MELLIN TRANSFORM FORMULAS FOR DRINFELD MODULES

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Abstract. We introduce formulas for the logarithms of Drinfeld modules using a framework recently developed by the second author. We write the logarithm function as the evaluation under a motivic map of a product of rigid analytic trivializations of t-motives. We then specialize our formulas to express special values of Goss L-functions as Drinfeld periods multiplied by rigid analytic trivializations evaluated under this motivic map. We view these formulas as characteristic-p analogues of integral representations of Hasse-Weil type zeta functions. We also apply this machinery for Drinfeld modules tensored with the tensor powers of the Carlitz module, which serves as the Tate twist of a Drinfeld module.

1. Introduction

1.1. **Motivation.** The main result of this paper gives a positive-characteristic function field analogue of certain integral representations of Hasse-Weil type zeta functions. In order to make a comparison with our new results, we remind the reader first some of the classical theory. The starting point is one of the original proofs of the functional equation and analytic continuation of the Riemann zeta function. The classical theta function, for $t \in \mathbb{C}$ with $\Re(t) > 0$

$$\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t},$$

satisfies the functional equation

$$\Theta(t) = t^{-1/2}\Theta(1/t). \tag{1.1}$$

We also recall the definition of the Mellin transform for a real-valued function $f(x)$ with suitable decay conditions at $x = 0$ and $x = \infty$,

$$M(f)(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1.2}$$

for suitable $s \in \mathbb{C}$. If we take the Mellin transform of a normalized version of $\Theta(t)$ (and account correctly for convergence, which is nontrivial), we get

$$\xi(s) = M\left(\frac{\Theta(t)-1}{2}\right)(s/2), \tag{1.3}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta function. Further, if we take the Mellin transform of (1.1) then we recover the functional equation for the Riemann zeta function,

$$\xi(s) = \xi(1-s).$$

These derivations also establish the analytic continuation of the Riemann zeta function. We refer the reader to [27, §7.1] for details on such constructions.

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With this theory as our base point, there are several important directions we can generalize these ideas. First, if we replace the classical theta function with theta series involving characters, then the same theory gives the functional equation and analytic continuation of Dirichlet L-functions with characters. A further generalization to higher dimensional theta series then gives the same theory for Dedekind zeta functions. Again, all this theory is detailed in [27, §7.2-7.5].

On the other hand, we can instead investigate Hasse-Weil zeta functions attached to algebraic varieties (the previous case of Dedekind zeta functions can be seen as a special case in this setting — that discussion is outside the scope of this introduction). In this setting, at least for elliptic curves defined over the rational numbers, Wiles's modularity theorem [34] shows that such zeta functions are given as the Mellin transform of special meromorphic functions, in this case modular forms. There are vast generalizations of this theory to motives and profound conjectures that come with them, such as Beilinson's conjectures (see [8]) and various aspects of the Langlands program (see [25]).

Our results in this paper establish an analogy to those described above in the positive characteristic function field setting. We prove that certain special values of L-functions can be realized as an algebraic interpolation of a Mellin transform of certain special functions. On the one hand, these L-values are certainly of Hasse-Weil type, because they have an Euler product representation given by the characteristic polynomial of the Frobenius acting on certain modules (see (1.6) and (1.7)). On the other hand, our formulas indicate that these L-values can be represented as a Mellin-type transform, not of Drinfeld modular forms as one might expect, but rather of rigid analytic trivializations of Drinfeld modules, which bear several similarities to classical theta function. Thus the results we present here should be viewed as a hybrid between the two generalizations given above: They express Hasse-Weil type L-values in terms of a Mellin transform of an analogue of the classical theta function. Whether there is a connection between the constructions in this paper and Drinfeld modular forms is an open question. We provide a few comments on this question in Remark 3.11.

