Complex Numbers

Bernd Schröder
Introduction
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4. This is not a fully formal introduction, but it can serve to facilitate the transition to doing proofs.
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2. We will *formally* construct entities that work like this.
3. We will assume the real numbers “work like they always did”.
4. This is not a fully formal introduction, but it can serve to facilitate the transition to doing proofs.
5. For a formal introduction of the number systems, consider my class “Fundamentals of Mathematics.”
If You Like “That Abstract Stuff”
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Definition.
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Complex numbers are also written in the form

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The number $b$ is also called the imaginary part of $z$, denoted $\Im(z)$.
Complex Numbers Visualized
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Although the above notation is not mandatory, it saves us unnecessary adjustments if we try to consistently use it. Be prepared for exceptions, though.
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What are Complex Numbers? Field Properties Absolute Value/Modulus The Complex Conjugate

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**What are Complex Numbers?**

**Field Properties**

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**Complex Numbers Visualized**

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\[ \Re(z) \]

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Complex Numbers
Complex Numbers

- $\mathbb{R}$ ($x$)
- $\mathbb{I}$ ($y$)

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Theorem.
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3. **There is a neutral element** 0 for addition, that is, there is an element 0 $\in \mathbb{C}$ so that for all $z \in \mathbb{C}$ we have $z + 0 = z$.
4. **For every element** $z \in \mathbb{C}$ there is an **additive inverse element** $(-z)$ so that $z + (-z) = 0$. 
<table>
<thead>
<tr>
<th>What are Complex Numbers?</th>
<th>Field</th>
<th>Properties</th>
<th>Absolute Value/Modulus</th>
<th>The Complex Conjugate</th>
</tr>
</thead>
</table>

### Complex Addition Visualized

The preceding properties should follow from the corresponding properties for the components.
Complex Addition Visualized

\[ y = \Im(z) \]

\[ x = \Re(z) \]

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What are Complex Numbers? 

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\[ y = \mathcal{I}(z) \]

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Complex Numbers
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Complex Numbers
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\begin{align*}
y &= \mathcal{I}(z) \\
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\[ z_1 \]

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Complex Numbers
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\[ z_1 + z_2 = (\Re(z_1) + \Re(z_2), \Im(z_1) + \Im(z_2)) \]

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8. For every element \( z \in \mathbb{C} \setminus \{0\} \) there is a **multiplicative inverse** element \( z^{-1} \) so that \( z \cdot z^{-1} = 1 \).
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Proof.
**Proof.** Good exercise.
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Additive inverses (number 4):
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z + (-z)
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\[
\begin{align*}
  z + (-z) &= (x, y) + (-x, -y) \\
           &= (x + (-x), y + (-y)) \\
           &= (0, 0) = 0
\end{align*}
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Proof. Good exercise. We will prove some of the properties, but not all of them. Throughout, let $z = x + iy = (x, y)$, $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$.

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$$z + (-z) = (x, y) + (-x, -y) = (x + (-x), y + (-y)) = (0, 0) = 0$$

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So “the usual formula” for the additive inverse (“the negative”) works because it gives us an object with the right properties.
Proof (cont.).
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\[ z_1 z_2 \]
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&= (x_2x_1 - y_2y_1, x_2y_1 + y_2x_1) = (x_2, y_2)(x_1, y_1) \\
&= z_2z_1
\end{align*}
\]

Multiplicative inverse (number 8): \(z^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)\).
Proof (cont.).

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Example.

\[
(3 + 5i) - 1 = 3 - 3\frac{1}{3} - i \frac{5}{3}
\]

(remember how computation was good for the soul?)
**Example.** Find the multiplicative inverse of $3 + 5i$. 

```math
(3 + 5i)^{-1} = \frac{3}{34} - \frac{5}{34}i
```

(remember how computation was good for the soul?)

**Definition.**

\[
\begin{align*}
\text{If } z_1, z_2 & \in \mathbb{C}, \\
\text{then } z_1 - z_2 : & = z_1 + (-z_2), \\
\text{and } z_1 \cdot z_2 : & = z_1 \cdot z_2^{-1}.
\end{align*}
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$$(3 + 5i)^{-1}$$
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**Example.** *Find the multiplicative inverse of* \(3 + 5i\).

