INTRODUCTION

In formulating and studying principles of valid reasoning, logicians have been guided not only by introspection and philosophical reflection, but also by an analysis of various rational procedures commonly employed by mathematicians and scientists. Because these principles have a multitude of disparate sources, efforts to consolidate them in a single coherent system have been unsuccessful. Instead, philosophers, logicians, and mathematicians have created a panoply of competing logical formalisms, each with its own domain of applicability. Among these formalisms are Boolean-based propositional and predicate calculi, modal and multivalued logics, intuitionistic logic, and quantum logic.

Our purpose in this article is to outline the history and present some of the main ideas of quantum logic. In what follows, it will be helpful to keep in mind that there are four levels involved in any exposition of logic and its relation to the experimental sciences.

1. Philosophical: Addresses the epistemology
of the experimental sciences. Guides and motivates the activities at the remaining levels while assimilating and coordinating the insights gained from these activities.

2. **Syntactic:** Emphasizes the formal structure of a general calculus of experimental propositions.

3. **Semantic:** Focuses on the construction of classes of mathematical models for a logical calculus.

4. **Pragmatic:** Concentrates on a specific mathematical model pertinent to a particular branch of experimental science.

For instance, studies regarding the logics associated with classical physics could be categorized as follows:

1. Philosophical writings extending back at least to Aristotle.
2. Propositional and predicate calculi.
3. The class of Boolean algebras.
4. The Boolean σ algebra of all Borel subsets of the phase space of a mechanical system.

Likewise, for quantum logic, we have

1. Philosophical writings beginning with Schrödinger, von Neumann, Bohr, Einstein, et al.
2. Quantum-logical calculi.
3. The class of orthoalgebras.
4. The lattice of projection operators on a Hilbert space.

For expository reasons, our survey proceeds roughly in the order 1, 4, 3, 2. Thus, we give a brief history of quantum logic in Sec. 1, outline the standard quantum logic of projections on a Hilbert space in Sec. 2, introduce orthoalgebras as models for quantum logic in Sec. 3, and discuss a general quantum-logical calculus of propositions in Sec. 4.

### 1. BRIEF HISTORY OF QUANTUM LOGIC

#### 1.1 The Origin of Quantum Logic

The publication of John von Neumann's *Mathematische Grundlagen der Quantenmechanik* (1932) was the genesis of a novel system of logical principles based on propositions affiliated with quantum-mechanical entities.

According to von Neumann, a quantum-mechanical system ℳ is represented mathematically by a separable (i.e., countable dimensional) complex Hilbert space ℳ, observables for ℳ correspond to self-adjoint operators on ℳ, and the spectrum of a self-adjoint operator is the set of all numerical values that could be obtained by measuring the corresponding observable. Hence, a self-adjoint operator with spectrum consisting at most of the numbers 0 and 1 can be regarded as a quantum-mechanical proposition by identifying 0 with "false" and 1 with "true." Since a self-adjoint operator has spectrum contained in [0,1] if and only if it is an (orthogonal) projection onto a closed linear subspace of ℳ, von Neumann (1935, p. 253) observed that

... the relation between the properties of a physical system on the one hand, and the projections on the other, makes possible a sort of logical calculus with these.

#### 1.2 The Work of Birkhoff and von Neumann

In 1936, von Neumann, now in collaboration with Garrett Birkhoff, reconsidered the matter of a logical calculus for physical systems and proposed an axiomatic foundation for such a calculus. They argued that the experimental propositions regarding a physical system ℳ should band together to form a lattice L (Birkhoff, 1967) in which the meet and join operations are formal analogs of the *and* and *or* connectives of classical logic (although they admitted that there could be a question of the experimental meaning of these operations). They also argued that L should be equipped with a mapping carrying each proposition \( a \in L \) into its negation \( a' \in L \). In present-day terminology, they proposed that L forms an orthocomplemented lattice with \( \land, \lor, \text{ and } a \rightarrow a' \) as meet, join, and orthocomplementation, respectively (Kalmbach, 1983; Pták and Pulmannová, 1991).

Birkhoff and von Neumann observed that the experimental propositions concerning a classical mechanical system ℳ can be identified with members of a field of subsets of the phase space for ℳ, (or, more accurately, with
elements of a quotient of such a field by an ideal). In any case, for a classical mechanical system $\mathcal{G}$, they concluded that $L$ forms a Boolean algebra.

An orthocomplemented lattice $L$ is a Boolean algebra if and only if it satisfies the distributive law:

$$x \land (y \lor z) = (x \land y) \lor (x \land z).$$

(1)

An example in which $a \in L$ denotes the observation of a wave packet on one side of a plane, $a' \in L$ its observation on the other side, and $b \in L$ its observation in a state symmetric about the plane shows that

$$b = b \land (a \lor a') \neq (b \land a) \lor (b \land a') = 0,$$

so that the distributive law of classical logic breaks down even for the simplest of quantum-mechanical systems. As Birkhoff and von Neumann observed,

... whereas logicians have usually assumed that properties of negation were the ones least able to withstand a critical analysis. The study of mechanics points to the distributive identities as the weakest link in the algebra of logic.

Invoking the desirability of an "a priori" thermo-dynamic weight of states," Birkhoff and von Neumann argued that $L$ should satisfy a weakened version of Eq. (1), called the modular law, and having the following form:

If $z \leq x$, then $x \land (y \lor z) = (x \land y) \lor z.$

(2)

If Eq. (1) holds, then so does Eq. (2) in view of the fact that $z \leq x$ implies $x \land z = z$. However, Eq. (2) is weaker than Eq. (1) since the projection operators for a Hilbert space of finite dimension $n \geq 2$ form a modular, but nondistributive, lattice. Thus, Birkhoff and von Neumann proposed an orthocomplemented modular lattice as a model for a quantum-mechanical calculus of logic, although they admitted that it would be satisfying if one could interpret the modular law in Eq. (2) "by simpler phenomenological properties of quantum physics."

Birkhoff and von Neumann also gave an example to show that the projection operators on an infinite-dimensional Hilbert space fail to satisfy the modular law. Evidently, von Neumann considered this to be a possible serious flaw of the Hilbert-space formulation of quantum mechanics as proposed in his own Grundlagen. Much of von Neumann's work on continuous geometries (1960) and rings of operators (Murray and von Neumann, 1936) was motivated by his desire to construct concrete complemented modular lattices carrying an "a priori" thermo-dynamic weight of states," that is, a continuous dimension or trace function.

1.3 The Orthomodular Law

Although the projection lattice of an infinite-dimensional Hilbert space fails to satisfy the modular law in Eq. (2), it was discovered by Husimi (1937) that it does satisfy the following weaker condition, now called the orthomodular law:

If $z \leq x$, then $x = (x \land z') \lor z.$

(3)

If Eq. (2) holds, then so does Eq. (3) in view of the fact that $x = x \land (z' \lor z)$. The same condition was rediscovered independently by Loomis (1955) and Maeda (1955) in connection with their work on extension of the Murray-von Neumann dimension theory of rings of operators to orthocomplemented lattices. An orthocomplemented lattice satisfying Eq. (3) is called an orthomodular lattice.

In 1957, Mackey published an expository article on quantum mechanics in Hilbert space based on notes for lectures that he was then giving at Harvard. These notes were later published in the form of a monograph (Mackey, 1963) in which the basic principles of quantum mechanics were introduced in terms of a function

$$p = \text{Prob}(A, \psi, E)$$

(4)

interpreted as the probability $p$ that a measurement of the observable $A$ in state $\psi$ results in a value in a set of $E$ of real numbers. The square $A^2$ of $A$ is then defined by the condition

$$\text{Prob}(A^2, \psi, E) = \text{Prob}(A, \psi, F),$$

where $F$ is the set of all real numbers $x$ such that $x^2 \in E$. If $A = A^2$, then $A$ is called a
question. Under certain more or less reasonable hypotheses, it can be shown that the set of all questions forms an orthomodular lattice \( L \).

The generality of Mackey's formulation and the natural way in which Mackey's questions give rise to an orthomodular lattice engendered the idea of a universal logical calculus for all of the experimental sciences. Such a calculus would be based on the class of all orthomodular lattices—including the Boolean algebras that would serve as models for the logics affiliated with classical mechanical systems. Would this be the realization of Leibniz's dream of a calculus ratiocinator? This captivating thought helped to motivate an ongoing study of the theory of orthomodular lattices by a relatively small but devoted group of researchers. An authoritative account of the resulting theory of orthomodular lattices as developed up to about 1983 can be found in Kalmbach (1983).

1.4 The Interpretation of Meet and Join

In spite of the appeal of a general scientific logic based on orthomodular lattices, a nagging question raised in the 1936 paper of Birkhoff and von Neumann was still unresolved. If, for a quantum-mechanical system, most pairs of observations are incompatible and cannot be made simultaneously, what experimental meaning can one attach to the meet \( p \land q \) of two propositions? Two and a half decades after his initial paper with von Neumann, Birkhoff returned to this question (Birkhoff, 1961), calling for an autonomous quantum logic that draws its authority directly from experiments. (A similar question arises in connection with the logic of relativistic physics where the traditional notion of simultaneity is meaningless for spatially separated events.) After all, simultaneity is an indispensable constituent of classical propositional conjunction.

An obvious way to avoid the interpretation issue for \( p \land q \) is to replace the assumption that \( p \land q \) always exists with the weaker assumption that it exists if \( p \) and \( q \) are compatible in the sense that they can be simultaneously tested by means of a single experiment. In this connection, Birkhoff and von Neumann were careful to point out that "... one may regard a set of compatible measurements as a single composite 'measurement'."

Thus, for compatible propositions, experimental meaning can be bestowed upon the meet and join by regarding these connectives as the conjunction and disjunction in the usual sense of classical logic.

Although the mainstream effort to develop a viable quantum logic has concentrated on the use of orthomodular lattices as the basic models (Jauch, 1963; Piron, 1976; Mittelstaedt, 1978; Beltrametti and Cassinelli, 1981), alternative models have been introduced that avoid the interpretation issue for meet and join by invoking the notion of compatibility. Among these are the orthomodular posets introduced in the early 1960s (Foulis, 1962) and the orthoalgebras proposed in the late 1970s (Randall and Foulis, 1978; Hardegree and Frazer, 1981; Lock and Hardegree, 1984a,b). Thus, the evolution of quantum logic from the 1930s to the present has been the story of a slow retreat from Boolean-algebra-based logic and the concurrent development of more and more general mathematical models.

2. STANDARD QUANTUM LOGIC

2.1 The Orthomodular Lattice of Projections on a Hilbert Space

As a mathematical model for a calculus of quantum logic, the orthomodular lattice \( L \) of projection operators on a Hilbert space \( \mathcal{H} \) is called a standard quantum logic. Wilbur (1977) has given a purely lattice-theoretic characterization of the standard quantum logics.

In the present section, we sketch the theory of standard quantum logics, considering only the special case in which \( \mathcal{H} \) is a separable Hilbert space of dimension at least three over the complex number field \( \mathbb{C} \). Thus, we leave aside real or quaternionic Hilbert spaces as well as the generalized Hilbert spaces of Gross and Keller (Keller, 1980). We regard \( \mathcal{H} \) as the Hilbert space corresponding to a quantum-mechanical system \( \mathcal{S} \). (For the time being, we do not consider superselection rules.)

If \( A \) is a bounded operator on \( \mathcal{H} \), we denote by \( A^* \) the adjoint of \( A \). Thus, \( \langle A\psi|\phi \rangle = \langle \psi|A^*\phi \rangle \) for all \( \psi, \phi \in \mathcal{H} \). A bounded operator
$P$ on $\mathcal{H}$ is called a projection if $P = P^* = P^2$, and we define $L = L(\mathcal{H})$ to be the set of all such projection operators. If $P \in L$ and

$$\mathcal{M} = P(\mathcal{H}) = \{P(\psi) | \psi \in \mathcal{H}\}$$

(5)

is the range of $P$, then $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$; conversely, every closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is of the form given in Eq. (5) for a uniquely determined $P \in L$. If $P$ and $\mathcal{M}$ are related as in Eq. (5), we say that $P$ is the projection onto $\mathcal{M}$. The zero operator 1 is the projection onto $0$, and the identity operator $I$ is the projection onto $\mathcal{H}$.

If $P$ is the projection onto $\mathcal{M}$ and $Q$ is the projection onto $\mathcal{N}$, we write $P \leq Q$ if and only if $\mathcal{M}$ is a linear subspace of $\mathcal{N}$. Thus, $L$ is a partially ordered set (poset) under $\subseteq$. If $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$, we write $\mathcal{M}^\perp$ for the set of all vectors in $\mathcal{H}$ that are orthogonal to every vector in $\mathcal{M}$. Then $\mathcal{M}^\perp$ is again a closed linear subspace of $\mathcal{H}$. If $P$ is the projection onto $\mathcal{M}$, we write the projection onto $\mathcal{M}^\perp$ as $P'$. Note that

$$P' = 1 - P, \quad (P')' = P, \quad 0' = 1, \quad \text{and} \quad 1' = 0.$$ 

Furthermore, if $P, Q \in L$ with $P \leq Q$, then $Q' \leq P'$.