Before continuing we say a few words about the difficulty and significance of our results. In the classical setting, one uses analytic ideas (cycle integration) to connect theta functions and related objects directly to L-functions and zeta functions. Our setting occurs in characteristic p, where it is cumbersome to work with characteristic-p valued measures and integration (see Remark 1.3 for the comparison of our results with the already existing literature). Our proofs here instead provide an algebraic alternative to this integration theory which takes a round-about path to connect special values of L-functions with the analogue of the classical theta function. Namely, we connect L-function values to logarithm values using the work of Taelman [31] and the first author [11]. Our new formulas in this paper then connect values of the logarithm to rigid analytic trivializations of Anderson t-motives. Works of Maurischat [26], Pellarin [29] and others then allow us to connect rigid analytic trivializations to periods and Anderson generating functions, which (as we explain below) are an analogue of theta functions. The main new technical advances in this paper include modifying a crucial construction from the work of the second author [19] to a tensor product of motives (this is our (2.15)), a very careful analysis of the convergence properties of (3.10) carried out in §3.3, as well as a particular choice of t-motive bases (discussed in §2.3-2.4) to account for the $\Theta_{\phi,\tau}$ matrix in (2.14).

1.2. **The Mellin transform of Drinfeld modules and L-functions.** We now briefly describe our main results, after which we will make some more precise comparisons to the classical theory. Let $q = p^r$ be a prime power, and let $A := \mathbb{F}_q[\theta]$ and $K := \mathbb{F}_q(\theta)$. Let K_∞ be the completion of K at the infinite place with respect to the norm $|\cdot|$, normalized so that $|\theta| = q$. This completion equals the formal Laurent series ring $\mathbb{F}_q((1/\theta))$ with coefficients in $\mathbb{F}_q$. Let $\mathbb{C}_\infty$ be a completion of an algebraic closure of K_∞. Consider the non-commutative polynomial ring $\mathbb{C}_\infty[\tau]$ which is defined subject to the condition $\tau c = c^q\tau$ for all $c \in \mathbb{C}_\infty$. We define *a Drinfeld module ϕ of rank r* to be an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathbb{C}_\infty[\tau]$ given by

$$\phi_\theta := \phi(\theta) := \theta + k_1\tau + \cdots + k_r\tau^r, \quad k_r \neq 0. \tag{1.4}$$

We also consider $\exp_\phi$ and $\log_\phi$ to be the exponential and logarithm functions associated to ϕ (see (2.2) for details). The function $\exp_\phi$ has a kernel Λ_ϕ which is a free A-module of rank r, called the period lattice of ϕ. Let us denote a set of generating periods as $\lambda_1, \ldots, \lambda_r$. The comparison is often made between a Drinfeld module ϕ and an elliptic curve E defined over $\mathbb{C}$. The periods $\lambda_1, \ldots, \lambda_r$ should then be compared with the Weierstrass periods of E and the exponential function $\exp_\phi$ should be compared to the Weierstrass-$\wp$ function.

We now briefly define Anderson generating functions which are intimately connected with periods. For a given period λ_i, define

$$f_i := \sum_{i=0}^{\infty} \exp_\phi\left(\frac{\lambda_i}{\theta^{i+1}}\right) t^i \in \mathbb{C}_\infty[[t]],$$

where t is a commuting variable (in fact, f_i is in a Tate algebra, see §2.2). We then define the matrix

$$\Upsilon := \begin{pmatrix} f_1 & \cdots & \cdots & f_r \\ f_1^{(1)} & \cdots & \cdots & f_r^{(1)} \\ \vdots & & & \vdots \\ f_1^{(r-1)} & \cdots & \cdots & f_r^{(r-1)} \end{pmatrix},$$

where $\cdot^{(k)}$ is the k-fold application of a Frobenius twisting automorphism (again, see §2.2). The matrix Υ is constructed to be a rigid analytic trivialization for the Drinfeld module ϕ. Namely, there is a naturally defined matrix $\Theta \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty)$ coming from the t-motive associate to ϕ such that we have the functional equation

$$\Theta\Upsilon = \Upsilon^{(1)}.$$

Let $V \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty)$ be a matrix of constants defined in (2.9) and let

$$\Psi := V^{-1}((\Upsilon^{(1)})^{\mathrm{tr}})^{-1}. \tag{1.5}$$

We explain all this theory more extensively in §2.3.

