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Splitter theorems for 3- and 4-regular graphs

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Abstract.

Let $\mathcal{G}_{k,g}$ be the class of k-connected 3-regular (or 4-regular) graphs of girth at least g. For several choices of k and g, we determine a set $\mathcal{O}_{k,g}$ of graph operations, for which, if G and H are graphs in $\mathcal{G}_{k,g}$, $G \neq H$, and G contains H topologically (or contains H as an immersion), then some operation in $\mathcal{O}_{k,g}$ can be applied to G to result in a smaller graph G' in $\mathcal{G}_{k,g}$ such that, on one hand, G' is contained in G topologically (or is immersed in G), and on the other hand, G' contains H topologically (or contains H as an immersion). We also investigate the class of k-connected 4-regular planar graphs of girth at least g, and determine a set $\mathcal{O}_{k,g}$.

1 Introduction

This is a summary of my dissertation. In this summary, we follow the notation and terminology of West [25].

A *graph* G with n *vertices* and m *edges* consists of a *vertex set* $V(G) = \{v_1, v_2, ..., v_n\}$, and an *edge set* $E(G) = \{e_1, e_2, ..., e_m\}$, where each edge is an unordered pair of vertices. Here, n and m are non-negative finite integers. So, we may allow the *empty graph*, which has no vertex and no edge. We write uv for the edge $\{u, v\}$. If $uv \in E(G)$, then we say that u and v are *adjacent*, or the edge uv is *incident* with u and v. The vertices contained in an edge e are its *endpoints*. So, u and v are endpoints of uv. We allow repeated edges or edges with both endpoints the same. These are *multiple edges* and *loops*, respectively. If a graph G contains no loop, then we call G *loopless*. The term *simple graph* prohibits multiple edges and loops.

In particular, the *degree* of a vertex v is the number of non-loop edges incident with v plus twice the number of loops incident with v. The minimum degree of a graph G is denoted by $\delta(G)$ and the maximum degree is $Md(G)$. A graph G is *regular* if $\delta(G) = Md(G)$; *r-regular* if $\delta(G) = Md(G) = r$. We concentrate to investigate 3-regular graphs and 4-regular graphs in the dissertation. Some authors call the next theorem the *shake hands lemma*. In other words, there is no graph having an odd number of vertices of odd degrees.

1.1. Theorem. (Degree-Sum Formula) *If G is a graph with vertex degrees $d_1, d_2, ..., d_n$, then* $\sum d_i = 2|E(G)|$.

A *walk of length* k is a sequence $v_0, e_1, v_1, e_1, ..., v_{k-1}, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ is an edge for all i. A *trail* is a walk with no repeated edge. A *path* is a walk with no repeated vertex. A *u,v-walk* is a walk with first vertex u and last vertex v; these are its *endpoints*, and it is *closed* if $u = v$. A *cycle* is a closed trail of length at least one in which "first = last" is the only vertex repetition. A loop is a cycle of length 1. Also, a multiple edge is contained in a cycle of length 2.

Here, note that *deleting edges* simply means to eliminate the edges, however *deleting vertices* means to eliminate the vertices together with edges incident with the vertices. If S is a set of edges or vertices which are deleted from a graph G, then the resulting graph is denoted by $G\backslash S$. A graph G is *connected* if it has u, v-path for each pair $u, v \in V(G)$. Otherwise, we say that G is *disconnected*. The *components* of a graph G are its maximal connected subgraphs. A *separating set* or a *vertex cut* of a graph G is a set S contained in $V(G)$ such that $G\backslash S$ has more than one component. If G has a pair of non-adjacent vertices and every vertex cut of G contains at least k vertices, then G is said to be *k-connected*. If G has a spanning complete subgraph and $k - 1 \leq |V(G)|$, then G is also said to be *k-connected*.

The *girth* of graph G is the minimum length of all cycles contained in G, denoted by $g(G)$.

2 Overview of the dissertation

The content of the dissertation consists of three parts, the first is to prove the splitter theorems for 3-regular graphs, and the second is to prove the splitter theorems for 4-regular graphs, and the last is to investigate the splitter theorems for 4-regular planar graphs. Since splitter theorems are main themes of the dissertation, we will see what splitter theorems mean, and what their backgrounds and applications are.

