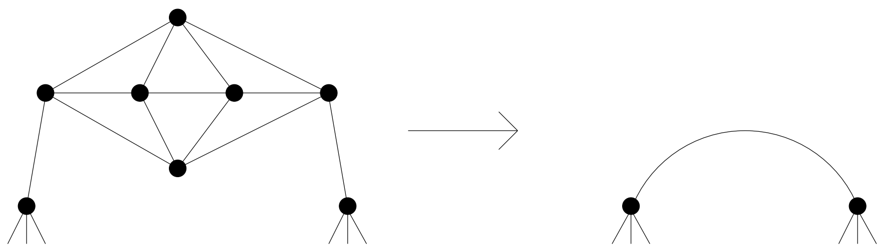

FIGURE 4.2. The operation $O_2(Oct)$.

Since we have proved a lemma for the operation PS, let us move to prove lemmas for the other graph operations used in this section. Since that the octahedron (see Figure 1.14), denoted by Oct, is the smallest 4-regular simple planar graph, we will use $O_i(Oct)$ with $i = 0, 2, 4$. Figure 4.2 shows $Oct_2(Oct)$. Also we will use $O_4(K_5)$ (see Figure 4.3). Each of four operations above is equivalent to applying a sequence of PS, but we need these operations to maintain simplicity. We will prove lemmas related with each graph operation one by one. In the following lemmas, let S be an induced subgraph of G to which we will apply each graph operations (see Section 1.5). The following lemma will be proved by (4.4.1(a)).

Lemma 4.4.3. *Let $i = 0$ or 2. Applying $O_i(Oct)$ to a subgraph S of a plane graph G results in a graph in $P\Phi_{0,1}(H)$ unless five or more vertices of S are from $V(H)$. This operation does not increase the number of crossing points in G.* $\square$

Next, let us investigate graph operations $O_i(K)$ with $i = 4$. We will see $K = K_5$ and $K = Oct$ in this order. Recall S is an induced subgraph isomorphic to $K - v$ if $i = 4$. Since $S = K_5 - v = K_4$, we will study K_4 in G. Suppose $G \in \Phi_{0,3}(H)$

contains K_4 as a subgraph. Let $T = \{a, b, c, d\}$ be edges in $G \backslash E(K_4)$ incident with $V(K_4)$. Since G is simple, edges of T are pairwise distinct. Hence, T is a 4-edge-cut of G, and K_4 is a component of $G \backslash T$. Moreover, since G is a plane graph, the set T consists of two 2-edge cuts of G (see Figure 4.3). Thus, we can contract K_4 to a point so that the resulting graph is a 4-regular planar graph. Notice that this is the same as $O_4(K_5)$ in Figure 1.13 but pictures are different because of planarity.

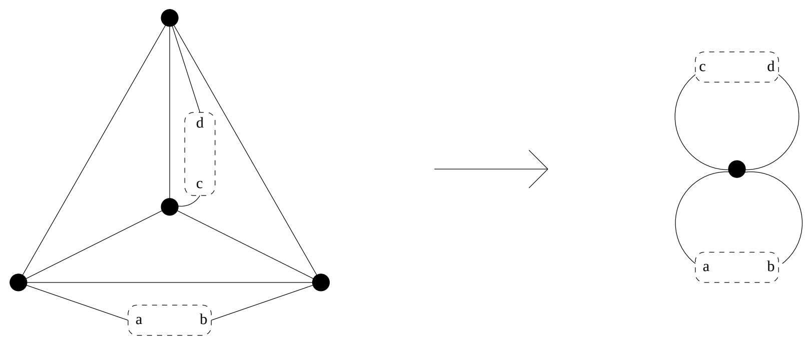

FIGURE 4.3. The operation $O_4(K_5)$ for plane graphs.

Note that by (4.4.1b), the resulting graph after applying $O_4(K_5)$ keeps H unless all four vertices of K_4 are from $V(H)$. Also applying $O_4(K_5)$ does not increase the number of crossing points in G. Therefore, the following (4.4.4) holds. Let G and H be a 4-regular simple graph and G be a plane graph with $H \propto G$ in the following.

Lemma 4.4.4. *Applying $O_4(K_5)$ to S in a plane graph G results in a graph in $P\Phi(H)$ unless all vertices of S are from $V(H)$. This operation does not increase the number of crossing points in G.* □

We will prove a useful lemma related with K_4.

Lemma 4.4.5. *If a 4-regular planar simple graph G contains a K_4 and not all vertices of the K_4 are from $V(H)$, then G is reducible in $P\Phi_{0,3}(H)$ and the operation does not increase the number of crossing points.*

Proof. Without loss of generality, the graph G contains the left-hand side graph in the Figure 4.3. By (4.4.1b), we only need to check simplicity. Suppose that applying $O_4(K_5)$ results in a non-simple graph. Then the resulting graph has a multiple edge, say xy. Since G is simple, x or y is a new vertex occurring from the contraction, say x. By (4.4.2), the vertex y has a planar splitting that releases the 2-cycle containing xy. Observe that y is a cut vertex in G, which implies the resulting graph is simple. Thus, we can apply the releasing PS to y instead of applying $O_4(K_5)$ to K_4 and the resulting graph is in $P\Phi_{0,3}$. □

To investigate $O_4(Oct)$, we will see $S = Oct - v$ isomorphic to a *wheel*. The wheel with five vertices, isomorphic to $K_1 \vee C_4$, will be denoted by W_4. Note that applying $O_4(Oct)$ (see Figure 1.15) is equivalent to contract $S = W_4$. We can see that the resulting graph after applying $O_4(Oct)$ to S in a 4-regular planar simple graph is a 4-regular planar graph. The following lemma is an analog to (4.4.4).

Lemma 4.4.6. *Applying $O_4(Oct)$ to S in a plane graph G results in $P\Phi(H)$ unless four or more vertices of S are from $V(H)$. This operation does not increase the number of crossing points.*

Proof. Let $T = \{a_i\}$ with $0 \leq i \leq 3$ be the set of all edges in $G \backslash E(S)$ incident with $V(S)$. Then, by (4.4.1b), we only need to show that T is a 4-edge cut. To show this we must prove that T consists of four distinct edges. Suppose a_i is numbered clockwise in the plane. Then clearly $a_i \neq a_{i+1}$ with modulo 4 holds because G is simple. Moreover, if $a_i = a_{i+2}$, then G contains K_4. By (4.4.5), the result holds. □

We will prove the following three lemmas analogous to (4.4.5) to complete the proof of splitter theorems in this section.

Lemma 4.4.9. *If G contains a subgraph S isomorphic to $Oct \backslash e$, then G is reducible in $P\Phi_{0,3}(H)$ and that graph operation does not increase the number of crossing points in G, unless five or more vertices of S are from $V(H)$.*

Lemma 4.4.10. *If G contains a subgraph S isomorphic to W_4, then G is reducible in $P\Phi_{0,3}(H)$ and that graph operation does not increase the number of crossing points in G unless four or more vertices of S are from $V(H)$.*

Lemma 4.4.11. *If G contains a triangle ladder $L_{3,2}$ and there is a triangle splitting at y_2 with the triangle $x_2y_2x_3$, then G is reducible in $P\Phi_{0,3}(H)$, or G contains a cyclic ladder.*

To prove (4.4.9), we need the following two lemmas. Let x be a vertex of a 4-regular simple plane graph G. If x is a vertex of an n-cycle, say C_n, we call two edges incident with x and contained in $G \backslash E(C_n)$, *unknown edges of x with C_n*. By (1.4.1), every n-cycle in the plane separates the plane into two areas: interior and exterior of the n-cycle, denoted by $int(C_n)$ and $ext(C_n)$, respectively. Observe that if there is a triangle splitting PS at x, then the unknown edges of x with the triangle, say C, are in $int(C)$ or $ext(C)$. In other words, the unknown edges can not be separated into two areas.