The final ingredient to state our first main theorem comes from a recent paper of the second author [19]. There, the second author develops a new map $\delta_{1,\mathbf{z}}^M$ for a parameter $\mathbf{z} \in \mathbb{C}_\infty$ from M_ϕ, the t-motive associated to ϕ, to $\mathbb{C}_\infty$ which recovers the structure of the Drinfeld module ϕ (see (2.13) for a precise definition). This map $\delta_{1,\mathbf{z}}^M$ should be viewed as an algebraic interpolation of cycle integration; in [19, Cor 5.10] the second author proves an algebraic analogue of the Mellin transform formula which relates the exponential function with *the Carlitz zeta values* $\zeta_A(n)$ given by

$$\zeta_A(n) := \sum_{\substack{a\in A \\ a \text{ monic}}} \frac{1}{a^n} \in K_\infty.$$

Our first main theorem relates the logarithm function of a Drinfeld module of rank r to this motivic map $\delta_{1,\mathbf{z}}^{M}$ evaluated at a product of rigid analytic trivializations.

Theorem 1.1. *Let ϕ be a Drinfeld module given by*

$$\phi_\theta = \theta + k_1\tau + \cdots + k_r\tau^r$$

so that $|k_i| \leq 1$ for each $1 \leq i \leq r-1$ and $k_r \in \mathbb{F}_q^\times$. Let $\mathbf{z} \in \mathbb{C}_\infty$ be an element in the domain of convergence of $\log_\phi$ and let $\mathfrak{e}_k$ be the k-th standard basis vector. Then, we have

$$\log_\phi(\mathbf{z}) = \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{t-\theta}\mathfrak{e}_1^{\mathrm{tr}} V^{\mathrm{tr}} \Upsilon^{(1)}_{|t=\theta} (\Psi^{\mathrm{tr}})^{(-1)}\right).$$

To describe our next result, in what follows, we briefly describe Goss L-functions attached to ϕ introduced by Goss [16], inspired by the ideas of Gekeler [10, Rem. 5.10]. For a given monic irreducible polynomial $w \in A$, we set K_w to be the completion of K at the place corresponding to w. We let (ρ_w) be a family of continuous representations of the Galois group of K^{sep}/K which is strictly compatible in the usual sense, meaning that the characteristic polynomial

$$P_v(X) := \det(1 - X \cdot \rho_w(\mathrm{Frob}_v))$$

of the Frobenius at a place $v \neq w$ of K acting on the w-adic Tate module of ϕ is independent of the choice of prime w and has coefficients in A (along with a ramification condition - see [18, §8.10] for full details). We further let $P_v(X) = (1-a_1X)\cdots(1-a_rX)$ for some $a_1, \ldots, a_r$ lying in a fixed algebraic closure of K and set

$$P_v^\vee(X) := (1 - a_1^{-1}X)\cdots(1 - a_r^{-1}X).$$

We then define *the L-function of ϕ* to be

$$L(\phi, n) := \prod_v P_v(v^{-n})^{-1}, \tag{1.6}$$

and *the dual L-function of ϕ* by

$$L(\phi^\vee, n) := \prod_v P_v^\vee(v^{-n})^{-1}, \tag{1.7}$$

where the product runs over all the finite places of A. In this definition, by [7, Cor. 3.6], we know that $L(\phi, n)$ converges in K_∞ for all $n \in \mathbb{Z}_{\geq 1}$ and $L(\phi^\vee, n)$ converges in K_∞ for all $n \in \mathbb{Z}_{\geq 0}$ (there is a way to extend the domain of such L-functions to an analogue of the upper half plane — since we do not use that here, we refer the reader to [18, §8.1]). We also note that when ϕ is *the Carlitz module* given by $C_\theta := \theta + \tau$, we have, for any positive integer n, $L(C^\vee, n-1) = \zeta_A(n)$. We refer the reader to [18] and [11] for full details on these constructions.