Suppose a graph G "contains" another graph H. Then how can G be built up from H in such a way that certain properties of G and H are preserved during the construction process? Probably the best-known results to answer this kind of question are the one by Seymour [19], for general matroids, and Negami [16], for graphs only, which explains the construction when the containment relation is the minor relation and the property to preserve is the 3-connectedness. Results of the same type, which are known as splitter theorems, can also be found in [10], [11], and [18]. In the dissertation, we will investigate splitter theorems for 3-regular graphs and 4-regular graphs. We will use two different types of graph containment relation: *topological minor* for 3-regular graphs and *immersion* for 4-regular graphs.

First, let us clarify the graph containment relation used for 3-regular graphs. A *subdivision* of a graph H is a graph H' obtained from H by replacing each $e \in E(H)$ with a path P_e of positive length (P_e actually is a cycle when e is a loop). We will call each P_e an *edge-path* of H'. For any two given graphs G and H, if a subgraph of G is isomorphic to a subdivision of H, then we say that H is *contained topologically* in G and we write $H \preceq G$. If H is cubic, then $H \preceq G$ if and only if H is a minor of G.

Next, let us make it clear what we mean by building G from H and also maintaining certain properties. In fact, we will talk about reducing G to H, which is clearly equivalent to building G from H yet it is much more convenient for stating our results. Suppose both G and H belong to a family Γ of graphs. Then we say that G can be *reduced to H within Γ by a set $\mathcal{O}$ of graph operations* if there is a sequence $G_0, G_1, ..., G_t$ of graphs in Γ such that $G_0 = G$, $G_t = H$, and each G_i is obtained from G_{i-1} by applying a single operation in $\mathcal{O}$. Moreover, in the sequence, $G_i \preceq G_{i-1}$ holds for each i, and so clearly $H \preceq G_i \preceq G$. Under this terminology, a *splitter theorem* is a result that claims the existence of $\mathcal{O}$ such that every $G \in \Gamma$ can be reduced within Γ by $\mathcal{O}$ to any $H \in \Gamma$ with $H \preceq G$.

The graph properties that we try to maintain are connectivity and girth. Let $\Gamma_{k,g}$ be the family of k-connected cubic graphs of girth at least g. Since only cubic graphs are considered, it is natural for us to assume that $k \leq 3$. It is also natural to assume $g > 0$ since every cubic graph has a cycle. In addition, notice that $\Gamma_{k,g} = \Gamma_{k,k}$ for all $g < k$, thus we also assume $g \geq k$. We prove the splitter theorems for $\Gamma_{k,g}$, for $k =$ 0,1,2, and 3, and $g =$ 1,2,3, and 4, respectively. In addition, we will also determine the $\preceq$-minimal graphs in each $\Gamma_{k,g}$. Then, combining with the corresponding splitter theorems, we will obtain results on how to generate all graphs in each $\Gamma_{k,g}$ from its minimal graphs.

Now, let us clarify the graph containment relation for 4-regular graphs. An *immersion* of H to G is a one-to-one map of the vertices of H into the vertices of G, together with a map of the edges of H into the trails of G, where the image trails are pairwise edge disjoint and only the endpoints of each such trail are in the image of the vertices of H (see [1]). Equivalently, we can define an *immersion* as follows. A *trail graph* of a graph H is a graph H' obtained from H by replacing each $e \in E(H)$ with a trail T_e of positive length (T_e actually is a closed trail or a circuit when e is a loop). For any two given graphs G and H, if a subgraph of G is isomorphic to a trail graph of H, then we say that H is *immersed* in G and we write $H \propto G$. We suppose that a 4-regular graph H is immersed in a 4-regular graph G with a trail graph H'.

For 4-regular graphs, the meaning of splitter theorems is exactly the same as one for 3-regular graphs except the graph containment relation. We prove the splitter theorems for $\Phi_{k,g}$, for $g = 1$, $g = 2$, and $g = 3$, respectively. For 4-regular planar graphs, we use the same graph containment relation, however we need more complicated preparation to state the splitter theorems because of the planarity.