If $\mathcal{M}$ and $\mathcal{N}$ are closed linear subspaces of $\mathcal{H}$, then so is the set-theoretic intersection $\mathcal{M} \cap \mathcal{N}$. If $P$ is the projection onto $\mathcal{M}$ and $Q$ is the projection onto $\mathcal{N}$, we define $P \land Q$ to be the projection onto $\mathcal{M} \cap \mathcal{N}$, noting that $P \land Q$ is the meet of $P$ and $Q$ in the poset $L$. If we define $P \lor Q = (P' \land Q')'$, we find that $P \lor Q$ is the join of $P$ and $Q$ in $L$. Also, $P \land P' = 0$ and $P \lor P' = 1$, and so $L$ forms a lattice that is orthocomplete by $P \rightarrow P'$. Furthermore, $L$ satisfies Eq. (3), and hence it is an orthomodular lattice.

If $P$ is the projection onto $\mathcal{M}$ and $Q$ is the projection onto $\mathcal{N}$, then $\mathcal{M}$ is a linear subspace of $\mathcal{N}$ if and only if $P \leq Q'$. If $P \leq Q'$, we say that $P$ and $Q$ are orthogonal to each other and write $P \perp Q$. It can be shown that $P \perp Q$ if and only if $P + Q$ is again a projection operator, in which case, $P + Q = P \lor Q$.

If $(\mathcal{M}_\alpha)$ is a family of closed linear subspaces of $\mathcal{H}$, then the set-theoretic intersection $\bigcap \mathcal{M}_\alpha$ is again a closed linear subspace of $\mathcal{H}$. If $P_\alpha$ is the projection onto $\mathcal{M}_\alpha$ for all $\alpha$ and $P$ is the projection onto $\bigcap \mathcal{M}_\alpha$, then $P$ is the greatest lower bound in $L$ of the family $(P_\alpha)$, and we write $\bigwedge_\alpha P_\alpha = P$. Likewise, $(\bigvee_\alpha P_\alpha)'$ is the least upper bound on $L$ of the family $(P_\alpha)$, and we write $\bigvee_\alpha P_\alpha = (\bigwedge_\alpha P_\alpha)'$. Consequently, the standard quantum logic $L$ is actually a complete orthomodular lattice.

The interpretation of $L$ as a model for a logic of quantum mechanics is based on the following premise:

The two-valued (true/false), experimentally testable propositions for the quantum-mechanical system $\mathcal{G}$ are represented by the projections in the standard quantum logic $L$ for the Hilbert space $\mathcal{H}$ corresponding to $\mathcal{G}$.

Furthermore, if each experimental proposition for $\mathcal{G}$ is identified with its corresponding projection $P \in L$, it is assumed that

if $P, Q \in L$, then $P \leq Q$ holds if and only if $P$ and $Q$ are simultaneously testable and, whenever they are both tested and $P$ is found to be true, then $Q$ will also be true.

### 2.2 Observables

As is customary, we assume that an observable or dynamical variable for the quantum-mechanical system $\mathcal{G}$ is represented by a (not necessarily bounded) self-adjoint operator $A$ on the Hilbert space $\mathcal{H}$. In particular, then, each projection operator $P \in L$ represents an observable that, when measured, can only produce the values 1 (true) or 0 (false). As we shall see, the connection between general observables and projection observables is effected by the celebrated spectral theorem.

The smallest collection of subsets of the real numbers $\mathbb{R}$ that contains all open intervals and is closed under the formation of complements and countable unions is called the $\sigma$ field of real Borel sets. A spectral measure is a mapping $E \rightarrow P_E$ from real Borel sets into projections such that $P_\emptyset = 0$, $P_\mathbb{R} = 1$, and, for every pairwise disjoint sequence $E_1, E_2, E_3, \ldots$ of real Borel sets,

$$\bigvee_{k=1}^\infty P_{E_k} = P_{E_1 \cup E_2 \cup E_3 \cup \ldots}.$$ 

If $E \rightarrow P_E$ is a spectral measure, $\lambda \in \mathbb{R}$, and $J$
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\[ A = \int_\lambda \lambda \, dP_A. \]

The projections in the family \((P_E)\) are called the spectral projections for the observable \(A\).

We can now be quite explicit about the connection between observables in general and projection observables in particular. Suppose that \(A\) is an observable and that \((P_E)\) is the corresponding family of spectral projections. Then

\[ P_E \text{ represents the experimental proposition asserting that a measurement of the observable } A \text{ yields a result } r \text{ that belongs to the real Borel set } E \subseteq \mathbb{R}. \]

In quantum mechanics it is understood that a family of observables \((A_a)\) is compatible (that is, jointly or simultaneously observable) if and only if \(A_\alpha A_\beta = A_\beta A_\alpha\) for all \(\alpha, \beta\) (that is, if and only if the observables commute with each other). On the basis of this understanding, we can state the following:

A family of projections \((P_E) \subseteq L\) is compatible (that is, simultaneously testable) if and only if \(P_\alpha P_\beta = P_\beta P_\alpha\) for all projections in the family.

It can be shown that two observables commute with each other if and only if their spectral projections commute with each other.

It is interesting to note that the question of whether or not two projections commute can be settled in purely lattice-theoretic terms. Indeed, for \(P, Q \in L\)

\[ PQ = QP \text{ if and only if } P = (P \wedge Q) \lor (P \wedge Q'). \]

The equation stating the condition is a special case of the distributive law in Eq. (1); hence, in a standard quantum logic, the failure of the distributive law is a direct consequence of the fact that there are incompatible pairs of quantum-mechanical observables.

2.3 States

A bounded self-adjoint operator \(W\) on \(\mathcal{H}\) is said to be nonnegative if \(\langle W\psi\psi \rangle \geq 0\) for all \(\psi \in \mathcal{H}\). A nonnegative operator \(W\) belongs to the trace class if the series

\[ \text{tr}(W) = \sum_{\psi \in \mathcal{B}} \langle W\psi\psi \rangle \]

converges for an orthonormal basis \(B \subseteq \mathcal{H}\). Convergence on any one orthonormal basis implies convergence on all orthonormal bases.

A (von Neumann) density operator on \(\mathcal{H}\) is a bounded, self-adjoint, nonnegative, trace-class operator \(W\) on \(\mathcal{H}\) such that \(\text{tr}(W) = 1\). Denote by \(\Omega = \Omega(\mathcal{H})\) the set of all density operators on \(\mathcal{H}\). One of the basic assumptions of statistical quantum mechanics is the following:

There is a one-to-one correspondence between the possible states of the system \(\mathcal{F}\) and the density operators \(W \in \Omega\) such that, for every experimental proposition \(P \in L\), \(\text{tr}(WP)\) is the probability that \(P\) will be true when tested in the state corresponding to \(W\).

In accordance with this assumption, we shall identify each possible state of the system \(\mathcal{F}\) with the corresponding density operator \(W\). In particular, if \(A\) is an observable with spectral family \((P_E)\), the probability function, Eq. (4) in Mackey's formulation, is realized as

\[ \text{Prob}(A, W, E) = \text{tr}(WP_E). \]  \hspace{1cm} (6)

Equation (6), one of the fundamental equations of quantum mechanics, says that

the probability that a measurement of the observable \(A\) in the state \(W\) yields a result \(r\) in the Borel set \(E\) is given by \(\text{tr}(WP_E)\).

By a countably additive probability measure on the orthomodular lattice \(L\) is meant a function \(\omega : L \to [0, 1] \subseteq \mathbb{R}\) such that, for every sequence \(P_1, P_2, P_3, \ldots\) of pairwise orthogonal projections in \(L\),

\[ \omega(\bigvee_i P_i) = \sum_i \omega(P_i). \]

By a celebrated theorem of Gleason (1957),
\( \omega \) is a countably additive probability measure on \( L \) if and only if there is a (uniquely determined) density operator \( W \in \Omega \) such that
\[
\omega(P) = \text{tr}(WP) \quad \text{for all } P \in L.
\]

2.4 Superposition of States

If \( W_1, W_2, W_3, \ldots \in \Omega \) is a sequence of density operators and \( t_1, t_2, t_3, \ldots \) is a corresponding sequence of nonnegative real numbers such that \( \sum t_k = 1 \), then \( W = \sum t_k W_k \) is again a density operator, which is referred to as a mixture or an incoherent superposition of the states \( W_k \). For instance, \( W \) could be regarded as the state of a statistical ensemble of systems for which \( t_k \) is the fraction of the systems that are in the state \( W_k \).

A state \( W \) is called a pure state if it cannot be obtained as a mixture of other states. It is customary to assume that individual physical systems are always in a pure state and that mixed states apply only to statistical ensembles of systems each of which is in a pure state, or to physical systems that are interactively coupled with other physical systems.

It can be shown that \( W \in \Omega \) is a pure state if and only if it is a projection onto a one-dimensional linear subspace \( \mathcal{M} \) of \( \mathcal{H} \).

Thus, any normalized vector \( \psi \in \mathcal{H} \) determines a unique pure state, namely the projection onto the linear subspace of complex multiples of \( \psi \). Such a state is called a vector state, and two normalized vectors determine the same vector state if and only if each can be obtained from the other by multiplying by a complex number of modulus 1 (a phase factor). Every state \( W \) is a mixture of pure (that is, vector) states.

We define the support of \( W \in \Omega \), in symbols \( \text{supp}(W) \), to be the set of all \( P \in L \) such that \( \text{tr}(WP) \neq 0 \). This is the same as the set of all \( P \in L \) for which \( WP \neq 0 \). If \( W = \sum t_k W_k \) is an incoherent superposition of the sequence \( \{W_k\} \), then it is clear that \( \text{supp}(W) \) is contained in the set-theoretic union \( \cup_k \text{supp}(W_k) \).

More generally, if \( \{W\}_a \) is a family of states, we say that the state \( W \) is a superposition of the states \( W_a \) if and only if
\[
\text{supp}(W) \subseteq \bigcup_a \text{supp}(W_a)
\]
(Bennett and Foulis, 1990). If \( W \) as well as every \( W_a \) is a pure state, then \( W \) is said to be a coherent superposition of the states \( W_a \). For instance, if \( W \) is the vector state determined by \( \psi \in \mathcal{H} \), each \( W_a \) is the vector state determined by \( \psi_a \in \mathcal{H} \), and \( \psi \) differs from a normalized linear combination of the \( \psi_a \) by a phase factor, then \( W \) is a coherent superposition of the \( W_a \).

2.5 Dynamics

By dynamics is meant a study of the way in which the states (Schrödinger picture) or the observables (Heisenberg picture) of a system change or evolve in time. The Schrödinger and Heisenberg pictures are mathematically equivalent. For definiteness, we adopt the Schrödinger picture. Thus, if the space \( \Omega \) of density operators represents the state space of the quantum-mechanical system \( \mathcal{G} \), then the dynamical evolution of the system is represented by a function \( f(t,W) \) of the time \( t \in \mathbb{R} \) and the state \( W \in \Omega \) such that
\[
f(t,W) \in \Omega, \quad f(0,W) = W \quad \text{and} \quad f(t + s,W) = f(t,f(s,W)). \tag{7}\]

The understanding in Eq. (7) is that \( f(t,W) \) represents the state of the system after a time interval \( t \) if it is in state \( W \) at time 0. The function \( f \) is called the dynamical law for the system \( \mathcal{G} \).

If the dynamical law \( f \) in Eq. (7) preserves superpositions, and is continuous in a suitable sense, it can be shown (Mackey, 1963) that there is a family \( \{U_t\} \) of unitary operators continuously indexed by real numbers such that
\[
f(t,W) = U_t W U_t^{-1}
\]
holds for all \( t \in \mathbb{R} \). Hence, by a celebrated representation theorem of Stone (1932), it follows that there is a self-adjoint operator \( H \) on \( \mathcal{H} \) such that
\[
U_t = e^{-itH} \tag{8}
\]
for all \( t \in \mathbb{R} \). Equation (8) is the operator form of the Schrödinger equation and \( H \) is the Hamiltonian operator for the system.
2.6 Combinations of Standard Quantum Logics

Suppose that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are complex separable Hilbert spaces with corresponding standard quantum logics \( L_1 \) and \( L_2 \). There are two natural ways to combine \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) to form a composite Hilbert space \( \mathcal{H} \) with its own standard quantum logic \( L \): We can form either the direct sum \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) or the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) (Foulis, 1989).

In neither case is the structure of the resulting standard quantum logic \( L \) easy to describe in terms of the structures of \( L_1 \) and \( L_2 \). If \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are quantum-mechanical systems represented by corresponding Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), it is customary to regard the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) as the Hilbert space corresponding to the "combined system" \( \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 \) (Jauch, 1968). If this is so, then in the combination \( \mathcal{S}_1 + \mathcal{S}_2 \) the systems can be tightly correlated, but they cannot exert instantaneous influences on each other (Klay et al., 1987).

If \( W \) is a state for the combined system \( \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 \), there exist uniquely determined states \( W_1 \) for \( \mathcal{S}_1 \) and \( W_2 \) for \( \mathcal{S}_2 \) such that for all \( p \in L_1 \) and all \( p^\prime \in L_2 \),

\[
\text{tr}(W_1 P_1) = \text{tr}(W(P_1 \otimes 1)) \quad \text{and} \quad \text{tr}(W_2 P_2) = \text{tr}(W(1 \otimes P_2)).
\]

The states \( W_1 \) and \( W_2 \) are called reduced states. In general, \( W \) is not determined by \( W_1 \) and \( W_2 \), but depends on the details of the coupling between \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \). However, if \( W \) is a pure state and either \( W_1 \) or \( W_2 \) is pure, then both \( W_1 \) and \( W_2 \) are pure and \( W = W_1 \otimes W_2 \). Therefore, if \( \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 \) is a pure state and if \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are correlated in any way, then neither \( \mathcal{S}_1 \) nor \( \mathcal{S}_2 \) can be in a pure state.

