Flow Past a Vortex

Consider a uniform stream, U_∞ flowing in the x direction past a vortex of strength K with the center at the origin. By superposition the combined stream function is

$$\psi = \psi_{stream} + \psi_{vortex} = U_\infty r \sin\theta - K \ln r$$

The velocity components of this flow are given by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta \qquad v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta + \frac{K}{r}$$

Setting v_r and v_θ = 0, we find the stagnation point at θ = 90°, r = a = K/ U_∞ or (x,y) = (0,a).

At this point the counterclockwise vortex velocity, K/r, exactly cancels the free steam velocity, U_∞. Figure 8.6 in the text shows a plot of the streamlines for this flow.

An Infinite Row of Vortices

Consider an infinite row of vortices of equal strength K and equal spacing a as shown in Fig. 8.7a. A single vortex, i , has a stream function given by $\Psi_i = -K \ln r_i$ and the total infinite row has a combined stream function of

$$\Psi = -K \sum_{i=1}^{\infty} \ln r_i$$

This infinite sum can also be expressed as

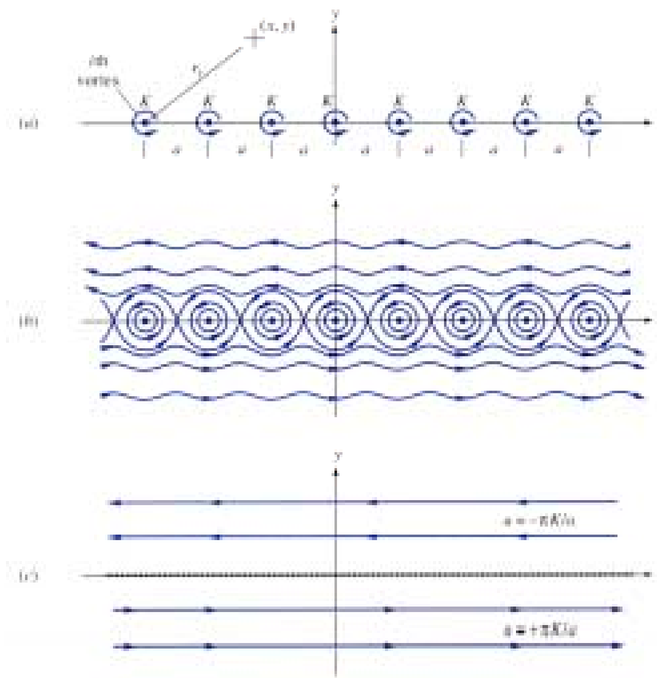
$$\psi = -\frac{1}{2} K \ln\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cosh\frac{2\pi x}{a}\right)\right]$$

Fig. 8.7 Superposition of vortices

(a) an infinite row of equal strength vortices;

(b) the streamline pattern for part a;

(c) vortex sheet, part a viewed from afar.



The resulting left and right flow above and below the row of vortices is given by

$$u = \left. \frac{\partial \psi}{\partial y} \right|_{|y|>a} = \pm \frac{\pi K}{a}$$

The Vortex Sheet

The flow pattern of Fig. 8.7b when viewed from a long distance will appear as the uniform left and right flows shown in Fig. 8.7c. The vortices are so closely packed together that they appear to be a continuous sheet. The strength of the vortex sheet is given by

$$\gamma = \frac{2\pi K}{a}$$

Since, in general, the circulation is related to the strength, γ, by $d\Gamma = \gamma\, dx$, the strength, γ, of a vortex sheet is equal to the circulation per unit length, $d\Gamma/dx$.

Plane Flow Past Closed-Body Shapes

Various types of external flows over a closed-body can be constructed by superimposing a uniform stream with sources, sinks, and vortices.

Key Point: The body shape will be closed only if the net source of the outflow equals the net sink inflow. Two examples of this are presented below.

The Rankine Oval

A Rankine Oval is a cylindrical shape which is long compared to its height. It is formed by a source-sink pair aligned parallel to a uniform stream.