A *drawing* of a graph G is a function that maps each vertex $v \in V(G)$ to a point $f(v)$ in the plane and each edge uv to a $f(u), f(v)$-curve in the plane, denoted by f_G. The images of vertices are distinct. For two edges e and e', a point in $f(e) \cap f(e')$ other than a common end of e and e' is a *crossing*. So, no two curves are tangent (or touch). Moreover, $f(e)$ neither crosses nor touches itself, and crosses at most once with $f(e')$ if $e \neq e'$. If e and e' are adjacent, then $f(e)$ does not cross with $f(e')$. Two drawings f_G and f'_G of G are equivalent if there is a homeomorphism g of the plane such that $f'_G = g \circ f_G$. For convenience, we also call the image $f(G)$ a drawing of G. A *pinched drawing* of a graph G is a graph obtained from $f(G)$ by replacing each crossing with a vertex, denoted by G^p. We call the replaced vertex a *crossing vertex.* A graph G is *planar* if and only if G has a drawing without crossings. A *plane graph* is a particular drawing of a planar graph in the plane with no crossings. So, a drawing $f(G)$ is not always planar, but a pinched drawing G^p is a plane graph because G^p can be obtained from $f(G)$ by replacing every crossing with a vertex.

We say a vertex x of G is a *crossing point of* G if x is a crossing point of two edge-trails of H or only one edge-trail crosses with edges in $E(G)\backslash E(H')$. Let G^p be a pinched drawing of a graph G. Then we can say $G \propto G^p$.

Next, let us make clear what we mean by building G from H^p and also preserving certain properties. In fact, we will talk about reducing G to H^p, which is clearly equivalent to building G from H^p yet *reducing* is much more convenient for stating our results. Suppose G is a planar graph in a class $P\Phi$. Then we say that G can be *reduced to* H^p *within* $P\Phi$ *by a set* $\mathcal{O}$ *of graph operations* if there is a sequence $G_0, G_1, ..., G_t$ of graphs in $P\Phi$ such that $G_0 = G$, $G_t = H^p$, and each G_i is obtained from G_{i-1} by applying a single operation in $\mathcal{O}$. Moreover, in the sequence, $G_i \propto G_{i-1}$ holds for each i, and so clearly $H \propto G_i \propto G$. We also would like to reduce G without increasing the number of crossing points of G. Under this terminology, a *splitter theorem* in the dissertation is a result that claims the existence of $\mathcal{O}$ such that every $G \in P\Phi$ can be reduced by $\mathcal{O}$ to H^p with $H \propto G$, not only within $P\Phi$ but also without increasing the number of crossing points of G.

As before, let $\Phi_{k,g}$ be the family of k-edge-connected 4-regular graphs of girth at least g, and let $P\Phi_{k,g}$ be all planar graphs in $\Phi_{k,g}$. Here, note that $k \leq 2g$ because all graphs in $P\Phi_{k,g}$ are 4-regular. Since only 4-regular graphs are considered, it is natural for us to assume that $k \leq 4$. It is also natural to assume $g > 0$ since every 4-regular graph has a cycle. In addition, notice that $P\Phi_{2k-1,g} = P\Phi_{2k,g}$ because there is no odd-edge-cut in an even graph from (1.1). So, we only need to work for $k = 0$, 2, 4 instead of working k=0, 1, 2, 3, and 4. We prove the splitter theorems for $P\Phi_{k,g}$, for $g = 1$, $g = 2$, and $g = 3$, respectively. In addition, we will also determine the $\propto$-minimal graphs in each $P\Phi_{k,g}$. Then, combining the $\propto$-minimal graphs with the corresponding splitter theorems, we will obtain results on how to generate all graphs in each $P\Phi_{k,g}$ from its minimal graphs. We can show that our splitter theorems for 4-regular planar graphs can not make better if we allow only a finite number of graph operations.

3 Main results

We will see three splitter theorems, which are ones for 3-regular graphs, 4-regular graphs, and for 4-regular planar graphs, respectively. First, let $\Gamma_{3,4}$ be the family of 3-connected 3-regular graphs of girth at least 4. Let G and H be in $\Phi_{2,2}$, and let G contain H topologically.

3.1. Theorem. *If G and H are in $\Gamma_{3,4}$ and $H \preceq G$, then G can be reduced to H within $\Gamma_{3,4}$ by $\mathcal{R}$, $O_3(K_{3,3})$, and $O_3(Q)$.*

Secondly, let $\Phi_{2,2}$ be the family of 2-edge-connected loopless 4-regular graphs. Let G and H be in $\Phi_{2,2}$, and let G contain H as an immersion.