Lemma 4.4.7. *If there is a triangle splitting PS at x with a triangle $C = xyz$, then G is reducible in $P\Phi_{0,3}(H)$, or all the unknown (four) edges of y and z are in $int(C)$ or $ext(C)$.*

Proof. Suppose that G is irreducible in $P\Phi_{0,3}(H)$. Let $\alpha = int(C)$ and let $\beta = ext(C)$. We only need to show that all unknown edges of y and z with xyz are in α or in β. Without loss of generality, let β contain the unknown edges of x with xyz. Suppose the unknown edges of y with xyz are in α and β, say yy_α and yy_β, respectively. Let G' be the resulting graph after applying the triangle splitting at x. Then, y is a u-vertex, which implies that z is also a u-vertex by (1.2.1). Thus, the unknown edges of z are in α and β, say zz_α and zz_β, respectively.

Since G' contains the multiple edge yz and $H \propto G'$, there are two releasing planar splittings at y by (4.4.2), including a PS pairing yz and yy_α. By our assumption, we must have a triangle containing yz and yy_α, which implies $y_\alpha = z_\alpha$.

Then by (4.4.2), there is a releasing PS at y_α, which results in a simple graph because yy_α and zz_α form a 2-edge-cut. It contradicts our assumption. Thus, all unknown edges of y with C must be in α or β. By symmetry, the unknown edges of z must be in an area.

Without loss of generality, we can suppose all unknown edges of y are in α and all unknown edges of z are in β. By (4.4.2), there is a releasing splitting PS at y, which results in a simple graph because two unknown edges of y with C form a 2-edge-cut. This contradiction completes the proof. □

We need one more lemma to prove (4.4.9). Let P_i be a x_1x_i-path containing vertices x_1, x_2, ..., x_i in this order. Similarly, let Q_j be a y_1y_j-path containing vertices y_1, y_2, ..., y_j in this order. Here, $i = j$ or $j + 1$. Let $L_{i,j}$ be the graph obtained from $P_i \cup Q_j$ by adding edges between x_m and $y_{m'}$ if $m = m'$ or $m' + 1$. So, all vertices are of degree four in $L_{i,j}$ except that x_1 and y_j (or x_{j+1}) are of degree two, and y_1 and x_j (or y_j) are of degree three. We call $L_{i,j}$ a *triangle ladder*. Note that W_4 contains $L_{3,2}$ as a subgraph, and Oct contains $L_{3,3}$ as a subgraph.

Lemma 4.4.8. *If applying a type of PS to a vertex x in G results in in $P\Phi(H)$, then G is reducible in $P\Phi_{0,3}(H)$, or G contains a triangle ladder $L_{3,2}$ containing x and there is a triangle splitting at y_2 in $L_{3,2}$ with the triangle $x_2y_2x_3$.*

Proof. Suppose G is irreducible in $P\Phi_{0,3}(H)$. Let x be the vertex of G where applying a type of PS results in $P\Phi(H)$. Since G is irreducible, the planar splitting PS must be a triangle splitting with a triangle, say $xyz = C$. Without loss of generality, we can assume that $ext(C)$ contains the unknown edges of x with C.

Then by (4.4.7) there are two cases; all the unknown edges of y and z with C are in $ext(C)$ or $int(C)$.

Let us investigate the first case. By (4.4.2), there are two releasing planar splittings at y, which are isomorphic to each other because y is not a u-vertex. Since G is irreducible, the unique releasing PS at y must be a triangle splitting. By symmetry, we can assume that the triangle splitting contains xy, say $xyw = C'$ is the triangle. Thus, G contains a $L_{2,2}$, say $x_1 = z, x_2 = x, y_1 = y, y_2 = w$. By (4.4.7), the unknown edges of w with C' are in $ext(C')$. By (4.4.2) again, w has a releasing splitting PS because there is a triangle splitting at y. Since G is irreducible, it must be a triangle splitting at w, say xwx_3, which implies that there is a triangle splitting at $y_2 = w$ with the triangle $x_2y_2x_3$. Note that by (4.4.5), $x_3 \neq x_1$ because G contains no K_4. Thus G contains a $L_{3,2}$ with the desired PS.

In the second case, all unknown edges of y and z with C are in $int(C)$. By (4.4.2), there is the unique releasing PS at y. It must be a triangle splitting because G is irreducible. The only chance to have a triangle containing an unknown edge of y is to make a triangle containing yz. Let $yzv = D$ be the triangle. By (4.4.7), the unknown edges of v with D are in $ext(D)$. Note that G contains a $L_{2,2}$, say $x_1 = x, x_2 = z, y_1 = y, y_2 = v$. By (4.4.2) again, there is the unique releasing PS at v, and it must be a triangle splitting. By symmetry, without loss of generality, there is a triangle splitting at $v = y_2$ with a triangle containing vz, say $vzx_3 = D'$. The unknown edges of x_3 with D' are in $ext(D')$. Moreover, $x_3 \neq x$ or $x_3 \neq x_1$ by (4.4.5). Therefore, G contains $L_{3,2}$ with the desired PS at y_2. □

Now, we will prove the following lemma by (4.4.7) and (4.4.8).

Lemma 4.4.9. *If G contains a subgraph S isomorphic to $Oct\backslash e$, then G is reducible in $P\Phi_{0,3}(H)$ and that graph operation does not increase the number of crossing points in G, unless five or more vertices of S are from $V(H)$.*

Proof. If S is not an induced graph, we can apply $O_0(Oct)$ and the result holds by (4.4.3). Hence let S be an induced subgraph of G and G be irreducible in $P\Phi_{0,3}(H)$. By (4.4.3), applying $O_2(Oct)$ to S results in $P\Phi(H)$ and it does not increase the number of crossing points in G. Since S is an induced subgraph, there is a 2-edge-cut, say $T = \{t_1, t_2\}$, such that $G \backslash T$ contains the component S. Let G' be the resulting graph, which is not simple by our assumption.

There are two cases: G' has a loop or a multiple edge. In the first case, we see that the loop is produced by applying $O_2(Oct)$ to S. This means that T has a common vertex, say v. In this case, by (4.3.3), G is reducible and applying PS to v does not increase the number of crossing points in G.