If we set $\mathbf{z} = 1$ in the previous theorem and choose a Drinfeld module ϕ as in Theorem 1.1 so that $k_i \in \mathbb{F}_q$ for each $1 \leq i \leq r-1$, then the logarithm value becomes a special value of the (dual) Goss L-function of ϕ. As a corollary to Theorem 1.1, we get the following.

Corollary 1.2. *Let ϕ be a Drinfeld module as in Theorem 1.1 so that each $k_i \in \mathbb{F}_q$ and let $\overline{\pi} = (\lambda_1, \ldots, \lambda_r)$ be a vector of fundamental periods of ϕ. Then we have*

$$L(\phi^\vee, 0) = \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{\theta - t}\overline{\pi}(\Psi^{\mathrm{tr}})^{(-1)}\right).$$

Remark 1.3. It is appropriate to make a brief comparison between our formulas and the results in [15] and [17] on the Mellin transform in the function field setting. Let A_v be the completion of A at v. Inspired by the construction of formal p-adic Mellin transform, in [15, §3], Goss developed the theory of A_v-valued measures on A_v and defined the Mellin transform of the Carlitz zeta value $\zeta_A(n)$ to be an element in *the divided power series ring* (see [14, §5] for the details on divided power series). Although its coefficients are arithmetically interesting and related to the Carlitz zeta values (see [33, Thm. VII]), there is no immediate relation to $\zeta_A(n)$ as in Corollary 1.2. Hence our construction seems to be better-suited in this direction. Later on, using the seminal work of Teitelbaum [32] relating v-adic measures to Drinfeld cusp forms, Goss [17, §4] defined the Mellin transform of a Drinfeld cusp form f as a continuous function L_f on $\mathbb{Z}_p$ whose values are attained in a finite extension K_∞. However, several aspects of the theory is still missing such as the link between f and the functional equation of L_f as well as the appearance of L_f as a Dirichlet series summed over the monic polynomials in A, which could be more parallel to the classical setting. It would be interesting to relate our construction in the present paper to the setting of Drinfeld modular forms to have a better understanding of the Mellin transformation (see Remark 3.11 for the discussion on a potential link to Drinfeld modular forms).

1.3. **Comparison with the classical theta functions.** Having stated our first two main theorems, we now make some precise comparisons between our setting and the classical theory discussed above. Fixing a $(q-1)$-st root of $-\theta$, we define *the Carlitz fundamental period* by

$$\tilde{\pi} := \theta(-\theta)^{1/(q-1)} \prod_{j=1}^{\infty} \left(1 - \theta^{1-q^j}\right)^{-1} \in \mathbb{C}_\infty^\times.$$

In the case of the Carlitz module C, our main results discussed above reduce to a formula from [19, Cor. 5.10]

$$L(C^\vee, 0) = \zeta_A(1) = \delta^M_{1,\mathbf{z}}(-\widetilde{\pi}\Omega), \tag{1.8}$$

where $\Omega := 1/\omega_C^{(1)}$ is defined in (2.6). In this context, the function Ω should be viewed as an analogue of the theta function $\Theta(z)$ for two reasons:

1. Taking the function field Mellin transform of Ω produces zeta values similar to formula (1.3).
2. It satisfies a similar functional equation to the classical theta function. Namely,

$$t \cdot \Omega = C_\theta^*(\Omega), \tag{1.9}$$

   where C_θ^* is the adjoint Carlitz operator $C_\theta^*(z) := \theta z + z^{1/q}$ (compare with (1.1)).