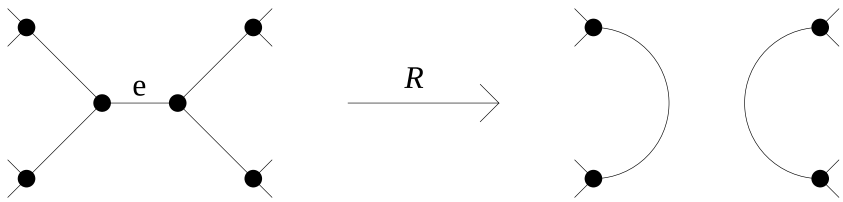

Figure 1: Operation $\mathcal{R}$.

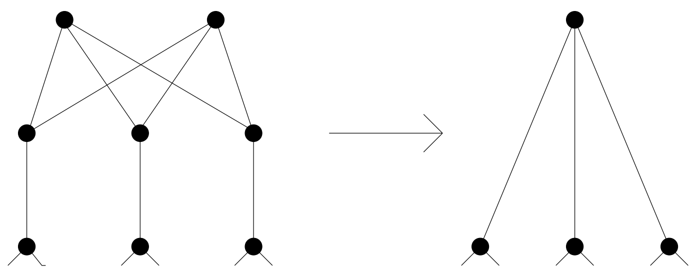

Figure 2: Operation $O_3(K_{3,3})$.

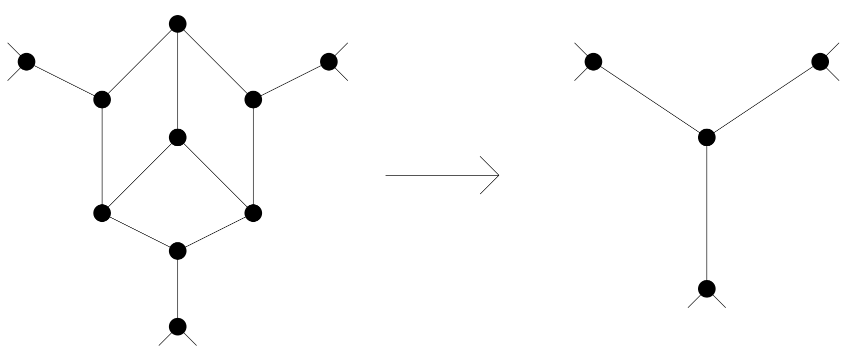

Figure 3: Operation $O_3(Q)$.

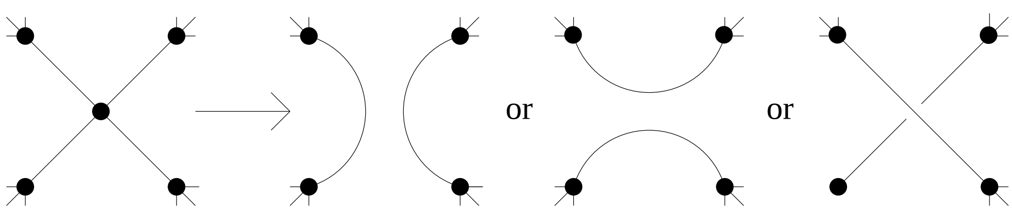

Figure 4: Operation Sp.

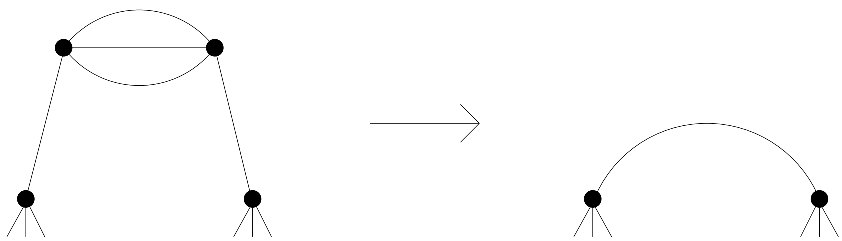

Figure 5: Operation $O_2(4K_2)$.