In the second case, t_1 and t_2 have a common adjacent edge, say xy and x is an endpoint of t_1. Thus xy is a multiple edge in G'. Let C be the 2-cycle containing xy in G'. Then, by (4.4.2), there are two releasing planar splittings at x. There are two cases depending on whether x is a u-vertex in G' or not. If x is a u-vertex, then y is also a u-vertex. Since x is a u-vertex, the two unknown edges of x with C are separated into $int(C) = \alpha$ and $ext(C) = \beta$, say xx_α and xx_β, respectively. Similarly, let yy_α and yy_β be the unknown edges of y with C in α and β, respectively. By (4.4.2), there are two releasing planar splittings at x in G. Since G is irreducible, the two releasing PS's must be triangle splittings. So, $x_\alpha = y_\alpha$ and $x_\beta = y_\beta$. By (4.4.2) again, we must have triangle releasing PS's at x_α and x_β. However, this is impossible because x_α and x_β are cut-vertices, which implies that x is not a u-vertex.

Now, xy is a multiple edge in G' and x is not a u-vertex. By (4.4.2), there is a planar splitting at x. Since G is irreducible, by (4.4.8), G contains a triangle ladder $L_{3,2}$ containing x. From the structure of G, we can say that there is the $L_{3,2}$ with $x_1 = x$ and $y_1 = y$. By (4.4.2) and (4.4.8), the vertex x_3 has the unique releasing

PS, which must be a triangle splitting. Then, there are two cases: x_1 and x_3 are adjacent or not. In the first case, G contains W_4 and a 4-edge-cut T' with $T \subset T'$. Note that no vertex in S is from $V(H)$ because the assumption and (4.4.1a). Since at least one vertex in W_4 is not from H and $T'' = T' \backslash T$ is a 2-edge-cut, by (4.4.1a), no vertices of W_4 is from $V(H)$. By (4.4.6), applying $O_4(Oct)$ to this W_4 results in $P\Phi(H)$ and does not increase the number of crossing points in G. Since G is irreducible, applying $O_4(Oct)$ produces a multiple edge, which implies T'' has a common vertex,say z. Then, by (4.4.2), there are releasing planar splittings at z. Since z is a cut vertex, G is reducible.

In the last, suppose x_1 and x_3 are not adjacent. Then G contains a triangle ladder $L_{i,j}$ with $3 \leq i$ and $3 \leq j$ with $x_1 = x, y_1 = y$. Let $L_{i,j}$ be a longest triangle ladder containing x and y. Then, there is a releasing PS at the last vertex of $L_{i,j}$: x_i or y_j. Since G is irreducible, it must be a triangle splitting. But to make a new triangle we cannot use a new vertex because $L_{i,j}$ is the longest, which implies that we must use $x_1 = x$. However, since only one edge is free among edges incident with x, neither x_i nor y_j has a triangle splitting, which implies that G is reducible. □

By (4.4.9), we can assume that G contains no subgraph isomorphic to $Oct \backslash e$ unless G contains five or more vertices from $V(H)$. The following lemma implies that we can assume G contains no W_4 unless G contains four or more vertices from $V(H)$.

Lemma 4.4.10. *If G contains a subgraph S isomorphic to W_4, then G is reducible in $P\Phi_{0,3}(H)$ and that graph operation does not increase the number of crossing points in G unless four or more vertices of S are from $V(H)$.*

Proof. Suppose G is irreducible. Then, by (4.4.6), applying $O_4(Oct)$ results in a non-simple graph, say G'. Let xy be the multiple edge of G', say that y is a new vertex from the contraction. Let T be the 4-edge-cut that $G \backslash T$ contains the component S. There are three cases: two, three or four edges of T are incident with x in G'. First, if four edges of T are incident with x, then G' is isomorphic to $4K_2$, which implies that G is isomorphic to Oct. Notice that no vertices in G is from $V(H)$ because we could use $O_4(Oct)$ and H is simple. Thus, we can apply $O_0(Oct)$ to G instead of $O_4(Oct)$, which contradicts our assumption. Second, if three of S are incident with x, then G contains $Oct\backslash e$. In this case, by (4.4.9) G is reducible. Third, if two of T are incident with x, then by (4.4.2), there are two releasing planar splittings at x in G. Since G is irreducible, they must be triangle splittings, which implies that three of T must be incident with x. □

By (4.4.8) and the following lemma, we can prove a splitter theorem for $P\Phi_{0,3}$ by using the graph operations, PS, $O_0(Oct)$, $O_2(Oct)$, $O_4(K_5)$, and $O_4(Oct)$.

Lemma 4.4.11. *If G contains a triangle ladder $L_{3,2}$ and there is a triangle splitting at y_2 with the triangle $x_2y_2x_3$, then G is reducible in $P\Phi_{0,3}(H)$, or G contains a cyclic ladder.*

Proof. Suppose G is irreducible. Choose the longest ladder containing the $L_{3,2}$, say either $L_{n+1,n}$ or $L_{n,n}$ where $n \geq 3$ holds by (4.4.5) and (4.4.10). By the assumption, the longest ladder $L_{n+1,n}$ (or $L_{n,n}$) contains a triangle splitting at y_n (or x_n) with the triangle $x_ny_nx_{n+1}$ (or $x_{n-1}y_nx_n$), respectively.

In the former case, by (4.4.2), x_{n+1} must have the unique releasing PS, which is a triangle splitting because G is irreducible. However, to make a new triangle we can not use a new vertex because this is the longest. So, we must use x_1 to make a new triangle containing y_nx_{n+1} because x_1 has two free edges, but y_1 does not. We can connect x_1 with y_n and x_{n+1}. Then there are two free edges: at y_1 and x_{n+1}. If y_1 and x_{n+1} are adjacent, G contains a subdivision of $K_{3,3}$, which contradicts (1.4.2). If they are not adjacent, then each of the two edges is a 1-edge-cut in a 4-regular graph, which contradicts (1.2.1).

In the latter case, by (4.4.2), y_n has the releasing PS, which must be a triangle splitting by our assumption. To make a new triangle containing $x_n y_n$, we must connect x_1 with x_n and y_n. Then, since y_n has PS, by (4.4.2), the vertex x_1 has a releasing PS, which must be a triangle splitting. It implies that $n = 3$ and that y_1 and y_3 are adjacent. Then G contains Oct, which is a cyclic ladder. □

The following is the first and most important splitter theorem in this section.

Theorem 4.4.12. *If G is a 4-regular simple planar graph, and $H \in \Phi_{0,3}$ with $H \propto G$, then G can be reduced to H^P within $P\Phi_{0,3}$) by PS, $O_0(Oct)$, $O_2(Oct)$, $O_4(K_5)$ and $O_4(Oct)$ without increasing the number of crossing points in G, unless G contains a (not Oct) cyclic ladder having a vertex not from $V(H)$.*

Proof. Suppose that G does not contain any cyclic ladders. Then, by (4.4.8) and (4.4.11), if there is a planar splitting in G such that the resulting graph is in $P\Phi(H)$, then G is reducible in $P\Phi_{0,3}(H)$ by graph operations above. Hence, we can assume that there is no planar splitting in G. By (4.2.3) and (4.2.4), the graph G does not contain any white edges, any touching vertices of two redtrails and any touching vertices by a redtrail itself. Thus, we can assume that every vertex of G is from $V(H)$ or a crossing point of two redtrails. By (4.2.5), there is a suitable planar splitting if G is not H^P. Then, by (4.4.8) and (4.4.11), the graph G can be reduced to H^P within $P\Phi_{0,3}$.