If \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), a superselection rule (Wick et al., 1952) may be imposed, in which case the quantum logic \( L \) associated with \( \mathcal{H} \) is understood to consist only of projections that commute with the projections \( P_1 \) and \( P_2 \) of \( \mathcal{H} \) onto the subspaces \( \mathcal{H}_1 \oplus [0] \) and \( [0] \oplus \mathcal{H}_2 \), respectively. In this case, \( L \) is isomorphic to the Cartesian product \( L_1 \times L_2 \) of the standard quantum logics \( L_1 \) and \( L_2 \), but \( L \) is no longer a standard quantum logic. If such a superselection rule is imposed, it is assumed that the superselected observables are those with spectral projections in \( L \) and the superselected states are represented by density operators \( W \) that commute with both \( P_1 \) and \( P_2 \).

3. ORTHOALGEBRAS AS MODELS FOR A GENERAL QUANTUM LOGIC

3.1 Orthoalgebras

In this section, we present an axiomatic mathematical structure called an orthoalgebra (Foulis et al., 1992), which generalizes the standard quantum logics. The idea is to endow a generic orthoalgebra with an absolute minimum of mathematical structure so that it becomes possible to investigate the meaning and consequences of the special features that distinguish particular orthoalgebras—for instance, Boolean algebras, orthomodular lattices, or standard quantum logics—as models for a calculus of experimental propositions.

By definition, an orthoalgebra is a set \( L \) containing two special elements \( 0 \) and \( 1 \) and equipped with a relation \( \perp \) called orthogonality such that, for each pair \( p,q \in L \) with \( p \perp q \), an orthogonal sum \( p \oplus q \) is defined in \( L \) and subject to the following four axioms:

(Commutativity) If \( p \perp q \), then \( q \perp p \) and \( p \oplus q = q \oplus p \).

(Associativity) If \( p \perp q \) and \( (p \oplus q) \perp r \), then \( q \perp r \), \( p \perp (q \oplus r) \), and \( p \oplus (q \oplus r) = (p \oplus q) \oplus r \).

(Orthocomplementation) For each \( p \in L \) there is a unique \( p' \in L \) such that \( p \perp p' \) and \( p \oplus p' = 1 \).

(Consistency) If \( p \perp p \), then \( p = 0 \).

We note that every orthomodular lattice \( L \) becomes an orthoalgebra if we define \( p \oplus q = p \lor q \) whenever \( p \leq q' \). In particular, every Boolean algebra and every standard quantum logic is an orthoalgebra.

If \( L \) is an orthoalgebra and \( p,q \in L \), we define \( p \preceq q \) to mean that there exists \( r \in L \) with \( p \perp r \) such that \( p \oplus r = q \). It can be shown that \( L \) is partially ordered by \( \preceq \) if \( p \leq q \) whenever \( p \leq q' \) hold for all \( p \in L \); and, if \( p \leq q \), then \( q' \leq p' \). Also, if \( p \perp q \), then with respect to \( \preceq \), \( p \oplus q \) is a minimal upper bound for \( p \) and \( q \); that is,

\[
p q \preceq p \oplus q \quad \text{and there exists no} \quad r \in L \quad \text{with} \quad p q \preceq r < p \oplus q.
\]
However, \( p \oplus q \) may not be the least upper bound for \( p \) and \( q \); that is, the conditions \( r \in L \) and \( p, q \leq r \) do not necessarily imply that \( p \oplus q \leq r \).

If \( x, y \in L \) have a least upper bound (respectively, a greatest lower bound), we write it as \( x \lor y \) (respectively, as \( x \land y \)). By definition, an orthomodular poset is an orthoalgebra \( L \) satisfying the condition that \( p \oplus q = p \lor q \) whenever \( p \perp q \). An orthomodular lattice is the same thing as an orthoalgebra in which every pair of elements \( x, y \) has a meet \( x \land y \) and a join \( x \lor y \). A Boolean algebra is the same thing as an orthomodular lattice satisfying the condition that \( x \land y = 0 \) only if \( x \perp y \).

By a subalgebra of the orthoalgebra \( L \), we mean a subset \( S \subseteq L \) such that \( 0, 1 \in S \) and, if \( p, q \in S \) with \( p \perp q \), then \( p \oplus q \in S \) and \( p' \in S \). Evidently, a subalgebra of an orthoalgebra is an orthoalgebra in its own right under the operations inherited from the parent orthoalgebra. If, as an orthoalgebra in its own right, a subalgebra \( B \) of \( L \) is a Boolean algebra, we refer to \( B \) as a Boolean subalgebra of \( L \). If \( p \perp q \in L \), then

\[
B = [0,1,p,q,p',q',p \oplus q, (p \oplus q)']
\]

is a Boolean subalgebra of \( L \), so \( L \) is a set-theoretic union of Boolean subalgebras.

### 3.2 Compatibility, Conjunction, and Disjunction in an Orthoalgebra

We say that a subset of an orthoalgebra \( L \) is a compatible set if it is contained in a Boolean subalgebra of \( L \). A compatible set of pairwise orthogonal elements is called an orthogonal subset of \( L \). If \( L \) is a standard quantum logic, then a subset of \( L \) is compatible if and only if the projections in the subset commute with one another.

Let \( A = \{a_1, a_2, a_3, \ldots, a_n\} \) be a finite orthogonal subset of \( L \). Then, it can be shown that the least upper bound

\[
\bigvee_B A = a_1 \bigvee_B a_2 \bigvee_B a_3 \bigvee_B \cdots \bigvee_B a_n
\]

as calculated in any Boolean subalgebra \( B \) of \( L \) that contains \( A \) is independent of the choice of \( B \). Thus, we define the orthogonal sum

\[
\oplus_A = \bigvee_B A
\]

as calculated in any such \( B \). If \( C \) and \( D \) are finite orthogonal subsets of \( L \), then \( \oplus C \perp \oplus D \) if and only if \( C \cap D \subseteq \{0\} \) and \( C \cup D \) is an orthogonal set, in which case \( \oplus C \oplus \oplus D = \oplus (C \cup D) \). If \( A = \{a_1, a_2\} \), then \( \oplus A = a_1 \oplus a_2 \). Thus, if \( A = \{a_1, a_2, a_3, \ldots, a_n\} \), is an orthogonal set, we can define

\[
a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n = \oplus A
\]

without notational conflict.

If \( p, q \in L \) and both \( p \) and \( q \) belong to a Boolean subalgebra \( B \) of \( L \), then the greatest lower bound \( p \land_B q \) and the least upper bound \( p \lor_B q \) of \( p \) and \( q \) as calculated in \( B \) may well depend on the choice of \( B \). If \( p \land_B q \) is independent of the choice of \( B \), we define the conjunction \( p & q \) of \( p \) and \( q \) by

\[
p & q = p \land_B q.
\]

Likewise, if \( p \lor_B q \) is independent of the choice of \( B \), we define the disjunction \( p + q \) of \( p \) and \( q \) by

\[
p + q = p \lor_B q.
\]

It can be shown that the compatible elements \( p \) and \( q \) have a conjunction if and only if they have a disjunction. Furthermore, if \( p \land_B q \) and \( p \lor_B q \) exist, then \( p \land_B q \) and \( p \lor_B q \) is a maximal lower bound and \( p + q \) is a minimal upper bound for \( p \) and \( q \) in \( L \). If \( p \) and \( q \) are compatible and at least one of \( p \land q \) or \( p \lor q \) exists in \( L \), then \( p \land q \) and \( p + q \) exist, \( p \land q = p \lor q \), and \( p + q = p \lor q \). If \( p \land_B q \) exists, then so do \( p' \land q' \) and \( p' \lor q' \), and we have \( p + q = (p' \land q')' \) and \( p \land q = (p' + q')' \). If \( p \perp q \), then \( p \land q = p \oplus q \) and \( p \land q = 0 \).

If \( L \) is an orthomodular poset, then any two compatible elements \( p, q \in L \) have a conjunction \( p \land_B q = p \land q \) and a disjunction \( p + q = p \lor q \); however, there are orthoalgebras containing compatible pairs of elements that do not admit conjunctions or disjunctions. There are non-Boolean orthoalgebras in which every pair of elements forms a compatible set. There exist orthomodular posets containing three elements that are pairwise compatible, but that do not form a compatible set; however, in an orthomodular
3.3 Probability Measures on and Supports in an Orthoalgebra

By a probability measure on an orthoalgebra \( L \), we mean a mapping \( \omega: L \rightarrow [0,1] \subseteq \mathbb{R} \) such that, for \( p,q \in L \), with \( p \perp q \),

\[
\omega(p \oplus q) = \omega(p) + \omega(q).
\] (8)

It is possible to define \( \sigma \)-complete orthoalgebras and countably additive probability measures thereon, and thus extend Eq. (8) to sequences in \( L \), but we do not do so here. The set of all probability measures on \( L \) is denoted by \( \Omega = \Omega(L) \). Evidently, \( \Omega \) is a convex support of the vector space of all real-valued functions on \( L \).

If the elements of \( L \) are regarded as representing two-valued experimental propositions concerning a physical system \( \mathcal{F} \), then a probability measure \( \omega \in \Omega \) may be interpreted in any of the following ways:

(Frequency) \( \omega \) is a complete stochastic model for \( \mathcal{F} \) in the sense that \( \omega(p) \) is the "long-run relative frequency" with which the proposition \( p \in L \) will be true when repeatedly tested (D'Espagnat, 1971).

(Subjective) \( \omega \) is a model for coherent belief encoding all of our current information about the system \( \mathcal{F} \). Thus, if \( p \in L \), then \( \omega(p) \) measures our current "degree of belief," on a scale from 0 to 1, in the truth of the proposition \( p \) (Jaynes, 1989).

(Propensity) \( \omega(p) \) is a measure on a scale from 0 to 1 of the "propensity" of the system \( \mathcal{F} \) to produce the outcome 1 (= true) when the proposition \( p \) is tested (Popper, 1959).

(Mathematical) \( \omega \) is a mathematical artifact that may be of use in making inferences about \( \mathcal{F} \) using data secured by making measurements on \( \mathcal{F} \) (Kolmogorov, 1933).

For \( \omega \in \Omega \), we define the support of \( \omega \) by

\[ \text{supp}(\omega) = \{ p \in L | \omega(p) > 0 \} \]

If \( S = \supp(\omega) \), then \( 1 \in S \) and, for all \( p, q \in L \) with \( p \perp q \),

\[ p \oplus q \in S \quad \text{if and only if} \quad p \in S \text{ or } q \in S. \] (9)

A subset \( S \) of \( L \) such that Eq. (9) holds is called a support in \( L \). In general, there are supports \( S \subseteq L \) that are not of the form \( \supp(\omega) \) for \( \omega \in \Omega \); those that are of this form are called stochastic supports. The set-theoretic union of supports is again a support, and it follows that the collection of all supports in \( L \) forms a complete lattice under set-theoretic inclusion.

3.4 Cartesian and Tensor Products of Orthoalgebras

If \( L_1 \) and \( L_2 \) are orthoalgebras, the Cartesian product \( L_1 \times L_2 \) becomes an orthoalgebra under the obvious componentwise operations. If \( L_1 \) is identified with \( L_1 \times \{0\} \) and \( L_2 \) is identified with \( \{0\} \times L_2 \) in \( L_1 \times L_2 \), then every element in \( L_1 \times L_2 \) can be written uniquely in the form \( p \otimes q \) with \( p \in L_1 \) and \( q \in L_2 \). This construction generalizes the superselected direct sum of standard quantum logics. Just as is the case for standard quantum logics, \( \Omega(L_1 \times L_2) \) is isomorphic in a natural way to the convex hull of \( \Omega(L_1) \) and \( \Omega(L_2) \).

A construction for the tensor product \( L_1 \otimes L_2 \) of orthoalgebras based on Foulis and Randall (1981) can be found in Lock (1981). The factors \( L_1 \) and \( L_2 \) are embedded in the orthoalgebra \( L_1 \otimes L_2 \) by mappings \( p \rightarrow p \otimes 1 \) and \( q \rightarrow 1 \otimes q \) for \( p \in L_1 \), \( q \in L_2 \) in such a way that \( p \otimes 1 \) and \( 1 \otimes q \) are compatible and have a conjunction \( (p \otimes 1) \& (1 \otimes q) = p \otimes q \). Furthermore, elements of the form \( p \otimes q \) generate \( L_1 \otimes L_2 \). If \( \alpha \in \Omega(L_1) \) and \( \beta \in \Omega(L_2) \), there is a unique \( \gamma = \alpha \beta \in \Omega(L_1 \otimes L_2) \) such that \( \gamma(p \otimes q) = \alpha(p) \beta(q) \) for all \( p \in L_1, q \in L_2 \). A probability measure on \( L_1 \otimes L_2 \) of the form \( \alpha \beta \) is said to be factorizable, and a convex combination of factorizable probability measures is said to be separable (Kläh, 1988). The existence of probability measures on \( L_1 \otimes L_2 \) that are not separable seems to be a characteristic feature of the tensor product of non-Boolean orthoalgebras.
3.5 The Logic of a Physical System

We are now in a position to summarize the quantum logic approach to the study of physical systems (quantum-mechanical or not). The basic postulate of quantum logic for a physical system \( \mathcal{S} \) is as follows:

(Logic postulate) The set \( L \) of all two-valued, experimentally testable propositions for \( \mathcal{S} \) has the structure of an orthoalgebra such that every simultaneously testable set of propositions forms a compatible subset of \( L \) and every finite compatible subset of \( L \) is a simultaneously testable set of propositions. If \( p, q \in L \) with \( p \perp q \), and if \( p, q \), and \( p \oplus q \) are tested simultaneously, then at most one of the propositions \( p, q \) will be true, and \( p \oplus q \) will be true if and only if either \( p \) or \( q \) is true.