3.2. Theorem. *If G and H are in $\Phi_{2,2}$ (or $\Phi_{1,2}$), and $H \propto G$, then G can be reduced within $\Phi_{2,2}$ by Sp and $O_2(4K_2)$ to H.*

Let $\Phi_{2,3}$ (or $P\Phi_{2,3}$) be the family of 2-edge-connected simple (or 2-edge-connected simple planar) graphs. Let H be in $\Phi_{2,3}$, and let G be in $P\Phi_{2,3}$, and let G contain H as an immersion. Let us introduce the smallest 4-regular simple planar graph, only one on six vertices. It is the graph of an octahedron, denoted by *Oct*. Also, we can see Oct as the smallest member of the following set of 4-regular simple planar graphs. Let n be an integer with $n \leq 3$ and let C_{2n} be a cycle of length $2n$. Let C_{2n}^2 be a graph obtained from C_{2n} by connecting all two vertices whose distance is equal to 2. Then C_{2n}^2 is a 4-regular simple planar graph and C_6^2 is isomorphic to oct.

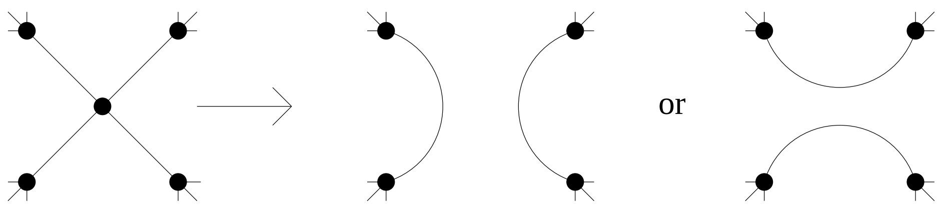

Figure 6: Operation PS.

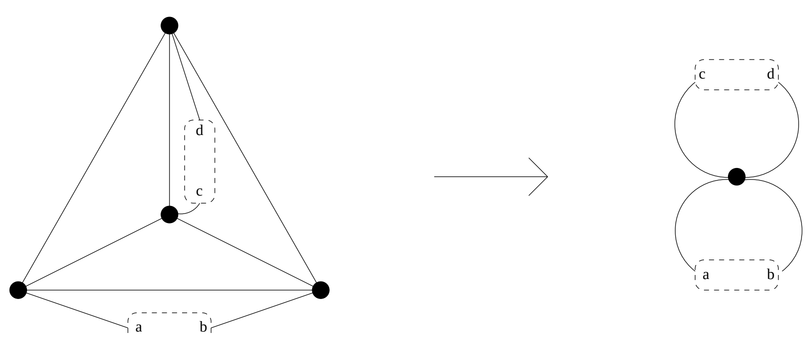

Figure 7: Operation $C(PK_4)$.

3.3. Theorem. *If G is a 2-edge-connected 4-regular simple planar graph (in $P\Phi_{2,3}$ or $P\Phi_{1,3}$),*

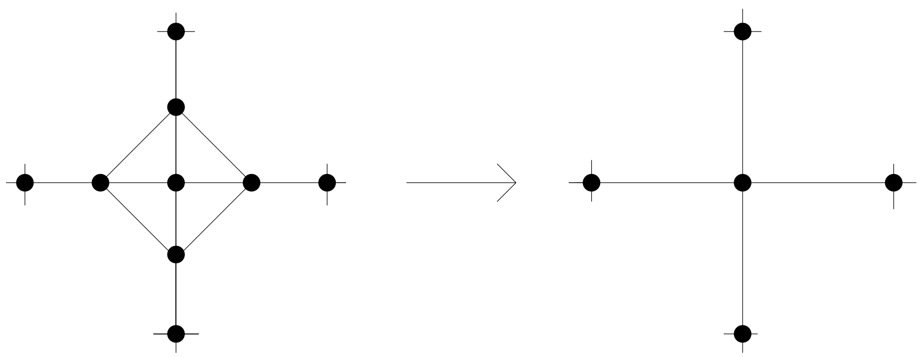

Figure 8: Operation $C(W_5)$.