Suppose that G contains a cyclic ladder and is irreducible in $P\Phi_{0,3}(H)$. Then if G contains an Oct, then all vertices of the Oct are from $V(H)$; otherwise none of them is from $V(H)$ and G is reducible by $O_0(Oct)$. Hence G contains a (not Oct) cyclic ladder having a vertex not from $V(H)$. □

The following splitter theorem for $P\Phi_{2,3}$ can be proved by (4.2.7) and (4.4.12).

Theorem 4.4.13. *If G is a connected 4-regular simple planar graph, and $H \in \Phi_{2,3}$ with $H \propto G$, then G can be reduced to H^P within $P\Phi_{2,3}$ by PS, $O_2(Oct)$, $O_4(K_5)$*

and $O_4(Oct)$, without increasing the number of crossing points in G, unless G is isomorphic to a cyclic ladder having a vertex not from $V(H)$. □

The following Corollary (4.4.14) is the same as J. Lehel [12] and P. Manca [13] except that they added one more operation, say O_α, and they reduced all connected 4-regular simple planar graphs to Oct by PS, $O_4(K_5)$, $O_4(Oct)$, $O_2(Oct)$, and O_α.

Corollary 4.4.14. *Every connected 4-regular simple planar graph can be reduced to a cyclic ladder within $P\Phi_{2,3}$ by PS, $O_2(Oct)$, $O_4(K_5)$ and $O_4(Oct)$.*

Proof. Let G be a connected 4-regular simple plane graph. Suppose G is not a cyclic ladder. We will prove that Oct is immersed in G. Let H be the empty graph. Then $H \propto G$ and (4.4.13) holds if we add $O_0(Oct)$ to the set of graph operations. Therefore, there is a sequence G_0, G_1,..., G_t of graphs in $P\Phi_{2,3}(H)$ such that $G_0 = G$, $G_t = H$, and each G_i is obtained from G_{i-1} by applying a single graph operation. Then, G_{t-1} is isomorphic to Oct because the operations that can produce the empty set are only $O_0(Oct)$. Thus, the graph Oct is immersed in G. Note that this immersion does not create any crossing points in G, which implies that, by (4.4.13), G can be reduced to Oct itself, instead of $(Oct)^P$. Hence, G is a cyclic ladder or can be reduced to Oct, which is a cyclic ladder within $P\Phi_{2,3}$ □

The following is a splitter theorem for 4-edge connected 4-regular simple planar graphs. Since a 4-edge connected 4-regular simple planar graph G does not contain any 2-edge-cuts, we do not need $O_2(Oct)$, nor do we need $O_4(K_5)$ (see Figure 4.3). By the same argument in (4.3.8), the following splitter theorem for $P\Phi_{4,3}$ can be proved.

Theorem 4.4.15. *If $G \in P\Phi_{4,3}$, $H \in \Phi_{4,3}$, and $H \propto G$, then G can be reduced to H^P within $P\Phi_{4,3}$ by applying a sequence of PS and $O_4(Oct)$ without increasing the number of crossing points in G, unless G is isomorphic to a cyclic ladder.* □

By using the same argument as (4.4.14), an Oct is immersed in a 4-edge connected 4-regular simple plane graph G without any crossing points. Therefore, by (4.4.15), the following holds.

Corollary 4.4.16. *Every 4-edge connected 4-regular simple plane graph can be reduced to a cyclic ladder within $P\Phi_{4,3}$ by applying a sequence of PS and $O_4(Oct)$.*

□

4.5 Negative Results

In the previous sections, we showed that we can reduce G to H^P, not to H, when G is a 4-regular plane graph and a 4-regular graph H is immersed in G. The next logical question is whether the result is best. The answer is yes, as long as we allow only finitely many operations. In this section, we will show the existence of infinitely many pairs of (G, H) such that a 4-regular graph H is immersed in G and that there is no planar graph between G and H. By this, we mean that there is no planar graph immersed in G and contains H as an immersion.

Let n be an integer with $n \geq 4$, and let H_n and G_n be the graphs in Figure 4.4. Both graphs are symmetric with respect to a vertical line L (see Figure 4.4). We call one side of L *side* A and the other side *side* B. Note that both H_n and G_n are 4-regular plane graphs, and G_n has $(4n + 2)$ more vertices than H_n. Also we notice that we can obtain G_n from H_n by pulling the edge $\alpha_6\beta_6$ in Figure 4.4, and making extra $(4n + 2)$ crossing points with $(4n + 2)$ edges of H_n. This describes one immersion, say ϕ_0, where each edge of H_n is mapped to an edge trail of length one or two except that $\alpha_6\beta_6$ is mapped to the xx'-trail containing $(4n+2)$ crossing points including z and z'.

We will prove the following theorem.

Theorem 4.5.15. *There is no planar graph (except G_n and H_n) that is immersed in G_n, and contains H_n as an immersion.*

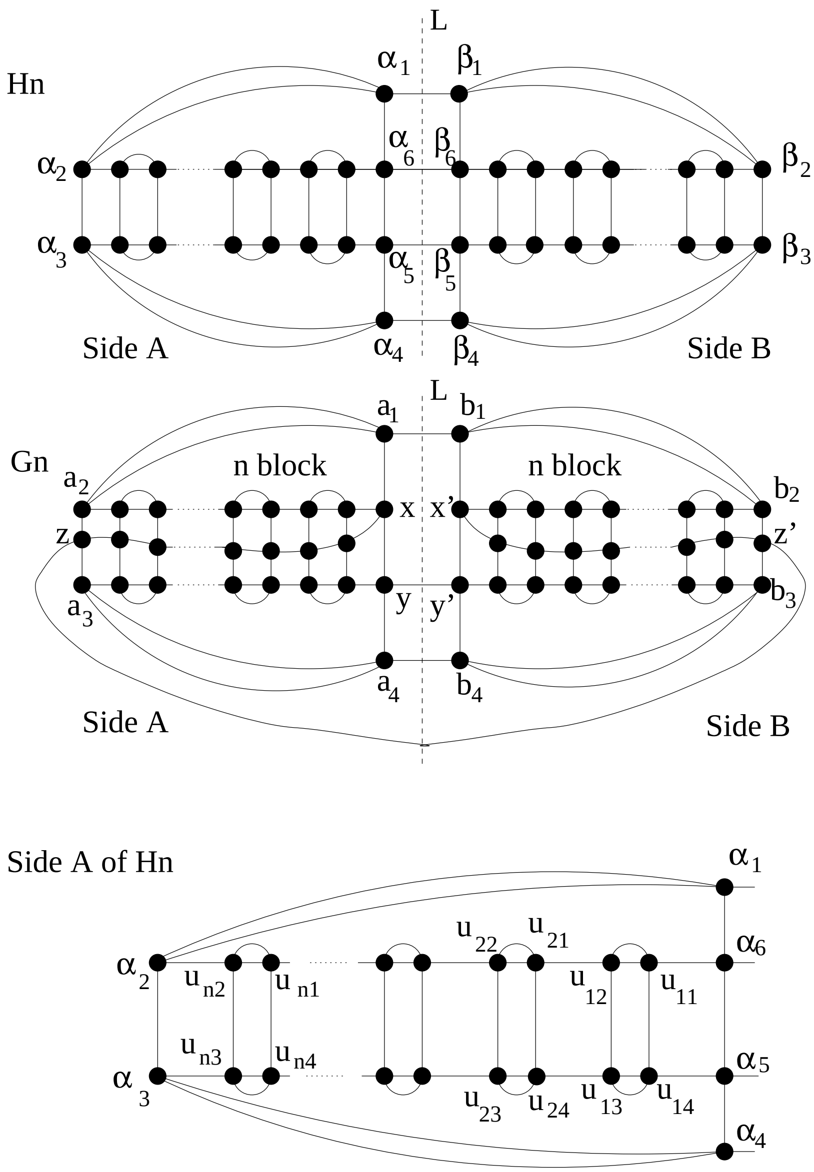

FIGURE 4.4. The graphs H_n and G_n.