We refer to \( L \) as the logic of the system \( \mathcal{S} \).

It is customary to assume that there is a state space \( \Psi \) associated with the physical system \( \mathcal{S} \). The elements \( \psi \in \Psi \) are called states, and, at any given moment, \( \mathcal{S} \) is presumed to be in one and only one state \( \psi \in \Psi \). A state is supposed to encode all available information about the consequences of performing tests or making measurements on \( \mathcal{S} \) when \( \mathcal{S} \) is in that state.

Whereas the truth or falsity of an experimental proposition \( p \in L \) can be determined by a suitable test, it may or may not be possible to determine the current state \( \psi \in \Psi \) of \( \mathcal{S} \) by a test or measurement; however, it may be possible to bring \( \mathcal{S} \) into a state \( \psi \) by means of a suitable state-preparation procedure. The state of the system \( \mathcal{S} \) can change under the action of a dynamical law, under a state collapse when an observer tests a proposition or measures an observable, because a state-preparation procedure is executed, or simply by virtue of a spontaneous state transition.

A connection between the state space \( \Psi \) for \( \mathcal{S} \) and its logic \( L \) is effected as follows:

(Stochastic postulate) Each state \( \psi \in \Psi \) determines a corresponding probability measure \( \omega_\psi \) on \( L \) in such a way that, for \( p \in L \), \( \omega_\psi(p) \) is the probability that the proposition \( p \) is true when tested with the system \( \mathcal{S} \) in the state \( \psi \).

In the stochastic postulate, the probability measure \( \omega_\psi \) can be interpreted in any of the four ways (frequency, subjective, propensity, mathematical) suggested in Sec. 3.3. In what follows, we denote by \( \Sigma \) the subset of \( \Omega(L) \) consisting of all probability measures of the form \( \omega_\psi \) for \( \psi \in \Psi \), and we refer to each \( \omega_\psi \in \Sigma \) as a probability state for the system \( \mathcal{S} \). It is customary to identify the state \( \psi \) with the corresponding probability state \( \omega_\psi \) and to speak of the elements in \( \Sigma \) as states for \( \mathcal{S} \). Although this custom can lead to philosophical and mathematical difficulties (what if \( \phi, \psi \in \Psi \), \( \phi \neq \psi \), and yet \( \omega_\phi = \omega_\psi \)?) we shall follow it in the interests of simplicity.

Let \( \omega \in \Sigma \) be a state and let \( p \in L \) be an experimental proposition for the physical system \( \mathcal{S} \). We say that \( p \) is possible, impossible, or certain in the state \( \omega \) if \( p \in \text{supp}(\omega) \), \( p \in \text{supp}(\omega) \), or \( p \in \text{supp}(\omega) \), respectively. If both \( p \) and \( p' \) are possible, we say that \( p \) is contingent in the state \( \omega \). The state space \( \Sigma \) is said to be initial if every nonzero \( p \in L \) is certain in at least one state \( \omega \in \Sigma \).

If \( \Lambda \subseteq \Sigma \) is a set of states, then a state \( \omega \in \Sigma \) is said to be a superposition of the states in \( \Lambda \) if

\[ \text{supp}(\omega) = \bigcup \{ \text{supp}(\lambda) | \lambda \in \Lambda \} \]

The superposition closure of \( \Lambda \) is defined to be the set \( \Lambda^\circ \) of all superpositions of states in \( \Lambda \). If \( \Lambda = \Lambda^\circ \), then \( \Lambda \) is called superposition closed. A state \( \omega \) is pure if the set \( \{ \omega \} \) is superposition closed. If \( \omega \) is a pure state, \( \Lambda \) is a set of pure states, and \( \omega \in \Lambda^\circ \), then \( \omega \) is a coherent superposition of the states in \( \Lambda \). In what follows, we denote by \( \mathcal{L} \) the set of all superposition-closed subsets of \( \Sigma \). Note that \( \mathcal{L} \) is closed under set-theoretic intersection, and hence, it forms a complete lattice under set-theoretic inclusion.

It has long been a tenet of natural philosophy that affiliated with a physical system \( \mathcal{S} \) is a class \( \mathcal{A} \) of attributes or properties. At any given moment, some of these attributes may be actual, while the others are only potential. The attributes of \( \mathcal{S} \) that are always actual are its intrinsic attributes; those that can be either actual or potential are its accidental attributes. The charge of an electron is one of its intrinsic attributes, whereas the attribute "spin up in the \( z \) direction" is accidental.

To each attribute \( A \in \mathcal{A} \) there corresponds a set \( \Lambda_A \subseteq \Sigma \) consisting precisely of
those states \( \omega \) such that \( A \) is actual whenever \( \mathcal{I} \) is in the state \( \omega \). A heuristic argument, which we omit here, indicates that \( A \) should be superposition closed, so that \( \Lambda_\alpha \subseteq \mathcal{I} \). Similar arguments suggest that every element of \( \mathcal{I} \) corresponds in this way to an attribute, and thus lead us to our third postulate:

\[
\text{(Attribute postulate) Each attribute } A \text{ determines a corresponding superposition-closed subset } \Lambda_\alpha \text{ of the state space } \Sigma \text{ such that } A \text{ is actual if and only if the system } \mathcal{I} \text{ is in a state } \omega \in \Lambda_\alpha; \text{ furthermore, every } \Lambda \in \mathcal{I} \text{ has the form } \Lambda_\alpha \text{ for some } \alpha \in \alpha.
\]

Just as we identified states with probability states, we propose to identify elements \( \Lambda \) of the complete lattice \( \mathcal{I} \) with attributes of the system \( \mathcal{I} \). (Note that, as a perhaps undesirable consequence, all of the intrinsic attributes of \( \mathcal{I} \) become identified with the superposition-closed subset \( \Sigma \) itself.) Thus, we shall refer to the complete lattice \( \mathcal{I} \) as the attribute lattice for the system \( \mathcal{I} \).

If \( \Lambda, \Gamma \in \mathcal{I} \) are attributes of \( \mathcal{I} \), then \( \Lambda \subseteq \Gamma \) if and only if \( \Gamma \) is actual whenever \( \Lambda \) is actual. Furthermore, the attribute \( \Lambda \cap \Gamma \in \mathcal{I} \) corresponds to a bona fide conjunction of the attributes \( \Lambda \) and \( \Gamma \) in the sense that \( \Lambda \cap \Gamma \) is actual if and only if both \( \Lambda \) and \( \Gamma \) are actual. However, the least upper bound of \( \Lambda \) and \( \Gamma \) in \( \mathcal{I} \) is \( (\Lambda \cup \Gamma)^c \), and it can be actual in states in which neither \( \Lambda \) nor \( \Gamma \) is actual. Following Aerts (1982), we say that the attributes \( \Lambda \) and \( \Gamma \) are separated by a superselection rule if \( \Lambda \cup \Gamma \) is superposition closed, so that the least upper bound of \( \Lambda \) and \( \Gamma \) in \( \mathcal{I} \) corresponds to a bona fide disjunction of the attributes \( \Lambda \) and \( \Gamma \).

3.6 The Canonical Mapping

We continue our discussion of the physical system \( \mathcal{I} \) subject to the logic, stochastic, and attribute postulates of Sec. 3.5.

Von Neumann (1955, p. 249) writes,

Apart from the physical quantities ..., there exists another category of concepts that are important objects of physics—namely the properties of the states of the system \( \mathcal{I} \).

Furthermore, he goes on to identify these properties (or attributes) with the projections in the standard quantum logic \( \mathcal{L} \) affiliated with the quantum-mechanical system \( \mathcal{I} \).

In the more general situation under discussion, it is also possible to relate propositions \( p \in \mathcal{L} \) and properties (i.e., attributes) \( A \in \mathcal{I} \). For \( p \in \mathcal{L} \), define

\[
[p] = \{\omega \in \Omega | \omega(p) = 1\}.
\]

We claim that \([p]\) is superposition closed. Indeed, suppose that \( \alpha \in [p]^c \), but that \( \alpha \in [p] \). Then \( \alpha(p) \neq 1 \), and so \( \alpha(p^c) > 0 \), \( p^c \in \text{supp}(\alpha) \), and so \( p^c \in \text{supp}(\omega) \) for some \( \omega \in [p] \). But, then, \( \omega(p^c) > 0 \), and so \( \omega(p) < 1 \), contradicting \( \omega \in [p] \). Thus \( p \rightarrow [p] \) provides a mapping from experimental propositions \( p \in \mathcal{L} \) to attributes \([p] \in \mathcal{I} \). We refer to \( p \rightarrow [p] \) as the canonical mapping (Foulis et al., 1983).

An attribute of the form \([p]\) is called a principal attribute; the principal attributes are those that can be identified with experimental propositions as von Neumann did. It is not difficult to show that every attribute is an intersection (i.e., a conjunction) of (possibly infinitely many) principal attributes. The state space \( \Sigma \) is unital if and only if \([p] = 0 \) implies that \( p = 0 \).

Evidently, \( p, q \in \mathcal{L} \) with \( p \leq q \) implies that \([p] \subseteq [q] \). If the converse holds, so that \([p] \subseteq [q] \) implies that \( p \leq q \), then \( \mathcal{I} \) is said to have a full set of states. If \( \mathcal{I} \) has a full set of states and every attribute is principal, then the logic \( \mathcal{L} \) is isomorphic to the attribute lattice \( \mathcal{I} \)—this is precisely what happens for a standard quantum logic and it accounts for von Neumann's identification of projections and properties.

3.7 Critique of Quantum Logic

Quantum logic is a relatively young subject, it is still under vigorous development, and many consequences of the epistemological and mathematical insights that it has already provided have yet to be exploited. Quantum-logical techniques involving the tensor product have already cast some light on the well-known Einstein–Podolsky–Rosen paradox (Kläy, 1988), and it is hoped that they will also clarify some of the other classical paradoxes (Wigner's friend, Schrödinger's cat, etc.). The problem of hidden vari-
able can be formulated, understood, and studied rigorously in terms of quantum logics (Greechie and Gudder, 1973). Quantumlogical techniques have enhanced our understanding of group-theoretic imprimivity methods and the role of superselection rules (Piron, 1976), and ideas related to quantum logic have been used to help unravel the measurement problem (Busch et al., 1991).

There is a strong possibility that unrestricted orthoalgebras are too general to serve as viable models for quantum logic. Some orthoalgebras are extremely "pathological" and thus may be suitable only for the construction of counterexamples. It seems likely that only an appropriately specialized class of orthoalgebras, e.g., unital orthoalgebras, might prove to be adequate as models for a general logic of experimental propositions.

The main drawback of quantum logic is already evident in the standard quantum logic $L$ of a Hilbert space $\mathcal{H}$: In the passage from the wave functions $\psi$ in $\mathcal{H}$ to the projections $P \in L$, all phase information is lost. The lost information becomes critical when sequential measurements—e.g., iterated Stern–Gehlach spin resolutions (Wright, 1978)—are to be performed. There are at least two ways to restore the lost information, both of which are currently being studied. One can introduce complex-valued amplitude functions on the logic $L$ (Gudder, 1988), or one can introduce a general mathematical infrastructure called a manual or test space (Randall and Foulis, 1973; Foulis, 1989) that can carry phase information and that gives rise to orthoalgebras as derived structures in much the same way that Hilbert spaces give rise to the standard quantum logics.

4. THE LOGICIAN'S APPROACH

We now present an approach to quantum logic more closely aligned with that of standard logical techniques. In the preceding section, we gave an axiomatic approach to orthoalgebras, the most general mathematical structures currently used as models for quantum logic. This section deals with quantum logics by using the methods of logical tradition. In so doing, we will speak of abstract quantum logics. As we have seen, standard quantum logic is identified with the complete orthomodular lattice of the projections on a separable Hilbert space of dimension at least three over the complex number field. Thus standard quantum logic is a particular kind of semantic model for a form of abstract quantum logic. Generally, a logic $L$ can be determined as a triple $\langle FL, \vdash, \models \rangle$, consisting of a formal language $FL$, a proof-theoretic consequence relation, and a semantic (or model-theoretic) consequence relation. For the sake of simplicity, we will consider only sentential languages, generated by an alphabet containing:

1. a denumerably infinite sequence of atomic sentences (i.e., sentences whose proper parts are not sentences),
2. a finite sequence of primitive logical connectives.

The set of the sentences of the language $FL$ is the smallest set that contains the atomic sentences and is closed under the logical connectives.

The proof-theoretic concept of consequence $\vdash$ for $L$ is defined by referring to a calculus (a set of axioms and of rules) that, in turn, determines a notion of proof from a set of premises to a conclusion. A sentence $\beta$ is called a proof-theoretic consequence of a sentence $\alpha$ ($\alpha \vdash \beta$) if and only if (hereafter abbreviated as iff) there is a proof where $\omega$ is the premise and $\beta$ the conclusion. The semantic-consequence relation $\models$ refers to a class of possible interpretations (models) of the language, which render any sentence "more or less" true or false. A sentence $\beta$ is called a semantic consequence of $\alpha$ ($\alpha \models \beta$) iff in any possible model of the language, $\beta$ is at least as true as $\alpha$.