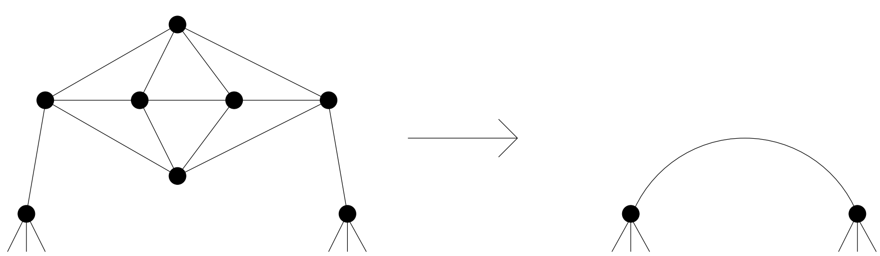

Figure 9: Operation $O_2(Oct)$.

and $H \in \Phi_{2,3}$ with $H \propto G$, then G can be reduced within $P\Phi_{2,3}$ by PS, $C(PK_4)$, $C(W_5)$, and $O_2(Oct)$, to H^p without increasing the number of crossing points, unless G is isomorphic to a cycle ladder.

In the previous sections, we showed that we can reduce G to H^p, not to H, when G is a planar 4-regular graph and a 4-regular graph H is immersed in G. People might ask us if the result is the best, or you can make the result better. The answer is negative. In this section, we will show the infinitely many pairs of (G, H) such that 4-regular graph H is immersed in G and there is no planar graph between G and H. We mean that there is no planar graph such that is immersed in G and contains H as an immersion. Therefore, if we would like to have a theorem which claims that G can be reduced to H, then we need infinitely many graph operations.

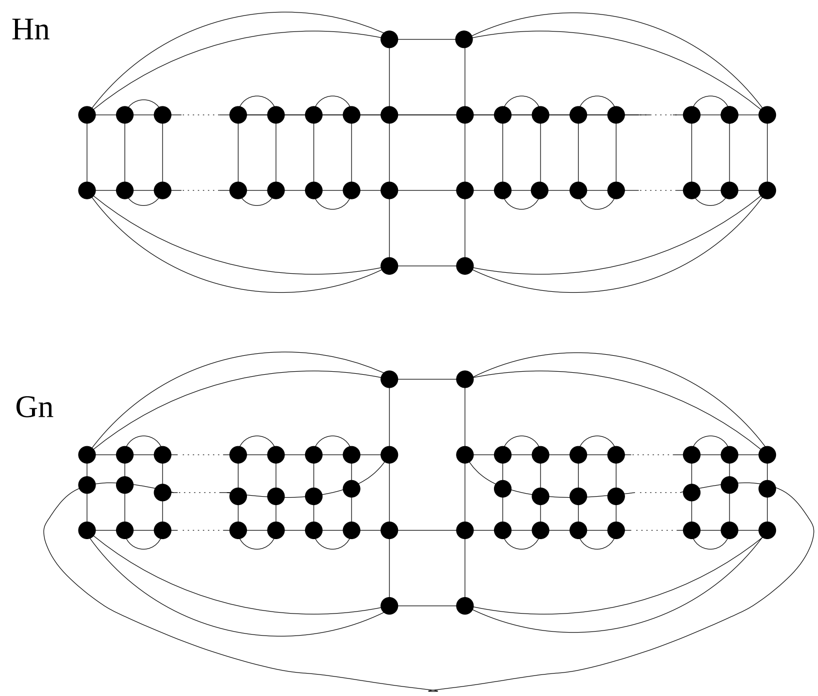

Figure 10: The graphs H_n and G_n.

Let n be an integer with $n \geq 4$, and let H_n and G_n be the following graphs in Figure 7. There is a vertical line L for each graph such that the graph is symmetric with respect to L. We say that subgraphs belonging to the same side of L as the side containing a_1 or b_1 *are in the A side* or *are in the B side*, respectively. Note that both of them are 4-regular planar graphs, and G_n has $(4n+2)$ more vertices than H_n does. Also we notice that we can obtain G_n from H_n by pulling the edge e in Figure 7, and making extra $(4n+2)$ crossing points with $(4n+2)$ edges of H_n when we put e back into H_n. Then, we make the crossing points vertices of G_n. So, G_n is a pinched graph of H_n, or G_n is isomorphic to a H_n^p. In other words, we can obtain H_n from G_n by applying non-planar splittings at all of the extra $(4n+2)$ vertices. We call this splitting the *trivial non-planar splitting*. We will see that applying the trivial non-planar splittings is the only way to obtain H_n from G_n.