To show this, we will prove that ϕ_0 is the unique immersion from H_n to G_n and that applying any non-planar splitting(s) to any vertex or any vertices results in a non-planar graph. Note that we need Sp in this section because H is immersed in G if and only if H can be obtained from G by applying a sequence of Sp by (3.2.4). The following are five key lemmas.

Lemma 4.5.2. *If there is an odd cycle of length at most nine in G_n, then we need to split at least one vertex of these odd cycles to obtain a graph containing H_n as an immersion.*

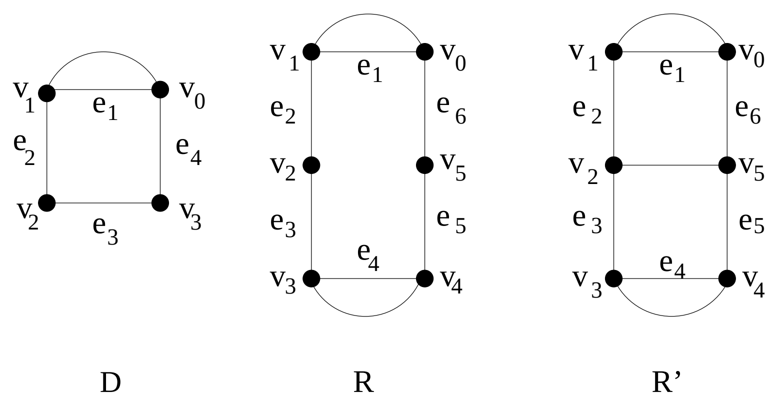

FIGURE 4.5. The graphs D, R and R'.

Side A of G_n (see Figure 4.4) contains a 7-cycle $a_1a_2za_3a_4yx$, and Side B contains a 7-cycle $b_1b_2z'b_3b_4y'x'$ each of which is called a *special* 7*-cycle*. See Figure 4.5 for an induced subgraph R' in G_n.

Lemma 4.5.7. *We must split exactly two vertices of $V(R')$ and split only one vertex in each of special* 7*-cycles.*

Lemma 4.5.10. *We cannot split the vertices, $a_1, a_2, a_3, a_4, b_1, b_2, b_3,$ and b_4 in G_n.*

By (4.5.7) and (4.5.10), we must split one of x, y, z and one of x', y', z'.

Lemma 4.5.14. *We must split z or z'.*

Lemma 4.5.13. *If we must apply a splitting Sp to either z or z', then we must apply the non-planar splittings to both z' and z.*

Let us prove these key lemmas. The first key lemma (4.5.2) will be proved by the following.

Lemma 4.5.1. *A shortest odd cycle in* H_n *with* $n \geq 4$ *has length* $(2n+3) \geq 11$.

Proof. We will use lemmas in Section 1.4. Note that multiple edges in H_n do not influence the length of a cycle if the length more than 2. Let H_n^- be the underlying simple graph of H_n, and let $J_n = H_n^- \setminus \{\alpha_1\alpha_2, \alpha_3\alpha_4, \beta_1\beta_2, \beta_3\beta_4\}$. Then, J_n does not contain any faces having an odd length. By (1.4.3), the unbounded face of J_n has also even length and by (1.4.4), J_n is bipartite.

Let X and Y be the two partite sets of J_n. Note that H_n^- can be obtained from J_n by adding four edges to $X \cup Y$. By (1.1.5), odd cycles in H_n must use an odd number of the four edges. So, odd cycles in H_n must use one or three edges of $\{\alpha_1\alpha_2, \alpha_3\alpha_4, \beta_1\beta_2, \beta_3\beta_4\}$. If only one edge is used, by symmetry, we can assume that $\alpha_1\alpha_2$ is used and $\alpha_3\alpha_4$ is not used. Similarly, if three edges are used, we may assume that $\alpha_1\alpha_2$ is used and $\alpha_3\alpha_4$ is not used. Thus, in either case, we can assume that a shortest odd cycle in H_n passes through $\alpha_1\alpha_2$, and does not pass through $\alpha_3\alpha_4$. This implies that we only need to find a shortest path between α_1 and α_2 in the graph $J_n \cup \{\beta_1\beta_2, \beta_3\beta_4\}$, which we call J_n^+.

Let $\mathcal{P}$ be the family of all paths between α_1 and α_2 in J_n^+ such that paths passing through $\beta_1\beta_2$ or $\beta_3\beta_4$ must pass through both $\beta_1\beta_2$ and $\beta_3\beta_4$. Then, a shortest path in $\mathcal{P}$ together with $\alpha_1\alpha_2$ gives us a shortest odd cycle in H_n. By using the method to find a shortest path (see section 2.3 in D. West [25]), we will see that a shortest path in $\mathcal{P}$ is $\alpha_1\alpha_6 u_{11}u_{12}u_{21}u_{22}...u_{n1}u_{n2}\alpha_2$, which has length $(2n+2)$. Hence, a shortest odd cycle in H_n has length $(2n+3)$. □

By (4.5.1), every odd cycle in H_n has length at least eleven because $(2n+3) \geq 11$ if $n \geq 4$. Hence, there are no odd cycles of length less than eleven in H_n. It implies (4.5.2).

Lemma 4.5.2. *If there is an odd cycle of length at most nine in G_n, then we need to split at least one vertex of the odd cycle to obtain a graph containing H_n as an immersion.* □

To prove the next key lemma (4.5.7), we use (4.5.6) which will be proved by (4.5.4) and (4.5.5). To prove (4.5.4) and (4.5.5), we will investigate the incidence relations among the vertices u_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq 4$ (see Figure 4.4). It is not difficult to see that the following Lemma (4.5.3) holds by looking at Figure 4.4.

Lemma 4.5.3. *Let i and k be an integer with $1 \leq i, k \leq n$, and let j and l be an integer with $1 \leq j, l \leq 4$. When $j = 1$ or 2, the vertex $u_{i,j}$ is adjacent with $u_{k,l}$ if and only if $k = i$ and $|l - j|$ is an odd integer, or $k = i \pm 1$ and $l = j \mp 1$. Also, when $j = 3$ or 4, the vertex $u_{i,j}$ is adjacent with $u_{k,l}$ if and only if $k = i$ and $|l - j|$ is an odd integer, or $k = i \pm 1$ and $l = j \pm 1$.* □

Let $I_i = H_n[u_{i1}, u_{i2}, u_{i3}, u_{i4}]$ with $i = 1, 2, ..., n$ in Side A of H_n. Note that Side A of H_n contains two types of multiple edges: one is in I_i and the other type is not, such as $\alpha_1\alpha_2$. We call multiple edges of the first type *I-type* and of the other type *O-type*. In each Side A and B, the graph H_n contains n distinct induced subgraphs each of which is isomorphic to I_i, and each I_i contains four multiple edges of I-type. So, H_n contains $8n$ multiple edges of I-type and eight multiple edges of O-type.