Remark 1.4. We note here that taking the Mellin transform of (1.1) (after some adjustments for convergence) gives the functional equation for the completed Riemann zeta function $\xi(s) = \xi(1-s)$. It is therefore natural to ask about what happens when we combine the functional equation (1.9) with our function field Mellin transform (1.8). We have the transformation

$$\delta^M_{1,z}(t\widetilde{\pi}\Omega) = \delta^M_{1,C_\theta(z)}(\widetilde{\pi}\Omega) = \delta^M_{1,\theta z}(\widetilde{\pi}\Omega) + \delta^M_{1,z^{(1)}}(\widetilde{\pi}\Omega) = \log_C(\theta) + \zeta_A(1)$$

(recall that for this application we set $z = 1$). On the other hand we also find that

$$\delta^M_{1,z}(t\widetilde{\pi}\Omega) = \delta^M_{1,z}(\widetilde{\pi}C_\theta^*(\Omega)) = \delta^M_{1,z}(\widetilde{\pi}\theta\Omega) + \widetilde{\pi}\delta^M_{1,z}(\Omega^{(-1)}) = \theta\zeta_A(1),$$

since one can show fairly quickly that $\Omega^{(-1)}$ is in the kernel of $\delta_{1,z}^M$ and since $\delta_{1,z}^M$ is $\mathbb{C}_\infty$-linear. After recalling Carlitz's formula that $\log_C(1) = \zeta_A(1)$, we arrive at

$$\log_C(C_\theta(1)) = \theta \log_C(1),$$

so we have recovered the functional equation for the Carlitz logarithm. We suspect that a similar phenomenon happens in the case of Drinfeld modules and more general t-modules. In fact, it seems possible that one could reverse the direction of these calculations to prove our logarithm formulas in §3 in an alternate way. However, there are many details to work out so we leave this as a question to be answered in future work.

In the case of Drinfeld modules of rank r discussed in the present paper, the matrix Ψ from Corollary 1.2 is a higher-rank generalization of Ω discussed above and should be viewed as a higher dimensional theta function. Indeed, it satisfies the functional equation

$$\Phi\Psi = \Psi^{(-1)}, \tag{1.10}$$

where $\Phi \in \mathrm{Mat}_{r\times r}(K[t])$ is defined in (2.5). Analyzing this functional equation shows that if we denote the top row of Ψ as $(g_1, \dots, g_r)$, then each g_i satisfies

$$t \cdot g_i = \phi_\theta^*(g_i), \tag{1.11}$$

where ϕ^* is the adjoint of the Drinfeld module ϕ given by $\phi_\theta^* := \theta + k_1^{1/q}\tau^{-1} + \cdots + k_r^{1/q^r}\tau^{-r}$ (see [18, §4.14] for more details). Our Corollary 1.2 then says that taking the function field Mellin transform of a vector of periods multiplied by this analogue of a theta function gives a Hasse-Weil type zeta value.

1.4. Tate twists of Drinfeld modules.

We also give a version of our main theorems for Drinfeld modules tensored with the positive powers of the Carlitz module. This is akin to taking the Tate twist of a motive, and shifts the value of the corresponding L-function allowing us to get formulas for values n larger than 1. Our result provides an interesting link between certain coordinates of the logarithms of Tate twists of Drinfeld modules and their periods as well as quasi-periods.

In this setting, let ϕ be a Drinfeld module of rank r given as in (1.4) without any restriction on the coefficients $k_1, \dots, k_r \in \mathbb{C}_\infty$. For any $1 \le \ell \le r-1$, we set $F_{\tau^\ell} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ to be the unique entire function satisfying

$$F_{\tau^\ell}(\theta z) - \theta F_{\tau^\ell}(z) = \exp_\phi(z)^{q^\ell}$$

for all $z \in \mathbb{C}_\infty$. Furthermore, for any $1 \le i \le rk+1$, we let $p_i : \mathbb{C}_\infty^{rk+1} \to \mathbb{C}_\infty$ be the projection onto the i-th coordinate.