The two consequence relations $\vdash$ and $\models$ are reciprocally adequate iff they are equivalent. In other words: for any sentences $\alpha, \beta$:

$\alpha \vdash \beta$ iff $\alpha \models \beta$.

The "if arrow" represents the soundness property of the logic, whereas the "only if arrow" is the semantic completeness property.

Naturally, a logic can be characterized by different consequence relations that turn out to be equivalent. A logic $L$ is called axiomatizable iff it admits a proof-theoretic relation, where the notion of proof is decidable. Fur-
5. ALGEBRAIC AND POSSIBLE-WORLD SEMANTICS

In the logical tradition, logics can be generally characterized by means of two privileged kinds of semantics: an algebraic semantics, or a possible-world semantics (called also Kripkean semantics).

These semantics give different answers to the question: What does it mean to interpret a formal language? In the algebraic semantics, the basic idea is that interpreting a language essentially means associating to any sentence an abstract truth value or, more generally, an abstract meaning: an element of an algebraic structure. Hence, generally, an algebraic model for a logic \( \mathbf{L} \) will have the form

\[
\mathcal{M} = (\mathcal{A}, v),
\]

where \( \mathcal{A} \) is an algebraic structure belonging to a class \( \mathcal{R} \) of structures satisfying a given set of conditions and \( v \) transforms sentences into elements of \( \mathcal{A} \), preserving the logical form (in other words, logical constants are interpreted as operations of the structure). We will consider only structures where a binary relation \( \leq \) (possibly a partial order) is defined. On this basis, the semantic-consequence relation is defined as follows:

**Definition 5.1**—\( \beta \) is a semantic consequence of \( \alpha \) (\( \alpha \vdash \beta \)) iff for any model \( \mathcal{M} = (\mathcal{A}, v) \), \( v(\alpha) \leq v(\beta) \) (in other words, the abstract meaning of \( \alpha \) precedes the abstract meaning of \( \beta \)).

In the possible-world semantics, instead, one assumes that interpreting a language essentially means associating to any sentence \( \alpha \) the set of the possible worlds (or situations) where \( \alpha \) holds: This set, that represents the extensional meaning of \( \alpha \), is called the proposition associated to \( \alpha \) (simply, the proposition of \( \alpha \)). Hence, generally, a Kripkean model for a logic \( \mathbf{L} \) will have the form:

\[
\mathcal{M} = (\mathfrak{I}, \mathcal{R}, \mathcal{O}, \Pi, v),
\]

where the meanings of the symbols are as follows.

1. \( \mathfrak{I} \) is a nonempty set of possible worlds possibly correlated by relations in the sequence \( \mathcal{R} \), and operations in the sequence \( \mathcal{O} \). In most cases, we have only one relation \( R \), called the accessibility relation.
2. \( \Pi \) is a set of sets of possible worlds, representing possible propositions of sentences. Any proposition and the total set of propositions \( \Pi \) must satisfy convenient closure conditions that depend on the particular logic.
3. \( v \) transforms sentences into propositions preserving the logical form.

A world \( i \) is said to verify a sentence \( \alpha \) (\( i \models \alpha \)) iff \( i \in v(\alpha) \).

On this basis, the Kripkean semantic-consequence relation is defined as follows:

**Definition 5.2**—\( \beta \) is a semantic consequence of \( \alpha \) (\( \alpha \vdash \beta \)) iff for any model \( \mathcal{M} = (\mathfrak{I}, \mathcal{R}, \mathcal{O}, \Pi, v) \) and for any world \( i \in \mathfrak{I} \),

\[
\text{if } i \models \alpha \text{ then } i \models \beta
\]

(in other words: whenever \( \alpha \) is verified, also \( \beta \) is verified).

In both semantics, a sentence \( \alpha \) is called a logical truth (\( \models \alpha \)) iff \( \alpha \) is the consequence of any sentence \( \beta \).

An interesting variant of Kripkean semantics is represented by the many-valued possible-world semantics, founded on a generalization of the notion of proposition. As we have seen, in the standard possible-world semantics, the proposition of a sentence \( \alpha \) is a set of worlds: the worlds where \( \alpha \) holds. This automatically determines the set of the worlds where \( \alpha \) does not hold (the "meaning" of the negation of \( \alpha \)). Intermediate truth values are not considered. In the many-valued possible-world semantics, instead, one fixes, at the very beginning, a set of truth values \( V \subseteq [0,1] \) and any proposition is represented as a function \( X \) that associates to any truth value \( r \in [0,1] \) a convenient set of possible worlds (the worlds where our proposition holds with truth value \( r \)). As a consequence, the total set of propositions \( \Pi \) turns out to behave like a family of fuzzy subsets of \( \mathfrak{I} \) (see THEORY AND APPLICATIONS OF FUZZY LOGIC).
Classical logic (CL) can be characterized both in the algebraic and in the Kripkean semantics. Algebraically, it is determined by the class of all algebraic structures \( \langle A, \cdot \rangle \), where \( A \) is a Boolean algebra and \( \cdot \) interprets the classical connectives (negation, conjunction, disjunction) as the corresponding Boolean operations (complement, meet, join). In the framework of Kripkean semantics, instead, CL is characterized by the class of all models \( \langle I, R, \Pi, v \rangle \), where

1. the accessibility relation \( R \) is the identity relation (in other words, any world is accessible only to itself);
2. the set of the possible propositions \( \Pi \) is the set of all subsets of \( I \);
3. \( v \) interprets the classical connectives as the corresponding set-theoretic operations.

### 6. ORTHODOX QUANTUM LOGIC

In the abstract quantum-logical universe, a privileged element is represented by orthodox quantum logic (QL), first described "as a logic" by Birkhoff and von Neumann (Birkhoff and von Neumann, 1936). QL is a singular point in the class of all logics that are weaker than classical logic. Many logical and metalogical problems concerning QL have been solved. However, some questions seem to be stubbornly resistant to being resolved.

#### 6.1 Semantic Characterizations of QL

Similarly to classical logic, QL can be characterized both in the algebraic and in the Kripkean semantics. The language of QL contains the two primitive connectives \( \neg \) (not), \( \otimes \) (and). Disjunction is supposed to be metalinguistically defined via De Morgan's law:

\[
\alpha \otimes \beta := \neg (\neg \alpha \otimes \neg \beta).
\]

A conditional connective can be defined as the "Sasaki hook":

\[
\alpha \rightarrow \beta := \neg \alpha \otimes (\alpha \otimes \beta).
\]

**Definition 6.1.1**—An algebraic model of QL is a pair \( \mathcal{M} = \langle A, \cdot \rangle \), where

1. \( A = \langle A, \leq, \cdot, 1, 0 \rangle \) is an orthomodular lattice;
2. \( v \) (the interpretation function) interprets the connective \( \cdot \) as the operation \( \cdot \), the connective \( \otimes \) as the lattice-meet \( \land \):
   a. \( v(\alpha) \in A \) for any atomic sentence \( \alpha \).
   b. \( v(\neg \beta) = v(\beta)' \).
   c. \( v(\beta \otimes \gamma) = v(\beta) \land v(\gamma) \).

**Definition 6.1.2**—A sentence \( \alpha \) is called true in a model \( \langle A, \cdot \rangle \) iff \( v(\alpha) = 1 \). Accordingly, we will have that \( \beta \) is a consequence of \( \alpha \) in the algebraic semantics of QL \( (\alpha \dashv \vdash QL \beta) \) iff \( v(\alpha) \leq v(\beta) \) in any model \( \langle A, \cdot \rangle \) based on an orthomodular lattice \( A \). Further, \( \alpha \) is a quantum-logical truth in the algebraic semantics \( (\vdash QL \alpha) \) iff \( \alpha \) is true in any algebraic model of QL.

As a consequence of the orthomodular property, a semantic version of a "deduction lemma" can be proved:

**Lemma 6.1.1**—\( \alpha \vdash QL \beta \) iff \( \vdash QL \alpha \rightarrow \beta \). In other words, \( \rightarrow \) represents a "good" conditional connective: \( \alpha \rightarrow \beta \) is logically true iff \( \beta \) is a consequence of \( \alpha \).

**Definition 6.1.3**—A Kripkean model of QL has the form \( \mathcal{M} = \langle I, R, \Pi, v \rangle \), where the following conditions held:

1. The accessibility relation \( R \) is reflexive and symmetric (we will also write \( i \sim j \) for \( Rij \); and \( i \perp j \) for not \( Rij \). Moreover, if \( X \subseteq I \), we will write \( i \sim X \) for \( \forall j \in X (i \sim j) \).

   A possible proposition of \( \mathcal{M} \) is a maximal set of worlds, which contains all and only those worlds whose accessible worlds are accessible to at least one element of \( X \). In other words, \( i \in X \) iff \( \forall j \in I \), \( \exists k \in X \) with \( j \in k \).

   For any \( X \subseteq I \), let \( X^\circ := \{ i \in I \mid i \sim X \} \). One can prove that \( X^\circ \) is a possible proposition for any \( X \subseteq I \); \( X \) is a possible proposition iff \( X = X^\circ \); \( \emptyset \) and \( I \) are possible propositions; if \( X, Y \) are possible propositions, then \( X \cap Y \) is a possible proposition.

2. \( \Pi \) is a set of possible propositions closed under \( I, \emptyset, \cap \).

3. \( \Pi \) is orthomodular: \( X \cap (X \cap (X \cap Y)^\circ)^\circ \subseteq Y \), for any \( X, Y \in \Pi \).
4. a. \( v(\alpha) \in \Pi \), for any atomic sentence \( \alpha \);
   b. \( v(\neg \beta) = v(\beta)^c \);
   c. \( v(\beta \circ \gamma) = v(\beta) \cap v(\gamma) \).

**Definition 6.1.4**—A sentence \( \alpha \) is called true in a model \( \mathcal{M} = (I, R, \Pi, v) \) iff \( \alpha \) is verified by any world \( i \in I \).

Accordingly, we will have that \( \beta \) is a consequence of \( \alpha \) in the Kripkean semantics of QL \( (\alpha \vdash^\Delta \beta) \) iff for any Kripkean model \( \mathcal{M} = (I, R, \Pi, v) \) of QL and for any world \( i \), if \( i \models \alpha \) then \( i \models \beta \). Further, \( \alpha \) is a quantum-logical truth in the Kripkean semantics for QL \( (\vdash_{QL} \alpha) \) iff \( \alpha \) is true in any Kripkean model of QL.

The algebraic and the Kripkean semantics for QL turn out to characterize the same logic:

**Theorem 6.1.1**—\( \alpha \vdash^\Delta \beta \) iff \( \alpha \models_{QL} \beta \). This permits us to write simply \( \alpha \models_{QL} \beta \) instead of \( \alpha \vdash^\Delta \beta \) and \( \alpha \models_{QL} \beta \).

Both the algebraic and the Kripkean models of QL admit of Hilbert-space exemplifications, which are the basis for the physical interpretations. Let \( \mathcal{H} \) be the separable complex Hilbert space associated to a physical system \( \mathcal{F} \). An algebraic model \( (\mathcal{A}, v) \) can be constructed by taking as \( \mathcal{A} \) the standard quantum logic based on \( \mathcal{H} \) (in other words, the orthomodular lattice of the projections on \( \mathcal{H} \)) whereas \( v \) will follow the intended physical meaning of the atomic sentences. At the same time, a Kripkean model \( \mathcal{M} = (I, R, \Pi, v) \) can be constructed by putting \( I = \) the set of the pure states (represented by normalized vectors \( \psi \) of \( \mathcal{H} \)), \( R = \) the non-orthogonality relation between pure states, and \( \Pi = \) the set of the possible propositions, which are uniquely determined by the closed subspaces of \( \mathcal{H} \). \( v \) will follow the physical meaning of the atomic sentences. It turns out that the propositions of the model correspond to superposition-closed subsets of the pure-state space. We will call this kind of models Hilbertian models of QL.

A question arises: Is QL characterized by the class of all algebraic Hilbertian models? The answer is negative as proved by Greechie (1981). For instance, there is a complicated sentence of QL (corresponding to the so-called orthoarguesian law) that is true in all Hilbertian models, and not true in some QL models. Let us call Hilbertian quantum logic (HQL) the logic that is semantically characterized by the class of all Hilbertian models. Apparently, HQL is stronger than QL. Hence, abstract quantum logic turns out to be definitely more general with respect to its physical and historical origin. The axiomatizability of HQL is still an open problem.

**Definition 6.1.5**—A sentence \( \alpha \) is called semantically consistent iff for any \( \beta \), \( \alpha \models_{QL} \beta \mathbin{\circ} (-\beta) \) (in other words, no contradiction is a semantic consequence of \( \alpha \)).

One can show that \( \alpha \) is semantically consistent iff there is at least one algebraic model \( (\mathcal{A}, v) \) such that \( v(\alpha) \neq 0 \) iff there exists at least one Kripkean model \( \mathcal{M} = (I, R, \Pi, v) \) and at least one world \( i \) such that \( i \models \alpha \).