Recall that $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k$ is the sequence of vertices and edges of a k-cycle C_k such that $e_i = v_{i-1}v_i$ is an edge for all i, and $v_k = v_0$ (see Section 1.1). We will use these notations in the following lemmas. Let D be the graph obtained from a 4-cycle C_4 by adding an edge so that e_1 becomes a multiple edge (see Figure 4.5).

Lemma 4.5.4. *The graph H_n does not contain D as an induced subgraph. More precisely, if H_n contains a subgraph isomorphic to D, then e_3 of D must be a multiple edge.*

Proof. Suppose that H_n contains an induced subgraph isomorphic to D. Then, the multiple edge e_1 of D must be one of two types. If it is O-type, without loss of generality, we may assume that e_1 of D is $\alpha_1\alpha_2$. By symmetry, we may assume that $v_0 = \alpha_1$ and $v_1 = \alpha_2$. Then, $v_2 = u_{n,2}$ or α_3, and $v_3 = \alpha_6$ or β_1. Clearly, in each case no two vertices corresponding v_2 and v_3 are adjacent in H_n; so, the multiple edge e_1 is not O-type. Thus, e_1 must be I-type.

If the multiple edge e_1 of D is I-type, then symmetry implies three cases: $v_0= u_{11}$ and $v_1= u_{12}$; $v_0= u_{i1}$ and $v_1= u_{i2}$ with $1 < i < n$; and $v_0= u_{n1}$ and $v_1= u_{n2}$. In the first case, $v_2 = u_{21}$ or u_{13}, and $v_3 = \alpha_6$ or u_{14}. The vertex u_{21} is not adjacent with α_6; otherwise, there is a 9-cycle $\alpha_6 u_{21} u_{22} u_{31} u_{32} u_{41} u_{42} \alpha_2 \alpha_1$ in H_4, which contradicts (4.5.1). By (4.5.3), u_{21} is not adjacent u_{14}. The vertex u_{13} is not adjacent with α_6 because they belong to different stories. Therefore, u_{13} is adjacent with u_{14}, but $u_{13}u_{14}$ is a multiple edge in H_n, which implies that e_3 of D is a multiple edge. Similarly, the lemma holds in the second and third cases by (4.5.3) and by using the same arguments as the first case. □

Let R be the graph obtained from a 6-cycle by adding two edges so that e_1 and e_4 are multiple edges in R (see Figure 4.5).

Lemma 4.5.5. *The graph H_n contains no subgraph isomorphic to R. More precisely, if H_n contains a 6-cycle, and an edge e_1 of the 6-cycle is a multiple edge, then e_4 is not a multiple edge.*

Proof. Suppose that H_n contains a subgraph isomorphic to R. Then, a multiple edge of R must be O-type or I-type. Without loss of generality, in the first case we can assume that the multiple edge e_1 of R is $\alpha_1\alpha_2$. Symmetry implies that we can assume that $v_0 = \alpha_1$ and $v_1 = \alpha_2$. Then, $v_2 = u_{n2}$ or α_3, and $v_5 = \alpha_6$ or β_1. Thus, by (4.5.3) and Figure 4.4, $v_3 = u_{n1}$, u_{n3}, or α_4, and $v_4 = u_{11}$, α_5, β_2, or β_6. Then, the only chance for the vertices v_3 and v_4 to be adjacent is if $v_3 = \alpha_4$ and $v_4 = \alpha_5$; the edge $e_4 = \alpha_4\alpha_5$ is not a multiple edge. Hence, the multiple edge e_1 in R is I-type.

If the multiple edge e_1 of R is I-type, then by symmetry, there are three cases: $v_0 = u_{11}$ and $v_1 = u_{12}$; $v_0 = u_{i1}$ and $v_1 = u_{i2}$ with $1 < i < n$; and $v_0 = u_{n1}$ and $v_1 = u_{n2}$. Note that by (4.5.4), neither v_0 and v_3 nor v_1 and v_4 of R are adjacent in H_n. In the first case, by (4.5.3) and Figure 4.4, $v_2 = u_{21}$ or u_{13}, and $v_5 = \alpha_6$ or u_{14}. By (4.5.4), we have that $v_3 = u_{22}$ or u_{24} and $v_4 = \alpha_1$, α_5, or β_6. Then, among all of the possible combinations, no two vertices corresponding v_3 and v_4 are adjacent in H_n. So, the first case is impossible.

In the second case, $v_2 = u_{i+1,1}$ or u_{i3}, and $v_5 = u_{i-1,2}$ or u_{i4}. Then, by (4.5.4), $v_3 = u_{i+1,2}$ or $u_{i+1,4}$, and $v_4 = u_{i-1,1}$ or $u_{i-1,3}$. By (4.5.3), the vertices corresponding to v_3 and v_4 can not be adjacent. Similarly, in the third case, we will see that $v_3 = \alpha_1$ or α_3 and $v_4 = u_{n-1,1}$ or $u_{n-1,3}$ by (4.5.3) and (4.5.4). Clearly, the vertices corresponding to v_3 and v_4 can not be adjacent: a contradiction. □

By (4.5.5), a R is not contained in H_n, but G_n contains a R as a subgraph of R', which is a induced subgraph in G obtained from R by adding edge v_2v_5 to $E(R)$ (see Figure 4.5). By (4.5.5), we know that at least one vertex of R' must be split. In fact, we can show that we need to split more than one vertex of R'.

Lemma 4.5.6. *We need to split at least two vertices of R'.*

Proof. Suppose that we need to split only one vertex of R', say v. If v is an endpoint of a multiple edge, we can say that by symmetry, applying releasing splitting to v produces a triangle containing the edge v_2v_5, and that applying non-releasing splitting to v produces a loop. Then, by (4.5.2), we need more splittings.

Hence, v is not an endpoint of multiple edges. Thus, v is v_2 or v_5. By symmetry, we may assume that $v = v_2$. There are three types of splitting at v_2. Two of them are planar splittings and the other is a non-planar splitting. Applying each of two planar splittings to v_2 produces a triangle, and applying the non-planar splitting to v_2 produces a 5-cycle. By (4.5.2), we must split at least two vertices of $V(R')$. □

Lemma 4.5.7. *We must split exactly two vertices of $V(R')$ and only one vertex of each special 7-cycle.*

Proof. Recall that we will split $(4n+2)$ vertices of G_n to obtain H_n. By (4.5.6), we need at least $4n$ splittings for $2n$ induced subgraphs R'. By (4.5.2), we need two splittings for each special 7-cycle. □

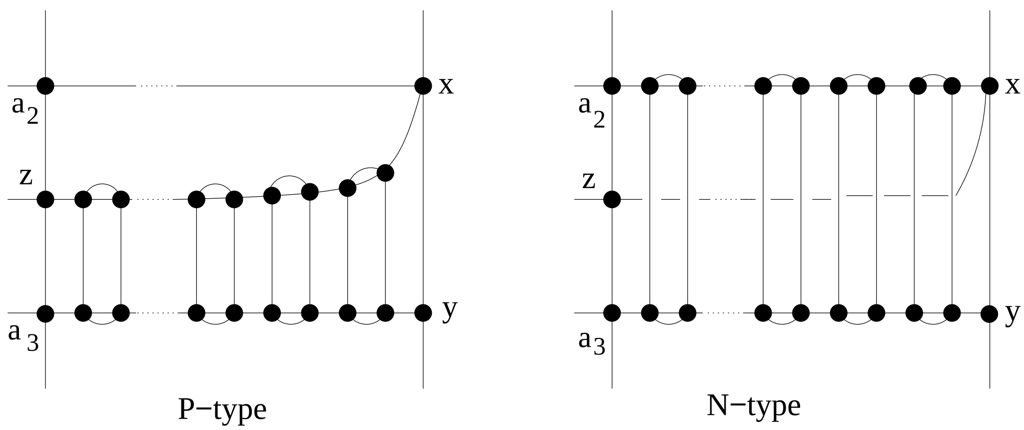

FIGURE 4.6. The P-type and N-type.