To see this, we would like to show that, for each n, there is no planar graph between G_n and H_n under the containment relationship of immersion. More precisely, there is no planar graph that contains H_n as an immersion and is immersed in G_n. In other words, if a graph contains H_n and is immersed in G_n, then the graph must be non-planar. To show this, there are two steps. In the first, we will show that the only way to obtain H_n from G_n is to apply the trivial non-planar splittings. In the second, applying a splitting of the trivial non-planar splittings produces a non-planar graph.

References

[1] T. Andreae, *On well-quasi-ordering-finite graphs by immersion*, Combinatorica 6(1986), no.4, 287-298.

[2] F. Bories, J-L. Jolivet, and J-L. Fouquet, *Construction of* 4*-regular graphs*, Combinatorial mathematics(1981), 99-118, North-Holland Math. Stud., 75, North-Holland, Amsterdam, 1983.

[3] H. J. Broersma, A. J. W. Duijvestijn, and F. Göbel, *Generating all* 3*-connected* 4*-regular planar graphs from the octahedron graph*, J. Graph Theory 17(1993), no.5, 613-620.

[4] G. Dirac, *A property of* 4*-chromatic graphs and some remarks on critical gaphs*, J. London Math. Soc. **27** (1952), 85-92.

[5] M. R. Fellows and M. A. Langston, *On well-partial-order theory and its application to combinatorial problems of VLSI design*, SIAM J. Discrete Math. 5(1) (1992), 117-126.

[6] R. Govindan and S. Ramachandramurthi, *A weak immersion relation on graphs and its applications*, Discrete Math. 230(2001), no.1-3, 189-206.

[7] B. Grünbaum, " Convex polytopes," John Wiley and Sons, London, 1967.

[8] H. Harborth, *Darstellungen von Graphen in der Ebene*, Graphen in Forschung und Unterricht(1985), 54-64.

[9] E. L. Johnson, *A proof of the four-coloring of the edges of a regular three-degree graph*, O. R. C. 63-28(R. R.) Mimeographed report 1963, Operations Research Center, Univ. of Calif.

[10] A. K. Kelmans, *Graph expansion and reduction*, in Algebraic methods in graph theory, Vol. 1, Colloq. Math. Soc. János Bolyai, (Szeged, Hungary, 1978) North-Holland **25** (1981), 318-343.

[11] A. K. Kelmans, *Graph planarity and related topics*, in Contemporary Mathematics, Vol. 147, (1993), 635-667.

[12] J. Lehel, *Generating all 4-regular planar graphs from the graph of the octahedron*, J. Graph Theory 5(1981), no.4, 423-426.

[13] P. Manca, *Generating all planar graphs regular of degree four*, J. Graph Theory 3(1979), 357-364.

[14] W. McCuaig, *Edge reductions in cyclically k-connected cubic graphs*, J. Combin. Theory Ser. B **56** (1992), 16-44.

[15] C. St. J. A. Nash-Williams, *A glance at graph theory–Part II*, Bulletin of the London Mathematical Society, **14** (1982), 294-328.

[16] S. Negami, *A characterization of 3-connected graphs containing a given graph*, J. Combinatorial Theory Series B **32** (1982), 69-74.

[17] O. Ore, " The four-color problem," Academic Press, New York, 1967.

[18] N. Robertson, P. D. Seymour, R. Thomas, Cyclically five-connected cubic graphs, unpublished.

[19] P. D. Seymour, *Decomposition of regular matroids*, J. Combinatorial Theory Series B **28** (1980), 305-359.

[20] E. Steinitz and H. Rademacher, " Vorlesungen über die Theorie der Polyeder," Verlag von Julius Springer, Berlin, 1934.

[21] S. Toida, *Construction of quartic graphs*, J. Combinatorial Theory Series B **16** (1974), 124-133.

[22] W. T. Tutte, *A theory of 3-connected graphs*, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 441-455.

[23] W. T. Tutte, " Connectivity in Graphs," University of Toronto Press, Toronto, 1966.

[24] W. T. Tutte, "Graph Theory," Cambridge University Press, Cambridge, 2001.

[25] D. B. West, "Introduction to Graph Theory," Prentice-Hall, Upper saddle River, 1996.

[26] N. C. Wormald, *Classifying k-connected cubic graphs*, Lecture Notes in Mathematics Vol. 748, 199-206, Springer-Verlag, New York, 1979.