### 6.2 An Axiomatization of QL

QL is an axiomatizable logic. Many axiomatizations are known: in the Hilbert–Bernays style (Hardegree, 1979), in the natural deduction, and in the sequent style (Gibbins, 1985; Nishimura, 1980). We will present here a calculus (Goldblatt, 1974; Dalla Chiara, 1986) that represents a kind of "logical copy" of orthomodular lattices. Our calculus (that has no axioms) is determined as a set of rules. Any rule has the form

\[
\begin{align*}
\alpha_1 & \vdash \beta_1, \ldots, \alpha_n \vdash \beta_n \\
\alpha & \vdash \beta.
\end{align*}
\]

(If \( \beta \) is inferred from \( \alpha_1, \ldots, \beta_n \), then \( \beta \) can be inferred from \( \alpha \).) The configurations \( \alpha_1 \vdash \beta_1, \ldots, \alpha_n \vdash \beta_n \) represent the premises of the rule, while \( \alpha \vdash \beta \) is the conclusion. An improper rule is a rule whose set of premises is empty. Instead of

\[
\frac{0}{\alpha \vdash \beta'}
\]

we will write \( \alpha \vdash \beta \).

The rules of QL are as follows:

- **R1** \( \alpha \vdash \alpha \) (identity).
- **R2** \( \alpha \vdash \beta, \beta \vdash \gamma \quad \alpha \vdash \gamma \) (transitivity).
- **R3** \( \alpha \mathbin{\circ} \beta \vdash \alpha \).
- **R4** \( \alpha \mathbin{\circ} \beta \vdash \beta \).
- **R5** \( \gamma \vdash \alpha, \gamma \vdash \beta \quad \gamma \vdash \alpha \mathbin{\circ} \beta \).
R6 $α \vdash \neg α$ (weak double negation).

R7 $\neg α \vdash α$ (strong double negation).

R8 $α \vdash β$ (contraposition).

R9 $α \circ \neg (α \circ (α \circ β)) \vdash β$ (orthomodularity).

**Definition 6.2.1**—A proof is a finite sequence of configurations $α \vdash β$ where any element of the sequence is either an improper rule or the conclusion of a proper rule whose premises are previous elements of the sequence.

**Definition 6.2.2**—$β$ is a proof-theoretic consequence of $α$ (or provable from $α$) ($α \vdash_{QL} β$) iff there is a proof whose last configuration is $α \vdash β$.

**Definition 6.2.3**—$β$ is a proof-theoretic consequence of a set of sentences $T$ (or provable from $T$) ($T \vdash_{QL} β$) iff $T$ includes a finite subset $\{α_1, \ldots, α_n\}$ such that $α_1 \circ \cdots \circ α_n \vdash_{QL} β$.

**Definition 6.2.4**—A set of sentences $T$ is called contradictory if $T \vdash_{QL} β \circ \neg β$ for some sentence $β$; noncontradictory, otherwise. A sentence $α$ is contradictory if $\{α\}$ is contradictory; noncontradictory, otherwise.

The proof-theoretic and the semantic-consequence relations turn out to be equivalent. Namely, a soundness and a completeness theorem can be proved:

**Theorem 6.2.1**—Soundness

If $α \vdash_{QL} β$ then $α \models_{QL} β$.

**Theorem 6.2.2**—Completeness

If $α \models_{QL} β$ then $α \vdash_{QL} β$.

As a consequence, one obtains the result that a sentence is noncontradictory iff it is semantically consistent.

A characteristic "anomaly" of QL is the violation of a metalogical condition, which is satisfied not only by CL but also by a large class of nonclassical logics. This condition is represented by the Lindenbaum property, according to which any noncontradictory set of sentences $T$ can be extended to a noncontradictory and complete set $T^*$ such that for any sentence $α$, either $α \in T^*$ or $\neg α \in T^*$.

The set $T := \{α \rightarrow (β \rightarrow α)\}$ (which contains the negation of the a fortiori principle) represents an example of a noncontradictory set that cannot be extended to a noncontradictory and complete set. The set $T$ is noncontradictory, because in some models $(s, ν)$: $ν(\neg(α \rightarrow (β \rightarrow α))) = 0$. For instance, take $(s, ν)$ based on the orthomodular lattice of the closed subspaces of $β^2$, where $ν(α)$ and $ν(β)$ are two nonorthogonal unidimensional subspaces. However, one can easily check that $ν(\neg(α \rightarrow (β \rightarrow α))) = 1$ is impossible. Hence, $\neg(α \rightarrow (β \rightarrow α))$ cannot belong to a noncontradictory and complete set $T^*$, which would trivially admit a model $(s, ν)$ such that $ν(β) = 1$, for any $β \in T^*$. From an intuitive point of view, the failure of the Lindenbaum property represents a very strong incompleteness result. The *tertium non datur* principle breaks down at the very deep level: There are theories that are intrinsically incomplete, even in *mente Dei*.

Among the questions that are still unsolved, let us mention at least the following:

1. Is QL decidable?

2. Does QL admit the finite-model property? In other words, if a sentence is not a quantum-logical truth, is there any finite model where our sentence is not verified? A positive answer to the finite-model property would automatically provide a positive answer to the decidability question, but not vice versa.

3. Is the set of all possible propositions in the Kripkean canonical model of QL orthomodular? (The worlds of the canonical model are all the noncontradictory and deductively closed sets of sentences $T$, whereas two worlds $T$ and $T'$ are accessible iff whenever $T$ contains a sentence $α$, $T'$ does not contain its negation $\neg α$.) This problem is correlated to the critical question whether any orthomodular lattice is embeddable into a complete orthomodular lattice. Only partial answers are known.

### 7. ORTHOLOGIC AND UNSHARP QUANTUM LOGICS

By dropping the orthomodular condition both in the algebraic and in the Kripkean semantics, one can characterize a weaker form...
of quantum logic, which is usually called orthologic or minimal quantum logic (MQL). This logic turns out to be more "tractable" from a metalogical point of view: It satisfies the finite-model property; consequently, it is decidable (Goldblatt, 1974). A calculus that represents an adequate axiomatization for MQL can be, naturally, obtained by replacing the orthomodular rule \( R_9 \) of our QL calculus with the weaker Duns Scotus rule \( \alpha \odot \neg \alpha \vdash \beta \) (ex absurdo sequitur quodlibet: Any sentence is a consequence of a contradiction).

A less investigated form of quantum logic is represented by paraconsistent quantum logic \( \text{(PQL)} \) (Dalla Chiara and Giuntini, 1989), which is a weak example of an unsharp quantum logic, possibly violating the noncontradiction and the excluded-middle principles. As we will see, unsharp quantum logics represent natural abstractions from the unsharp approaches to quantum theory. Algebraically, PQL is characterized by the class of all models based on an involutive lattice \((A, \leq, 1, 0)\), with smallest element 0 and largest element 1. Equivalently, in the Kripkean semantics, PQL is characterized by the class of all models \((I, R, \Pi, v)\), where \( R \) is a symmetric, not necessarily reflexive, relation, and \( \Pi \) behaves like in the MQL case. Differently from QL and MQL, a world \( i \) of a PQL model may verify a contradiction. Since \( R \) is generally not reflexive, it may happen that \( i \in v(\beta) \) and \( i \nsubseteq v(\beta) \). Hence: \( i \models \beta \odot \neg \beta \). An adequate axiomatization for PQL can be obtained by dropping the orthomodular rule \( R_9 \) in our QL calculus. Like MQL, also PQL satisfies the finite-model property and consequently is decidable.

Interesting unsharp extensions of PQL are the Brouwer-Zadeh logics first investigated by Cattaneo and Nistico (1989). A characteristic of these logics is a splitting of the connective "not" into two forms of negation: a fuzzylike negation, which gives rise to a paraconsistent behavior, and an intuitionisticlike negation. The fuzzy "not" represents a weak negation, which inverts the truth values truth and falsity, satisfies the double-negation principle, but generally violates the noncontradiction and the excluded-middle principles. The second "not" is a stronger negation, a kind of necessitation of the fuzzy "not." As a consequence, the language of the Brouwer-Zadeh logics is an extension of the QL language, with two primitive negations: \( \neg \) represents the fuzzy "not," whereas \( \sim \) is the intuitionistic "not." On this basis, a necessity operator can be defined in terms of the two negations:

\[
La := \sim \neg a.
\]

In other words: "necessarily \( a \)" means the intuitionistic negation of the fuzzy negation of \( a \). A possibility operator is then defined in terms of \( L \) and \( \neg \):

\[
Ma := L\neg a.
\]

We will consider two forms of Brouwer-Zadeh logics: BZL (weak Brouwer-Zadeh logic) and BZL\(^1\), which represents a form of three-valued quantum logic. Both logics admit of Hilbert-space exemplifications. Algebraically, BZL is characterized by the class of all models \( \mathcal{M} = (\mathcal{S}, v) \), where \( \mathcal{S} = (A, \leq, \sim, 1, 0) \) is a Brouwer-Zadeh lattice (simply a BZ lattice). In other words:

1. a. \((A, \leq, \sim, 1, 0)\) is an involutive lattice with smallest element 0 and largest element 1.
2. \( \neg \) behaves like an intuitionistic complement:
   \[
a \land a' = 0.
   \]
   \[
a \leq a''.
   \]
   \[
   If \ a \leq b, \ then \ b' \leq a''.
   \]
3. The following relation holds between the fuzzy and the intuitionistic complement:
   \[
a'' = a''.
   \]
4. The regularity condition holds:
   \[
a \land a' \leq b \lor b'.
   \]

The logic BZL, which can be equivalently characterized also by a Kripkean semantics, is axiomatizable and decidable (Giuntini, 1991). The modal operators of BZL behave...
similarly to the corresponding operators of the famous modal system \( \mathcal{S}_\alpha \). For instance, \( LL\alpha \) is equivalent to \( L\alpha \); and \( L\mathcal{M}\alpha \) is equivalent to \( M\alpha \).

The three-valued \( BZL^1 \) can be naturally characterized by a kind of many-valued possible-world semantics. The intuitive idea can be sketched as follows: One supposes that interpreting a language means associating to any sentence two domains of certainty: the domain of possible worlds where the sentence holds, and the domain of possible worlds where the sentence does not hold. All the other worlds are supposed to associate an intermediate truth value (indetermined) to our sentence. The models of this semantics will be called \textit{models with positive and negative domains} (shortly, ortho-pair models).

Briefly, an ortho-pair model has the form \( \mathcal{M} = \langle I, R, \Pi, \nu \rangle \), where

1. \( I \) is a nonempty set of worlds and \( R \) (the accessibility relation) is reflexive and symmetric (like in the Kripkean characterization of \( \mathcal{Q} \).

The possible propositions (in the sense of our definition of the Kripkean model for \( \mathcal{Q} \)) will be here called \textit{simple propositions}. The set \( \Sigma \) of all simple propositions gives rise to an ortholattice; let us indicate by \( \# \), \( \cap \), \( \sqcup \) the lattice operations defined on \( \Sigma \).

2. A \textit{possible proposition} of \( \mathcal{M} \) is any pair \( (X_1, X_0) \), where \( X_1, X_0 \) are simple propositions such that \( X_1 \subseteq X_0 \) (in other words: \( X_1, X_0 \) are orthogonal). The following operations and relations are defined on the set of all possible propositions:
   a. the fuzzy complement
   \[
   (X_1, X_0)' = (X_0, X_1);
   \]
   b. the intuitionistic complement
   \[
   (X_1, X_0)^\omega = (X_0, X_1);
   \]
   c. the propositional conjunction
   \[
   (X_1, X_0) \land (Y_1, Y_0) = (X_1 \cap Y_1, X_0 \sqcup Y_0);
   \]
   d. the order relation
   \[
   (X_1, X_0) \leq (Y_1, Y_0) \iff X_1 \subseteq Y_1 \text{ and } Y_0 \subseteq X_0.
   \]

3. \( \Pi \) is a set of possible propositions closed under \( \land, \lor, \cdot \), and \( 0 := (\emptyset, I) \).

4. \( \nu \) (the interpretation function) maps sentences into propositions in \( \Pi \) and interprets the connectives \( \oplus, \neg, \sim \) as the corresponding operations.

The other basic semantic definitions are like in the algebraic semantics. One can show that in any ortho-pair model the set of propositions has the structure of a \( BZ \) lattice. As a consequence, the logic \( BZL^1 \) is at least as strong as \( BZL \). In fact, one can prove that \( BZL^1 \) is properly stronger than \( BZL \). As a counterexample, let us consider an instance of the fuzzy excluded middle and an instance of the intuitionistic excluded middle applied to the same sentence \( \alpha \). One can easily check:

\[
\alpha \oplus \neg \alpha \models_{\text{bzl}} \alpha \oplus \sim \alpha \quad \text{and} \quad \alpha \oplus \sim \alpha \not\models_{\text{bzl}} \alpha \oplus \neg \alpha.
\]

However, generally

\[
\alpha \oplus \neg \alpha \not\models_{\text{bzl}} \alpha \oplus \sim \alpha.
\]

Also \( BZL^3 \) is axiomatizable (Cattaneo et al., 1993) and can be characterized by means of an algebraic semantics.