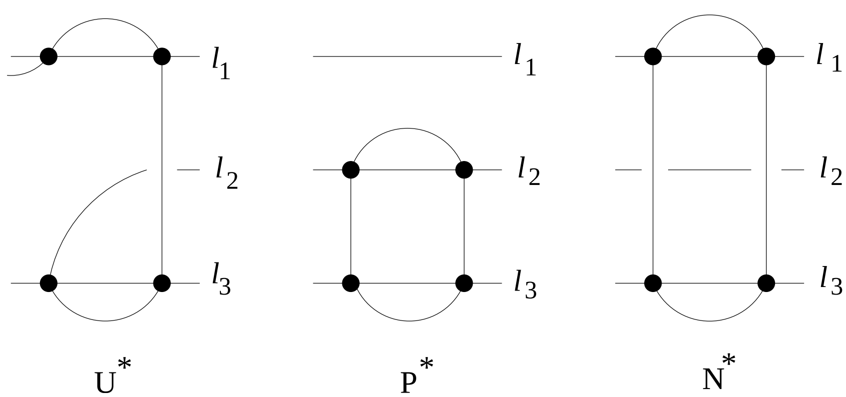

FIGURE 4.7. The graphs U^*, P^*, and N^*.

There are twenty-one non-isomorphic (containing no odd cycle) resulting graphs or parts after applying splittings to two vertices of $V(R')$. By symmetry, it is enough to investigate one side, say Side A. Let R'_i with $i = 1, 2, ..., n$ be n induced subgraphs isomorphic to R' in Side A. Let $M = G_n[R'_1, R'_2, ..., R'_n, x, y, z, a_2, a_3]$ and call a subgraph isomorphic to M an *n-block* (see Figure 4.4). We call the two resulting graphs after applying certain $2n$ splittings to an n-block *P-type* and *N-type*, shown in Figure 4.6.

Also, we notice that $\alpha_5, \alpha_6, \beta_5$, and β_6 in $H - n$ are the only four vertices which are not endpoints of multiple edges. We call these four vertices *exceptional vertices.* Note that exceptional vertices must be adjacent with two endpoints of two types of multiple edges: I-type and O-type.

Lemma 4.5.8. *If neither $x(x')$ nor $y(y')$ can be split in G_n, then an n-block must result in either P-type or N-type.*

Proof. After applying two splittings Sp, we have twenty-one non-isomorphic resulting graphs or parts. To prove (4.5.8), we can classify these resulting parts into five groups; we will call these five groups *type*-1*, type*-2*, type*-3*, type-D* and *type-E* exception group, which will be defined later. We have ten type-1, three type-2, three type-3, two type-D and three type-E graphs. In Side A, we can say that three lines pass through the n-block: two go from x to a_2 and z, one goes from y to a_3, denoted by l_1, l_2, and l_3, respectively (see Figure 4.7). Note that in each resulting part, we can recognize these three lines.

We will combine each resulting part with another resulting part, possibly with the isomorphic parts, and determine which combinations are permissible by comparing with our lemmas. Since we study Side A, we will go from the right to the left one by one and will call each part combined with a previous part the *first part, second part,....*

Looking the resulting parts from the right hand side to the left hand side, let v_1, v_2 or v_3 be the first vertices except x and y in l_1, l_2 and l_3, respectively. Note that in the first part, x is adjacent with v_1 and v_2, and y is adjacent with v_3. If v_1 and v_2 are adjacent, then we call *type*-1. If $v_2 = v_3$, then we call the resulting graph *type*-2. If $v_1 = v_2$ and v_1 is an endpoint of a multiple edge that is contained in the resulting graph, then we call the resulting graph *type*-3. The *type-D* graphs contain an induced subgraph isomorphic to D. The *type-E graphs* consist of three graphs in Figure 4.7, denoted by U^*, P^*, and N^*. By (4.5.4), we can have no type-D graph.

First, we will investigate the first part. Since v_1, v_2 are adjacent with x, type-1 produces a 3-cycle v_1v_2x, which contradicts (4.5.2) and (4.5.7). Similarly, type-2 produces a 3-cycle xyv_2. Moreover, type-3 produces a consecutive multiple edge, which is not contained in H_n. Thus, only chance is for type-E. If the first part contains U^*, then $xv_1 = e_3$ is in D, which contradicts (4.5.4). Therefore, P^* and N^* are only available as the first part.

If we have P^* or N^* as the first part, then we can conclude x is an exceptional vertex because x is not an endpoint of a multiple edge. Hence, x must have two other exceptional vertices and two endpoints of the two types of multiple edges (I-type and O-type) as neighborhoods. So, x can not have two vertices of two disjoint 2-cycles contained in l_1, l_2 or l_3 as neibourhoods because multiple edges involved in l_1, l_2 or l_3 are only I-type multiple edges. For convenience, we call this condition the *exceptional situation*.

Second, we can study the second part if the first part is P^*. Then, if we use either type-1 or -2 as the second part, then each resulting graph produces an small odd cycle, which contradicts (4.5.2) and (4.5.7). If we use either type-3 or U^* as the second part, each resulting graph produces an induced subgraph D, which contradicts (4.5.4). Also, if we use N^* as the second part, the resulting graph contradicts the exceptional situation. Hence, the only available graph for the second part is P^* if the first part is P^*.

In general, let k be an integer with $2 \leq k < n$. Then, if we use type-1 or -2 as the $(k+1)$-th part after having k P^*'s, we have an odd cycle of length $(2k+3)$, or a triangle which contradicts (4.5.2) and (4.5.7). If we use U^* after consecutive k P^*'s, then the resulting graph produces an induced subgraph isomorphic to D. Also, if we use either type-3 or the graph N^* after consecutive k P^*'s, then the resulting graph contradicts the exceptional situation. It implies that we have the P-type graph for the n-block of the side A if the first part is P^*.

Third, we will investigate the second part if the first part is N^* . Then, if we use either type-1 or -2 as the second part, then the resulting graph produces a 5-cycle, which contradicts (4.5.2) and (4.5.7). If we use type-3 as the second part, then the resulting graph produces an induced subgraph D, which contradicts (4.5.4). Also, if we use either U^* or P^* as the second part, then the resulting graph contradicts the special situation. Hence, the only permissible graph as the second part is the graph N^* if the first part is N^*.