Theorem 1.5. *Let $\rho = \phi \otimes C^{\otimes k}$ and let $\mathbf{z} \in \mathbb{C}_\infty^{rk+1}$ be an element in the domain of convergence of Log_ρ and $\lambda_1, \dots, \lambda_r$ be fundamental periods of ϕ. Then*

$$p_{rk+1-(j-1)}(\mathrm{Log}_\rho(\mathbf{z})) = \begin{cases} \delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{\tilde{\pi}^k}{\omega_C^k(\theta-t)}(\lambda_1, \dots, \lambda_r)(\Psi^{\mathrm{tr}})^{(-1)}\right) & \text{if } j = 1 \\ \delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(F_{\tau^{r-(j-1)}}(\lambda_1), \dots, F_{\tau^{r-(j-1)}}(\lambda_r))(\Psi^{\mathrm{tr}})^{(-1)}\right) & \text{if } 2 \le j \le r \end{cases}.$$

In our last result, we analyze the special values of Goss L-functions of Drinfeld modules defined over $\mathbb{F}_q$. Let ϕ be a Drinfeld module of rank 2 given as in (1.4) such that $k_1, k_2 \in \mathbb{F}_q$. Let us also consider the Drinfeld module $\tilde{\phi}$ given by

$$\tilde{\phi}_\theta := \theta - k_1 k_2^{-1}\tau + k_2^{-1}\tau^2.$$

There exists a particular relation between certain coordinates of logarithms of Anderson t-module $\tilde{\rho} := \tilde{\phi} \otimes C^{\otimes k}$ and $L(\phi, k+1)$ (see Corollary 4.5 for more details). Using this relation allows us to obtain the following corollary, restated as Corollary 4.5 later, of Theorem 1.5.

Corollary 1.6. *For $k \geq 1$, let $\mathbf{z}_i \in \mathrm{Mat}_{(2k+1)\times 1}(\mathbb{F}_q)$ be the i-th unit vector. We have*

$$L(\phi, k+1) = \det \begin{bmatrix} \delta_{1,\mathbf{z}_{2k}}^{M_{\tilde{\rho}}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(\tilde{F}_\tau(\lambda_1), \tilde{F}_\tau(\lambda_2))(\Psi_{\tilde{\phi}}^{\mathrm{tr}})^{(-1)}\right) & \delta_{1,\mathbf{z}_{2k+1}}^{M_{\tilde{\rho}}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(\tilde{F}_\tau(\lambda_1), \tilde{F}_\tau(\lambda_2))(\Psi_{\tilde{\phi}}^{\mathrm{tr}})^{(-1)}\right) \\ \delta_{1,\mathbf{z}_{2k}}^{M_{\tilde{\rho}}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(-\lambda_1, -\lambda_2)(\Psi_{\tilde{\phi}}^{\mathrm{tr}})^{(-1)}\right) & \delta_{1,\mathbf{z}_{2k+1}}^{M_{\tilde{\rho}}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(-\lambda_1, -\lambda_2)(\Psi_{\tilde{\phi}}^{\mathrm{tr}})^{(-1)}\right) \end{bmatrix}.$$

where $\Psi_{\tilde{\phi}}$ is the matrix defined as in (1.5) with respect to $\tilde{\phi}$ and $\tilde{F} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the unique entire function satisfying

$$\tilde{F}_\tau(\theta z) - \theta \tilde{F}_\tau(z) = \exp_{\tilde{\phi}}(z)^q$$

for all $z \in \mathbb{C}_\infty$.

1.5. **Outline of the paper.** In §2, we introduce Anderson t-modules, Anderson t-motives, dual t-motives and the formulas obtained by the second author in [19] for the logarithms of Anderson t-modules. In §3, after discussing the tensor construction for Drinfeld modules by using our results in §3.2, we provide a proof for Theorem 1.1 as well as Corollary 1.2 which will be restated as Theorem 3.9 and Corollary 3.10 respectively. Finally, in §4, we discuss the structure of a certain motivic map (see §4.2) and then, using our ideas established in §3.2, we prove Theorem 1.5 (restated as Theorem 4.4 later).