8. HILBERT-SPACE MODELS OF THE BROUWER-ZADEH LOGICS

Hilbert-space models of both BZL and \( BZL^1 \) can be obtained in the framework of the \textit{unsharp} (or \textit{operational}) approach to quantum theory that was first proposed by Ludwig (1954) and developed (among others) by Kraus (1983), Davies (1976), Busch et al. (1991), and Cattaneo and Laudisa (1994). One of the basic ideas of this approach is a "liberalization" of the mathematical counterpart for the intuitive notion of "experimentally testable proposition." As we have seen, in orthodox Hilbert-space quantum mechanics, experimental propositions are mathematically represented as projections \( P \) on the Hilbert space \( \mathcal{H} \) corresponding to the physical system \( \mathcal{S} \) under investigation. If \( P \) is a projection representing a proposition and \( W \) is a density operator representing a state of \( \mathcal{S} \), the number \( \text{tr}(WP) \) represents the probability value that the system \( \mathcal{S} \) in the state \( W \)
verifies $P$ (Born probability). However, projections are not the only operators for which a Born probability can be defined. Let us consider the class $\mathcal{E}(\mathcal{H})$ of all linear bounded operators $D$ such that for any density operator $W$, 

$$\text{tr}(WD) \in [0,1].$$

It turns out that $\mathcal{E}(\mathcal{H})$ properly includes the set $L(\mathcal{H})$ of all projections on $\mathcal{H}$. In a sense, the elements of $\mathcal{E}(\mathcal{H})$ represent a "maximal" possible notion of experimental proposition, in agreement with the probabilistic rules of quantum theory. In the framework of the unsharp approach, the elements of $\mathcal{E}(\mathcal{H})$ have been called effects. An important difference between projections and proper effects is the following: Projections can be associated to sharp propositions having the form "the value for the observable $A$ lies in the exact Borel set $F$," whereas effects may represent also fuzzy propositions like "the value of the observable $A$ lies in the fuzzy Borel set $F$." As a consequence, there are effects $D$ that are different from the null projection $0$ and that are verified with certainty by no state (for any $W$, $\text{tr}(WD) \neq 1$). A limit case is represented by the semitransparent effect $\frac{1}{2}1$ (where $1$ is the identity operator), to which any state $W$ assigns probability value $\frac{1}{2}$.

The class of all effects of $\mathcal{H}$ gives rise to a structure $(\mathcal{E}(\mathcal{H}),\leq,\cdot,\bot,1,0)$ which is a BZ poset (not a BZ lattice!). In other words, $\leq$ is a partial order with largest element $1$ and smallest element $0$, while the fuzzy and the intuitionistic complement (‘‘and’’) behave like in the BZ lattices. The relation and the operations of the effect-structure are defined as follows:

1. $D_1 \leq D_2$ iff for any density operator $W$, $\text{tr}(WD_1) \leq \text{tr}(WD_2)$.
2. $1 = 1$.
3. $D' = 1 - D$.
4. $D^*$ is the projection $P_{\text{Ker}(D)}$ into the subspace $\text{Ker}(D)$, consisting of all vectors that are transformed by the operator $D$ into the null vector.
5. $0 = 1'$.

In the particular case where $D$ is a projection, it turns out that $D' = D^*$. In other words, the fuzzy and the intuitionistic complement coincide for sharp propositions. One can show that any BZ poset can be embedded into a complete BZ lattice [for the MacNeille completion (Birkhoff, 1967) of a BZ poset is a complete BZ lattice (Giuntini, 1991)]. As a consequence, the MacNeille completions of the effect-BZ posets represent natural Hilbert-space models for the logic BZL.

As to BZL, Hilbert-space models $(I,R,\Pi,V)$ in the ortho-pair semantics can be constructed as follows:

1. $I$ and $R$ are defined like in the Kripkean Hilbertian models of QL. The simple propositions turn out to be in one-to-one correspondence to the set of the projections of $\mathcal{H}$.
2. $\Pi$ is the set of all possible propositions. Any effect $D$ can be transformed into a proposition $f(D) = (X^?_p,X^?_n)$, where

$$X^?_p := \{\psi \in I\text{tr}(P_\psi D) = 1\} \quad \text{and} \quad X^?_n := \{\psi \in I\text{tr}(P_\psi D) = 0\}$$

$(P_\psi$ is the projection onto the unidimensional subspace spanned by the vector $\psi$). In other words, $X^?_p$ and $X^?_n$ represent respectively the positive and the negative domain of $D$ (in a sense, the extensional meaning of $D$ in the model). The map $f$ turns out to preserve the order relation and the two complements.

3. The interpretation function $v$ follows the intuitive physical meaning of the atomic sentences.

9. PARTIAL QUANTUM LOGICS

So far we have considered only examples of abstract quantum logics, where conjunctions and disjunctions are supposed to be always defined. However, as we have seen, the experimental and the probabilistic meaning of conjunctions of incompatible propositions in quantum theory has been often put in question. How do we construct logics where we admit that conjunctions and disjunctions are possibly meaningless? For instance, how do we give a natural semantic characterization for a logic corresponding to the class of all orthoalgebras or to the class of all orthomodular posets? Let us call these logics respectively weak partial quantum logic (WPaQL) and strong partial quantum logic.
(SPaQL). Are WPaQL and SPaQL axiomatizable?

9.1 Algebraic Semantics for WPaQL

The language of WPaQL contains two primitive connectives: the negation \(-\cdot\) and the exclusive disjunction \(\oplus\) \((aut)\). A conjunction is metalinguistically defined, via De Morgan’s law:

\[
\alpha \otimes \beta := -(\neg \alpha \oplus \neg \beta).
\]

The intuitive idea underlying our semantics for WPaQL is the following: Disjunctions and conjunctions are considered “legitimate” from a mere linguistic point of view. However, semantically, a disjunction \(\alpha \ominus \beta\) will have the intended meaning only in the “well-behaved cases” (where the values of \(\alpha\) and \(\beta\) are orthogonal in the corresponding orthoalgebra). Otherwise, \(\alpha \ominus \beta\) will have any meaning whatsoever (generally not connected with the meanings of \(\alpha\) and \(\beta\)). A similar semantic “trick” is used in some standard treatments of the description operator \(\iota\) (“the unique individual that satisfies a given property”) in classical model theory.

**Definition 9.1.1**—An algebraic model of WPaQL is a pair \(\mathcal{M} = \langle \mathcal{A}, \nu \rangle\), where

1. \(\mathcal{A} = \langle A, \ominus, 1, 0 \rangle\) is an orthoalgebra.
2. \(\nu\) (the interpretation function) satisfies the following conditions:
   - \(\nu(\alpha) \in A\), for any atomic sentence \(\alpha\);
   - \(\nu(\neg \beta) = \nu(\beta)^\prime\), where ‘ is the orthocomplement operation that is defined in \(\mathcal{A}\);
   - \(\nu(\alpha \ominus \beta) = \{\nu(\alpha) \oplus \nu(\beta)^\prime\} \cup \nu(\beta)\), if \(\nu(\alpha) \oplus \nu(\beta)^\prime\) is defined in \(\mathcal{A}\), \(\alpha\) any element otherwise.

Accordingly, we will have that

\[
\alpha \vdash_{WPaQL} \beta \iff \text{in any WPaQL model } \mathcal{M} = \langle \mathcal{A}, \nu \rangle, \nu(\alpha) \leq \nu(\beta),
\]

where \(\leq\) is the partial order relation defined in \(\mathcal{A}\).

9.2 An Axiomatization of Partial Quantum Logics

The logic WPaQL is axiomatizable. We present here a calculus that is obtained as a natural transformation of our QL calculus.

The rules of WPaQL are as follows:

- **R1** \(\alpha \vdash \alpha\) (identity).
- **R2** \(\alpha \vdash \beta \beta \vdash \gamma\) \(\alpha \vdash \gamma\) (transitivity).
- **R3** \(\alpha \vdash \neg \neg \alpha\) (weak double negation).
- **R4** \(\neg \neg \alpha \vdash \alpha\) (strong double negation).
- **R5** \(\neg \beta \vdash \neg \alpha\) (contraposition).
- **R6** \(\alpha \oplus \neg \alpha\) (excluded middle).
- **R7** \(\alpha \vdash \neg \beta\) \(\alpha \oplus \neg \alpha \vdash \alpha \oplus \beta\) (unicity of negation).
- **R8** \(\alpha \vdash \neg \beta\) \(\alpha \vdash \alpha\), \(\alpha \vdash \beta\), \(\beta \vdash \beta\), \(\beta \vdash \beta\), \(\beta \vdash \beta\).
- **R9** \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\).

(R10–R13 require a weak associativity.)

The other basic proof-theoretic definitions are given like in the QL case. Some derivable rules of the calculus are the following:

- **D1** \(\alpha \vdash \beta\) \(\beta \vdash \gamma\) \(\alpha \vdash \gamma\).
- **D2** \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\).
- **D3** \(\alpha \vdash \neg \beta\) \(\beta \vdash \gamma\) \(\alpha \vdash \beta\) \(\beta \vdash \gamma\) \(\alpha \vdash \beta\) \(\beta \vdash \gamma\) \(\alpha \vdash \beta\) \(\beta \vdash \gamma\) \(\alpha \vdash \beta\) \(\beta \vdash \gamma\).
- **D4** \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\) \(\alpha \vdash \beta\).
As to strong partial quantum logic (SPaQL), an axiomatization can be obtained by adding to our WPaQL calculus the following rule:

\[
R_{14} \quad \frac{\alpha \vdash \neg \beta \quad \alpha \vdash \gamma \quad \beta \vdash \gamma}{\alpha \oplus \beta \vdash \gamma}
\]

Semantically, the models of SPaQL will be based on orthoalgebras \( \mathcal{A} = (\mathfrak{A}, \oplus, 1, 0) \), satisfying the following condition: If defined, \( a \oplus b \) is the sup of \( a \) and \( b \). As we have seen in Sec. 3.1, this condition is necessary and sufficient in order to make the orthoposet induced by the orthoalgebra \( \mathcal{A} \) an orthomodular poset. The soundness and the completeness theorems for SPaQL (with respect to this semantics) can be proved similarly to the case of WPaQL.

10. CRITIQUE OF ABSTRACT QUANTUM LOGICS

Do abstract quantum logics represent "real" logics or should they rather be regarded as mere extrapolations from particular algebraic structures that arise in the mathematical formalism of quantum mechanics? Different answers to this question have been given in the history of the logical-algebraic approach to quantum theory. According to our analysis, the logical status of abstract quantum logics can be hardly put in question. These logics turn out to satisfy all the canonical conditions that the present community of logicians require in order to call a given abstract object a logic: syntactical and semantical descriptions, proofs of soundness and completeness theorems, and so on.

Has the quantum-logical research definitely shown that "logic is empirical"? At the very beginning of the history of quantum logic, the thesis according to which the choice of the "right" logic to be used in a given theoretical situation may depend also on experimental data appeared a kind of extremist view, in contrast with the traditional description of logic as "an a priori and analytical science." These days, an empirical position in logic is no more regarded as a "daring heresy." At the same time, we are facing a new difficulty: As we have seen, quantum logic is not unique. Besides orthodox quantum logic, different forms of partial and unsharp quantum logics have been developed. In this situation, one can wonder whether it is still reasonable to look for the most adequate abstract logic that should faithfully represent the structures arising in the quantum world.

A question that has been often discussed concerns the compatibility between quantum logic and the mathematical formalism of quantum theory, based on classical logic. Is the quantum physicist bound to a kind of "logical schizophrenia"? At first sight, the presence of different logics in one and the same theory may give a sense of uneasiness. However, the splitting of the basic logical operations (negation, conjunction, disjunction, ...) into different connectives with different meanings and uses is now a well-accepted logical phenomenon that admits consistent descriptions. As we have seen, classical and quantum logic turn out to apply to different sublanguages of quantum theory that must be sharply distinguished.

GLOSSARY

Abstract Quantum Logic: A logic \( \langle FL, \vdash, \models \rangle \), where the proof-theoretic and the semantic-consequence relations violate some characteristic classical principles like the distributivity of conjunction and disjunction.

Adjoint: If \( A \) is a bounded operator on a Hilbert space, then the adjoint of \( A \) is the unique bounded operator \( A^* \) that satisfies \( \langle A\psi, \phi \rangle = \langle \psi, A^*\phi \rangle \) for all vectors \( \psi, \phi \) in the space.

Algebraic Model of a Language: A pair \( \langle \mathcal{A}, \nu \rangle \) consisting of an algebraic structure \( \mathcal{A} \) and of an interpretation function \( \nu \) that transforms the sentences of the language into elements of \( \mathcal{A} \), preserving the logical form.

Algebraic Semantics: The basic idea is that interpreting a formal language means associating to any sentence an element of an algebraic structure.

Attribute: One of a class of properties affiliated with a physical system. At any given moment some of the attributes of the system may be actual, while others are only potential.
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Axiomatizable Logic: A logic is axiomatizable when its concept of proof is decidable.

Boolean Algebra: An orthocomplemented lattice \( L \) that satisfies the distributive law \( p \land (q \lor r) = (p \land q) \lor (p \land r) \) for all \( p, q, r \in L \).

Borel Set: A (real) Borel set is a set that belongs to the smallest collection of subsets of the real numbers \( \mathbb{R} \) that contains all open intervals and is closed under the formation of complements and countable unions.

Brouwer–Zadeh Lattice (or Poset): A lattice (or poset) \( L \) with smallest element \( 1 \) and largest element \( 0 \) equipped with a regular involution \( \sim \) (a fuzzylike complement), and an intuitionistic like complement \( \neg \), subject to the following conditions for all \( p, q \in L \): (i) \( p \land q = q \land p \), (ii) \( p \land \neg p = 0 \), (iii) \( p \leq p' \), (iv) \( p'' = p' \).

Brouwer–Zadeh Logic: A logic that is characterized by the class of all models based on Brouwer–Zadeh lattices.

Compatible: A set \( C \) of elements in an orthoalgebra \( L \) is a compatible set if there is a Boolean subalgebra \( B \) of \( L \) such that \( C \subseteq B \).