Similarly, if we use type-1 or -2 as the $(k+1)$-th part after having consecutive k N^*'s, the resulting graph contains an odd cycle of length $(2k+3)$. If the $(k+1)$-th part is one of type-3, U and P^*, then the resulting graph contradicts the special situation. Thus, if we have the graph N^* as the first part, then we must have the N-type graph for the n-block in Side A. □

Lemma 4.5.9. *Let ab be a multiple edge in G_n. If we can not split a, then we can not split b.*

Proof. The proof is straightforward because G_n (see Figure 4.4) does not contain any u-vertices, which implies that the resulting graph after applying a releasing PS at an endpoint of a multiple edge ab is unique and it is isomorphic to a graph obtained by contracting the two multiple edges ab in G_n. Also, applying a non-releasing PS to a or b produces a loop after applying any endpoints of a multiple edge. □

Lemma 4.5.10. *We cannot split the vertices, $a_1, a_2, a_3, a_4, b_1, b_2, b_3$, and b_4 (see Figure 4.4).*

Proof. By (4.5.9) and symmetry, we only need to show that we can split neither a_2 nor a_3. Applying the non-releasing splitting to either a_2 or a_3 produces a loop at a_1 or a_4, respectively, which contradicts (4.5.2) and (4.5.7).

Applying the releasing splitting to either a_2 or a_3 implies that we can split neither x nor y by (4.5.7). Thus, we can assume that we have either P-type or N-type on Side A by (4.5.8).

Suppose that we have the P-type graphs. Then, applying the releasing splitting to either a_2 or a_3 produces a R or a 3-cycle a_1a_2x, which contradicts either (4.5.5) or (4.5.2) and (4.5.7), respectively. Next, suppose that we have the N-type graphs. Then, applying the releasing splitting to either a_2 or a_3 produces a 3-cycle a_1xz or a D, which also leads a contradiction, respectively. □

Lemma 4.5.11. *If we can split neither $x(x')$ nor $y(y')$, then an n-block must result in the N-type.*

Proof. By (4.5.8), an n-block must be all P-type or all N-type. If an n-block is P-type, then by (4.5.10), there is a 3-cycle, say a_1a_2x in Side A, which contradicts (4.5.2) and (4.5.7). □

Lemma 4.5.12. *If we must apply a splitting to z or z' in Figure 4.4, then the non-planar splitting must be applied to z or z', respectively.*

Proof. By symmetry, it is enough to show (4.5.12) for z. Since we apply a splitting to z, we can apply to neither x nor y by (4.5.7). Thus, we can apply (4.5.11), which implies that both n-blocks are N-type. Then, applying each of two types of PS to z produces a 3-cycle a_1a_2x or a D, which contradicts either (4.5.2) and (4.5.7) or (4.5.4), respectively. □

Lemma 4.5.13. *If we must apply a splitting Sp to either z or z', then we must apply the non-planar splittings to both z' and z.*

Proof. By (4.5.12) and symmetry, we can assume that we apply a non-planar splitting to z. Then, by (4.5.10) and (4.5.11), there is a 9-cycle $a_1a_2a_3a_4b_4b_3z'b_2b_1$. By (4.5.2), (4.5.7) and (4.5.10), only z' is available. Then, the graph operation applied to z' must be a non-planar splitting by (4.5.12). □

Let J be the graph obtained from C_6 by adding two edges so that e_1 and e_3 are multiple edges in J.

Lemma 4.5.14. *We must split z or z'.*

Proof. Suppose that neither z nor z' can be split. Let G'_n be the resulting graph after applying all suitable splittings to suitable $(4n+2)$ vertices of G_n. Then, from (4.5.10), G' contains an induced subgraph isomorphic to J, say $e_1 = b_1b_2$, $e_3 = a_1a_2$ and $e_5 = zz'$. Since G'_n must be isomorphic to H_n, a J must be isomorphic to an induced subgraph of H_n.

First, we will investigate which multiple edges of H_n correspond to e_1 and e_3 of J. We can see that e_1 or e_3 is not a multiple edge of I-type; otherwise, J is not an induced subgraph in H_n. Hence, both e_1 and e_3 must be O-type. Then, the only possible correspondence is that e_1 and e_3 are in the same side, say in Side A. Without loss of generality, we can say that $\{b_2, b_1, a_1, a_2, z, z'\}$ corresponds to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ in order. Note that the two sets of vertices, $\{a_1, z\}$ and $\{b_1, z'\}$ corresponds to $\{\alpha_3, \alpha_5\}$ and $\{\alpha_2, \alpha_6\}$, respectively. However, we can separate the two sets of vertices by four edges in G' and we need at least five edges to separate the two sets of vertices in H_n: a contradiction. Hence, we have to apply a splitting to z or z'. □

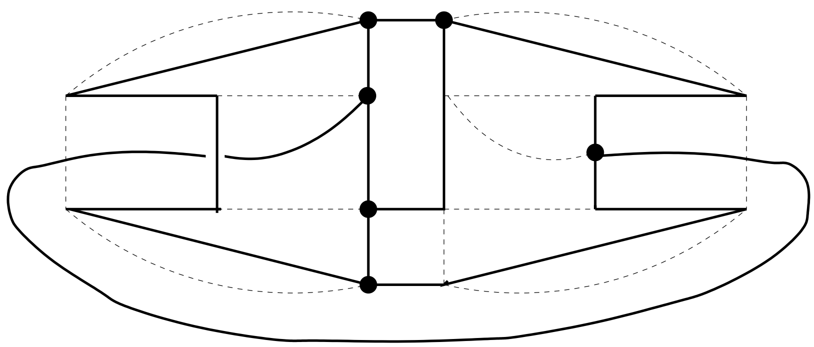

FIGURE 4.8. A subdivision of $K_{3,3}$.

Theorem 4.5.15. *There is no planar graph (except G_n and H_n) that is immersed in G_n, and contains H_n as an immersion.*

Proof. By (4.5.13) and (4.5.14), we must use a non-planar splitting to obtain H_n from G_n. Hence, we only need to show that applying every proper subset of the trivial non-planar splittings results in a non-planar graph. We can do this by showing that each resulting graph contains a subdivision of $K_{3,3}$ by (1.4.2). Since the subset of the trivial non-planar splittings is proper, without loss of generality, we can assume that Side A contains a vertex split and Side B contains a vertex not split by symmetry. Then, Figure 4.8 shows how to find a subdivision of $K_{3,3}$. ☐

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Vita

Jinko Kanno was born on October 28 1954, in Sôma City, Fukushima, Japan. She finished her undergraduate studies in the Department of the Foundations of Mathematical Sciences at Tokyo University of Education (currently known as Tsukuba University) March 1977. She earned a master of science degree in mathematics from Tsuda College in Tokyo March 1981 and studied Low Dimensional Topology for five years there. She became a math teacher at Shirayuri-Gakuen Junior and High School in Tokyo April 1984. She worked there until March 1998. Besides the work, she joined the Saturday Seminar of Graph Theory at Tokyo University of Science and started studying Graph Theory by herself. In April 1998 she arrived in Nashville TN and studied English at the ESL in Vanderbilt University. In January 1999 she came to Louisiana State University in Baton Rouge to pursue graduate studies in mathematics. She earned a second master degree of science in mathematics from Louisiana State University December 2000. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2003.