The individual flows used to produce the final result and the combined flow field are shown in Fig. 8.9. The combined stream function is given by

$$\psi = U_\infty y - m \tan^{-1} \frac{2 a y}{x^2 + y^2 - a^2}$$

or

$$\psi = U_\infty r \sin\theta + m(\theta_1 - \theta_2)$$

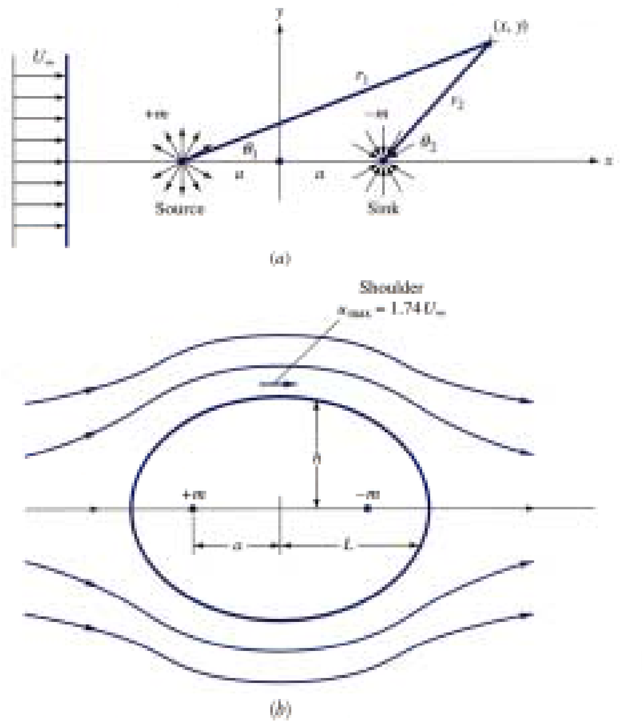


Fig. 8.9 The Rankine Oval

The oval shaped closed body is the streamline, $\psi = 0$. Stagnation points occur at the front and rear of the oval, $x = \pm L, y = 0$. Points of maximum velocity and minimum pressure occur at the shoulders, $x = 0, y = \pm h$. Key geometric and flow parameters of the Rankine Oval can be expressed as follows:

$$\frac{h}{a} = \cot \frac{h/a}{2m/(U_\infty a)} \qquad \frac{L}{a} = \left(1 + \frac{2m}{U_\infty a}\right)^{1/2}$$

$$\frac{u_{max}}{U_\infty} = 1 + \frac{2m/(U_\infty a)}{1 + h^2/a^2}$$

As the value of the parameter $m/(U_\infty a)$ is increased from zero, the oval shape increases in size and transforms from a flat plate to a circular cylinder at the limiting case of $m/(U_\infty a) = \infty$.

Specific values of these parameters are presented in Table 8.1 for four different values of the dimensionless vortex strength, $K/(U_\infty a)$.

Table 8.1 Rankine-Oval Parameters

$m/(U_\infty a)$	h/a	L/a	L/h	u_{max}/U_∞
0.0	0.0	1.0	∞	1.0
0.01	0.31	1.10	32.79	1.020
0.1	0.263	1.095	4.169	1.187
1.0	1.307	1.732	1.326	1.739
10.0	4.435	4.458	1.033	1.968
10.0	14.130	14.177	1.003	1.997
∞	∞	∞	1.000	2.000

Flow Past a Circular Cylinder with Circulation

It is seen from Table 8.1 that as source strength m becomes large, the Rankine Oval becomes a large circle, much greater in diameter than the source-sink spacing 2a. Viewed, from the scale of the cylinder, this is equivalent to a uniform stream plus a doublet. To add circulation without changing the shape of the cylinder we place a vortex at the doublet center. For these conditions the stream function is given by

$$\psi = U_\infty \sin\theta \left(r - \frac{a^2}{r} \right) - K \ln \frac{r}{a}$$

Typical resulting flows are shown in Fig. 8.10 for increasing values of non-dimensional vortex strength $K/U_\infty a$.