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Definition.

$z_1 - z_2 := z_1 + (-z_2)$

$\frac{z_1}{z_2} := z_1z_2^{-1}$
Example.

\[ \frac{2 - 3i}{5 + 4i} = \left(2 - 3i\right) \left(5 + 4i\right) \]

\[ = 2 \cdot 5 + 2 \cdot 4i - 3i \cdot 5 - 3i \cdot 4i \]

\[ = 10 + 8i - 15i - 12i^2 \]

\[ = 10 - 23i + 12 \]

\[ = 22 - 23i \]
Example. Simplify the quotient \[ \frac{2 - 3i}{5 + 4i} \].
Example. Simplify the quotient \( \frac{2 - 3i}{5 + 4i} \).

\[
\frac{2 - 3i}{5 + 4i} = \left( \frac{2 - 3i}{5 + 4i} \right) \left( \frac{5 - 4i}{5 - 4i} \right) = \frac{(2 - 3i)(5 - 4i)}{5^2 + 4^2} = \frac{10 - 12i - 15i + 12i^2}{25 + 16} = \frac{-2 - 23i}{41}
\]
Example. *Simplify the quotient* \( \frac{2 - 3i}{5 + 4i} \).

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\[
= -\frac{2}{41} - \frac{23}{41}i
\]
Theorem.
Theorem. $i^2 = -1$. 
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Proof.
Theorem. \( i^2 = -1 \).

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Proof. $i^2 = (0 + 1i) \cdot (0 + 1i)$
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Proof. $i^2 = (0 + 1i) \cdot (0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)i$
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Proof. $i^2 = (0 + 1i) \cdot (0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)i = -1$. 
Theorem. $i^2 = -1$.

**Proof.** $i^2 = (0 + 1i) \cdot (0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)i = -1$. □
### Proposition.

<table>
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**Proof 1.**

\[
z \cdot 0 = (x, y) \cdot (0, 0) = (x \cdot 0 - y \cdot 0, y \cdot 0 + x \cdot 0) = (0, 0) = 0.
\]

**Proof 2.**

Note that

\[
z = z + 0 z + (-0 z).
\]

Now

\[
0 = z + (-z).
\]

Proof 2 also applies in more abstract settings, so, although it is longer, it actually is preferred.

Idea:

Do you want to prove that anything times zero is zero in many abstract structures or do you want to prove once that it follows from their properties?
**Proposition.** Let $z \in \mathbb{C}$. 
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$$z \cdot 0$$
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So sometimes the abstract stuff works better than the concrete stuff. Choosing the right approach can almost be an art form.
Definition.

For a \( a + ib \in C \) we define

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and we call it the absolute value or the modulus of \( z \).

**Theorem.** Properties of the absolute value.

0. For all \( z \in C \), we have \( |z| \geq 0 \).

(Because \( |z| \in \mathbb{R} \), it is permissible to use inequalities here.)

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2. For all \( z_1, z_2 \in C \), we have

\[ |z_1z_2| = |z_1||z_2| \]

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What are Complex Numbers? Field Properties Absolute Value/Modulus The Complex Conjugate

Definition.

For $z = x + iy \in \mathbb{C}$, the complex conjugate of $z$ is $\bar{z} = x - iy$. 

$\Re(z) = x$ \hspace{2cm} $\Im(z) = y$ 

$z \overline{z} = (x + iy)(x - iy) = x^2 + y^2$ 

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![Complex Conjugate Diagram](attachment:image.png)
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**Diagram:**

- A complex number $z = x + iy$ is plotted on the complex plane.
- The real part $x$ is shown on the horizontal axis (Re(z)), and the imaginary part $y$ is shown on the vertical axis (Im(z)).
- The complex conjugate $\overline{z}$ is shown as a reflection of $z$ across the real axis.

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$\overline{z_1z_2}$
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\[ \bar{z_1 z_2} = (x_1, y_1)(x_2, y_2) \]
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**Proof.** Good exercise.
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**Proof.** Good exercise.