Complete Lattice: A lattice in which every subset has a least upper bound, or join, and a greatest lower bound, or meet.

Completeness: A logic \( \langle FL, \vdash, \models \rangle \) is (semantically) complete when all the semantic consequences are proof-theoretic consequences (if \( \alpha \vdash \beta \) then \( \alpha \models \beta \)).

Conjunction (in an Orthoalgebra): If \( L \) is an orthoalgebra and \( p, q \in L \), then \( p \land q \) is the join (or least upper bound) of \( p \) and \( q \) in \( L \) satisfying the following conditions for all \( p, q \in L \): (i) \( p \land q = q \land p \), (ii) \( p \land \neg p = 0 \), (iii) \( p \leq p' \), (iv) \( p'' = p' \).

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Decidable Logic: A logic is decidable when its proof-theoretic consequence relation is decidable.

Density Operator: A self-adjoint, nonnegative, trace-class operator \( W \) on a Hilbert space, such that \( \text{tr}(W) = 1 \).

Direct Sum (or Cartesian Product) of Hilbert Spaces: The direct sum of the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{I} \) is the Hilbert space \( \mathcal{H} \oplus \mathcal{I} \) consisting of all ordered pairs \((x, y)\) with \( x \in \mathcal{H} \), \( y \in \mathcal{I} \), and with coordinatewise vector operations. The inner product is defined by \( \langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \).

Disjunction (in an Orthoalgebra): If \( L \) is an orthoalgebra and \( p, q \in L \), then \( p \lor q \) have a disjunction \( p \lor q \) if there is a Boolean subalgebra \( B \) of \( L \) with \( p, q \in B \) and the join \( p \lor q \) of \( p \) and \( q \) in \( B \) is independent of the choice of \( B \), in which case \( p \lor q = p \lor q \).

Dynamics: The evolution in time of the state of a physical system.

Effect: A linear bounded operator \( A \) of a Hilbert space such that for any density operator \( W \), \( \text{tr}(WA) \in [0,1] \). In the unsharp approach to quantum mechanics, effects represent possible experimental propositions.

Experimental Proposition: A proposition whose truth value can be determined by conducting an experiment.

Greatest vs Maximal: If \( L \) is a partially ordered set (poset) and \( X \subseteq L \), then an element \( b \in X \) is a greatest element of \( X \) if \( x \leq b \) for all \( x \in X \). An element \( b \in X \) is a maximal element of \( X \) if there exists no element \( x \in X \) with \( b < x \).

Involutive Lattice (or Poset): A lattice (or poset) with smallest element \( 0 \) and largest element \( 1 \), equipped with an involution.

Join: If \( L \) is a partially ordered set (poset) and \( p, q \in L \), then the join (or least upper bound) of \( p \) and \( q \) in \( L \), denoted by \( p \lor q \) if it exists, is the unique element of \( L \) satisfying the following conditions: (i) \( p, q \in p \lor q \) and (ii) \( r \in L \) with \( p, q \leq r \Rightarrow p \lor q \leq r \).

Kripkean Model of a Language: A system \((L, R, ..., R_n, o_1, ..., o_n, \Pi, v)\) consisting of a set \( L \) of possible worlds, a (possibly empty) sequence of world relations \( R, ..., R_n \) and of world operations \( o_1, ..., o_n \), a family \( \Pi \) of subsets of \( L \) (called the propositions), and an interpretation function \( v \) that transforms the sentences of the language into propositions, preserving the logical form.

Kripkean Semantics: The basic idea is that interpreting a formal language means associating to any sentence the set of the possible worlds where the sentence holds. This set is called also the proposition associated to the sentence.

Lattice: A partially ordered set (poset) in which every pair of elements \( p, q \) has a least upper bound, or join, \( p \lor q \) and a greatest lower bound, or meet, \( p \land q \).

Least vs Minimal: If \( L \) is a partially ordered set (poset) and \( X \subseteq L \), then an element \( a \in X \) is a least element of \( X \) if \( a \leq x \) for all
x \in X. An element \( a \in X \) is a minimal element of \( X \) if there exists no element \( x \in X \) with \( x < a \).

Lindenbaum Property: A logic satisfies the Lindenbaum property when any noncontradictory set of sentences \( T \) can be extended to a noncontradictory and complete set \( T' \) (such that \( T' \) contains, for any sentence of the language, either the sentence or its negation). Abstract quantum logics generally violate the Lindenbaum property.

Logic: According to the tradition of logical methods, a logic can be described as a system \((FL, \vdash, =)\) consisting of a formal language, a proof-theoretic-consequence relation \( \vdash \) (based on a notion of proof), and a semantic-consequence relation \( = \) (based on a notion of model and of truth).

MacNeille Completion of a Brouwer-Zadeh Lattice: Let \( L \) be a Brouwer-Zadeh lattice (or poset). For \( X \subseteq L \), let \( X^\prime = \{ p \in L \mid \forall q \in X, p \leq q \} \), and \( \mathcal{P}(L) = \{ X \subseteq L \mid X = X^\prime \} \). The structure \( \mathcal{P}(L) = (\mathcal{P}(L), \subseteq, \wedge, \vee, \top, \bot) \) is called the MacNeille completion of \( L \). \( \mathcal{P}(L) \) is a complete Brouwer-Zadeh lattice and \( L \) is embeddable into \( \mathcal{P}(L) \) via the mapping \( p \rightarrow (p) \), where \( (p) = \{ q \in L \mid q \leq p \} \).

Meet: If \( L \) is a partially ordered set (poset) and \( p, q \in L \), then the meet (or greatest lower bound) of \( p \) and \( q \), denoted by \( p \wedge q \) if it exists, is the unique element of \( L \) satisfying the following conditions: (i) \( p \wedge q \leq p, q \) and (ii) \( r \in L \) and \( r \leq p, q \Rightarrow r \leq p \wedge q \).

Minimal Quantum Logic: A logic that is semantically characterized by the class of all models based on orthocomplemented lattices.

Modular Lattice: A lattice \( L \) satisfying the modular law: \( p \leq r \Rightarrow p \vee (q \wedge r) = (p \vee q) \wedge r \) for all \( p,q,r \in L \).

Nonnegative Operator: A self-adjoint operator \( A \) on a Hilbert space \( \mathcal{H} \) such that \( (A\psi, \psi) \geq 0 \) for all vectors \( \psi \in \mathcal{H} \).

Observables or Dynamical Variables: A numerical variable associated with a physical system the value of which can be determined by conducting an experiment, a measurement, or an experiment on the system.

Orthoaalgebra: A mathematical system consisting of a set \( L \) with two special elements \( 0, 1 \) and equipped with a relation \( \perp \) such that, for each pair \( p,q \in L \) with \( p \perp q \), an orthogonal sum \( p \oplus q \in L \) is defined subject to the following conditions for all \( p,q,r \in L \): (i) \( p \perp q \Rightarrow p \perp q \) and \( p \oplus q = q \oplus p \), (ii) \( p \perp q \) and \( (p \oplus q) \perp r \Rightarrow q \perp r, p \perp (q \oplus r) \) and \( p \oplus (q \oplus r) = (p \oplus q) \oplus r \), (iii) \( p \in L \Rightarrow \) there is a unique \( p' \in L \) such that \( p \perp p' \) and \( p \oplus p' = 1 \), and (iv) \( p \perp p \Rightarrow p = 0 \).

Orthocomplementation: A mapping \( p \rightarrow p' \) on a poset \( L \) with smallest element \( 0 \) and largest element \( 1 \) satisfying the following conditions for all \( p, q \in L \): (i) \( p \lor p' = 1 \), (ii) \( p \land p' = 0 \), (iii) \( p \leq q \Rightarrow q' \leq p' \), and (iv) \( p' = p \).

Orthocomplemented Lattice (or Poset): A lattice (or poset) equipped with an orthocomplementation \( p \rightarrow p' \).

Orthodox Quantum Logic: A logic that is semantically characterized by the class of all algebraic models based on orthomodular lattices. Standard quantum logic is a particular model of orthodox quantum logic.

Orthogonality: If \( L \) is an orthocomplemented poset and \( p, q \in L \), then \( p \) is orthogonal to \( q \) in symbols \( p \perp q \), if \( p \leq q' \).

Orthomodular Lattice: An orthomodular lattice \( L \) satisfying the orthomodular law: For all \( p,q \in L \), \( p \leq q \Rightarrow p = p \lor (q \land p') \).

Orthomodular Poset: An orthoalgebra \( L \) such that, for all \( p,q \in L \), \( p \perp q \Rightarrow p \oplus q = p \lor q \).

Paraconsistent Quantum Logic: A logic that is semantically characterized by the class of all models based on involutive lattices with smallest element \( 0 \) and largest element \( 1 \).

Partially Ordered Set (or Poset): A set \( L \) equipped with a relation \( \leq \) satisfying the following conditions for all \( p,q,r \in L \): (i) \( p \leq p \), (ii) \( p \leq q \) and \( q \leq p \Rightarrow p = q \), (iii) \( p \leq q \) and \( q \leq r \Rightarrow p \leq r \).

Probability Measure: A function \( \omega : L \rightarrow [0,1] \subseteq \mathbb{R} \) on an orthoalgebra \( L \) such that \( \omega(0) = 0, \omega(1) = 1 \), and, for all \( p,q \in L \) with \( p \perp q \), \( \omega(p \oplus q) = \omega(p) + \omega(q) \).

Projection: An operator \( P \) on a Hilbert space that is self-adjoint (\( P = P^* \)) and idempotent (\( P = P^2 \)).

Pure State: A state \( \psi \) is pure if the set \( \{ \psi \} \) consisting only of that state is superposition closed. In Hilbert-space quantum mechanics, the pure states are precisely the vector states.

Quantum Logic: The study of the formal structure of experimental propositions affiliated with a quantum physical system, or any mathematical model (e.g., an orthoaalgebra) representing such a structure.
**Regular Involution:** An involution * on a poset * that satisfies the regularity condition: For all * * and * , * implies * . If * is a lattice, then an involution is regular iff it satisfies the Kleene condition: For all * * and * , * implies * or * .

**Soundness:** A logic * is sound when all the proof-theoretic consequences are semantic consequences (if * then * ).

**Spectral Measure:** A mapping from real Borel sets into projection operators on a Hilbert space that maps the empty set into 0, maps the union of a disjoint sequence of real Borel sets into the least upper bound (join) of the corresponding projection operators.

**Spectral Theorem:** The theorem establishing a one-to-one correspondence between (not necessarily bounded) self-adjoint operators * on a Hilbert space * and spectral measures * such that, if * , then * .

**Spectrum:** If * is a (not necessarily bounded) self-adjoint operator on a Hilbert space and * is the corresponding spectral measure, then a real number * belongs to the spectrum of * if * for all * .

**Standard Quantum Logic:** The complete orthomodular lattice * of all projection operators on a Hilbert space. For * , * , * is defined to mean that * .

**State:** The state of a physical system encodes all information concerning the results of conducting tests or measuring observables on the system. It is usually assumed that, corresponding to each state * of the system, there is a probability measure * on the logic * of the system such that * is the probability that the experimental proposition * in * is true when the system is in the state * .

**State Space:** The set of all possible states of the physical system.

**Strong Partial Quantum Logic:** A logic that is semantically characterized by the class of all models based on orthomodular posets.

**Superselection Rule:** A rule that determines the possible states of a physical system. The usual quantum-mechanical superselection rules state that only vector states that commute with a certain set of pairwise orthogonal projections (i.e., projections onto superposition sectors) represent possible states of the systems.

**Support:** If * is a density operator on a Hilbert space * , then the support of * is defined to be the set * of all projection operators * on * such that * . More generally, if * is a probability measure on an orthoalgebra * , then * if * .

**Superposition Closed:** A set of states is superposition closed if it contains all of its own superpositions.

**Superposition in Hilbert Space:** For a Hilbert space * , if * is a family of vector states determined by the corresponding family ( * ) of normalized vectors in * , and if * is a normalized linear combination of the vectors in this family, then the vector state * determined by * is a (coherent) superposition of the family ( * ). If * is an arbitrary family of density operators on * and * is a corresponding family of nonnegative real numbers such that * , then * is an (incoherent) superposition (or mixtures) of the family ( * ).

**Superposition in an Orthoalgebra:** A probability measure * on an orthoalgebra * is a superposition of a family ( * ) of probability measures on * if * .

**Tensor Product of Hilbert Spaces:** If * and * are Hilbert spaces, then the tensor product * is a Hilbert space together with a mapping * with * for * , * and * that is separately linear in each argument and has the property that if * is an orthonormal basis for * and * is an orthonormal basis for * , then * is an orthonormal basis for * .

**Trace:** The trace of an operator * on a Hilbert space * is defined by * , where * is an orthonormal basis for * , provided that the series converges.

**Trace Class:** An operator * on a Hilbert space * belongs to the trace class if * converges absolutely, where * is an orthonormal basis for * .

**Unsharp Quantum Logics:** Examples of paraconsistent logics where the noncontradiction principle is generally violated.

**Vector State:** A probability measure on the standard quantum logic * of a Hilbert space * determined by a normalized vector
\[ \psi \in \mathcal{H} \] and assigning to each projection operator \( P \in \mathcal{L} \) the probability \( \langle \psi | P | \psi \rangle \).

**Weak Partial Quantum Logic:** A logic that is semantically characterized by the class of all models based on orthoalgebras.

### Works Cited