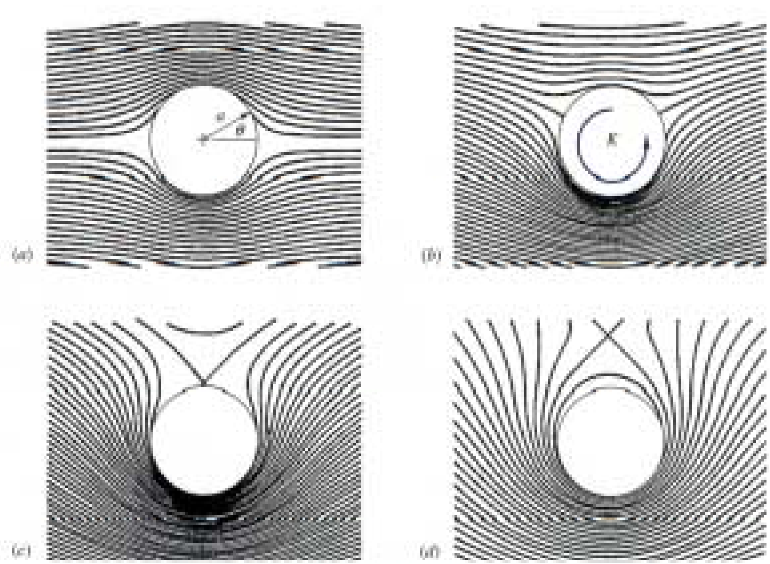


Fig. 8.10 Flow past a cylinder with circulation for values of $K/U_\infty a$ of (a) 0, (b) 1.0, (c) 2.0, and (d) 3.0

Again, the streamline $\psi = 0$ is corresponds to the circle r = a. As the counter-clockwise circulation $\Gamma = 2\pi K$ increases, velocities below the cylinder increase and velocities above the cylinder decrease (*could this be related to the path of a curve ball?*). In polar coordinates, the velocity components are given by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta \left(1 - \frac{a^2}{r^2}\right)$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta \left(1 + \frac{a^2}{r^2}\right) + \frac{K}{r}$$

For small K, two stagnation points appear on the surface at angles θ_s or for which

$$\sin\theta_s = \frac{K}{2U_\infty a}$$

Thus for K = 0, θ_s = 0 and 180°. For $K/U_\infty a = 1$, θ_s = 30 and 150°. Figure 8.10c is the limiting case for which with $K/U_\infty a = 2$, $\theta_s = 90°$ and the two stagnation points meet at the top of the cylinder.

The Kutta-Joukowski Lift Theorem

The development in the text shows that from inviscid flow theory,

> The lift per unit depth of any cylinder of any shape immersed in a uniform stream equals to $\rho U_\infty \Gamma$ where Γ is the total net circulation contained within the body shape. The direction of the lift is 90^o from the stream direction, rotating opposite to the circulation.

This is the well known Kutta-Joukowski lift theorem.

For the cylindrical flows shown in Fig. 8.10 b to d, there is a downward force, or negative lift, proportional to the free stream velocity and vortex strength. The surface pressure distribution is given by

$$P_s = P_\infty + \tfrac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta + 4\beta\sin\theta - \beta^2\right)$$

where $\beta = K / (U_\infty a)$ and P_∞ is the free stream pressure. For a cylinder of width b into the paper, the drag D is given by

$$\mathrm{D} = -\int_0^{2\pi}\left(\mathrm{P_s} - \mathrm{P_\infty}\right)\cos\theta\,\mathrm{b\,a}\,d\theta$$

The lift force L is normal to the free stream and is equal to the sum of the vertical pressure forces (for inviscid flow) and is determined by

$$\mathrm{L} = -\int_0^{2\pi}\left(\mathrm{P_s} - \mathrm{P_\infty}\right)\sin\theta\,\mathrm{b\,a}\,d\theta$$

Substituting $P_s - P_\infty$ from the previous equation the lift is given by

$$L = -\frac{1}{2}\rho U_\infty^2 \frac{4K}{aU_\infty} b a\int_0^{2\pi}\sin^2\theta\, d\theta = -\rho U_\infty\left(2\pi K\right)b$$

or

$$\frac{L}{b} = -\rho U_\infty \Gamma$$