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2. Preliminaries and background

Our goal in this section is to review the notion of Anderson t-modules, Anderson t-motives and dual t-motives as well as a formula for the logarithms of Anderson t-modules derived in [19]. The main references for our exposition are [1], [3], [19] and [21, §2.3–2.5].

2.1. **Anderson t-modules.** For any matrix $\mathfrak{M} = (m_{\mu\nu}) \in \mathrm{Mat}_{d_1\times d_2}(\mathbb{C}_\infty)$ and $i \in \mathbb{Z}$, we define *the i-th twist of* $\mathfrak{M}$ by $\mathfrak{M}^{(i)} := (m_{\mu\nu}^{q^i})$. Furthermore, we let

$$\mathrm{Mat}_{d_1\times d_2}(\mathbb{C}_\infty)[[\tau]] := \left\{ \sum_{i\geq 0} \mathfrak{M}_i \tau^i \;\middle|\; \mathfrak{M}_i \in \mathrm{Mat}_{d_1\times d_2}(\mathbb{C}_\infty) \right\}$$

and when $d = d_1 = d_2$, we define the non-commutative power series ring $\mathrm{Mat}_d(\mathbb{C}_\infty)[[\tau]]$ subject to the condition

$$\tau\mathfrak{M} = \mathfrak{M}^{(1)}\tau.$$

We also let $\mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$ be the subring of $\mathrm{Mat}_d(\mathbb{C}_\infty)[[\tau]]$ consisting of polynomials in τ.

Definition 2.1. (i) *An Anderson t-module G of dimension $d \geq 1$* is a tuple $(\mathbb{G}^d_{a/\mathbb{C}_\infty}, \phi)$ consisting of the d-dimensional additive algebraic group $\mathbb{G}^d_{a/\mathbb{C}_\infty}$ defined over $\mathbb{C}_\infty$ and an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$ given by

$$\phi_\theta := d[\theta] + A_1\tau + \cdots + A_\ell\tau^\ell \tag{2.1}$$

so that $\ell \in \mathbb{Z}_{\geq 1}$ and $d[\theta] := \theta\,\mathrm{Id}_d + \mathfrak{N}$ for some nilpotent matrix $\mathfrak{N}$.

(ii) The morphisms between Anderson t-modules $G_1 = (\mathbb{G}^{d_1}_{a/\mathbb{C}_\infty}, \phi)$ and $G_2 = (\mathbb{G}^{d_2}_{a/\mathbb{C}_\infty}, \psi)$ are defined to be the morphisms $g : \mathbb{G}^{d_1}_{a/\mathbb{C}_\infty} \to \mathbb{G}^{d_2}_{a/\mathbb{C}_\infty}$ of algebraic groups satisfying $g\phi_\theta = \psi_\theta g$.

We define $G(\mathbb{C}_\infty) := \mathrm{Mat}_{d\times 1}(\mathbb{C}_\infty)$ equipped with the A-module structure given by

$$\theta \cdot \mathbf{z} = \phi_\theta(z) := d[\theta]\mathbf{z} + A_1\mathbf{z}^{(1)} + \cdots + A_\ell\mathbf{z}^{(\ell)}, \quad \mathbf{z} \in \mathrm{Mat}_{d\times 1}(\mathbb{C}_\infty).$$

We also consider $\mathrm{Lie}(G)(\mathbb{C}_\infty) := \mathrm{Mat}_{d\times 1}(\mathbb{C}_\infty)$ which is equipped with the A-module action defined by

$$\theta \cdot \mathbf{z} := d[\theta]\mathbf{z}.$$

It is known, due to Anderson [1, §2], that there exists a unique infinite series $\mathrm{Exp}_G := \sum_{i\geq 0} Q_i\tau^i \in \mathrm{Mat}_d(\mathbb{C}_\infty)[[\tau]]$ satisfying $Q_0 = \mathrm{Id}_d$ and

$$\mathrm{Exp}_G\, d[\theta] = \phi_\theta\, \mathrm{Exp}_G\,.$$

Moreover, it induces an entire function $\mathrm{Exp}_G : \mathrm{Lie}(G)(\mathbb{C}_\infty) \to G(\mathbb{C}_\infty)$ given by

$$\mathrm{Exp}_G(\mathbf{z}) := \sum_{i=0}^{\infty} Q_i\mathbf{z}^{(i)}. \tag{2.2}$$

We let $\mathrm{Log}_G := \sum_{i\geq 0} P_i\tau^i \in \mathrm{Mat}_d(\mathbb{C}_\infty)[[\tau]]$ be the formal inverse of $\mathrm{Exp}_G \in \mathrm{Mat}_d(\mathbb{C}_\infty)[[\tau]]$. On a certain subset $\mathcal{D}$ of $G(\mathbb{C}_\infty)$, Log_G induces a vector valued function $\mathrm{Log}_G : \mathcal{D} \to \mathrm{Lie}(G)(\mathbb{C}_\infty)$ defined by

$$\mathrm{Log}_G(\mathbf{z}) := \sum_{i=0}^{\infty} P_i\mathbf{z}^{(i)}.$$

For further details on the exponential and the logarithm function, we refer the reader to [21, §2.5.1].

In what follows, we provide some examples of Anderson t-modules.

Example 2.2. (i) Any Drinfeld module ϕ is an Anderson t-module $(\mathbb{G}_{a/\mathbb{C}_\infty}, \phi)$ of dimension one.

(ii) Let $C : A \to \mathbb{C}_\infty[\tau]$ be the Carlitz module and $k \in \mathbb{Z}_{\geq 1}$. We consider *the k-th tensor power of the Carlitz module $C^{\otimes k} := (\mathbb{G}^k_{a/\mathbb{C}_\infty}, \psi)$* where $\psi : A \to \mathrm{Mat}_k(\mathbb{C}_\infty)[\tau]$ is given by (see [2])

$$\psi_\theta := \begin{pmatrix} \theta & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \theta \end{pmatrix} + \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \tau.$$

(iii) Let ϕ be a Drinfeld module of rank r given as in (1.4). We define *the tensor product ϕ and the k-th tensor power of the Carlitz module* as $\phi \otimes C^{\otimes k} := (\mathbb{G}_{a/\mathbb{C}_\infty}^{rk+1}, \rho)$ where

$$\rho : A \to \mathrm{Mat}_{rk+1}(\mathbb{C}_\infty)[\tau]$$

is given by

$$\rho_\theta := \begin{pmatrix} \theta & \cdots & 0 & \overbrace{1 \quad 0 \quad \cdots \quad 0}^{rk+1-r} \\ & \ddots & & \ddots \quad \ddots \qquad \vdots \\ & & \ddots & \qquad \ddots \quad \ddots \quad 0 \\ & & & \theta \quad \cdots \quad 0 \quad 1 \\ & & & \qquad \theta \quad \cdots \quad 0 \\ & & & \qquad\qquad \ddots \quad \vdots \\ & & & \qquad\qquad\qquad \theta \end{pmatrix} + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & & \vdots \\ & & 1 & \ddots & & & \vdots \\ k_1 & \cdots & \cdots & k_r & 0 & \cdots & 0 \end{pmatrix} \tau.$$

For more details on the tensor product of Drinfeld modules of arbitrary rank with the tensor powers of the Carlitz module, we refer the reader to [11, 12, 20, 22, 23].

Consider $\Lambda_G := \mathrm{Ker}(\mathrm{Exp}_G) \subset \mathrm{Lie}(G)(\mathbb{C}_\infty)$. By the work of Anderson [1, Lem. 2.4.1], we know that, under a certain condition on G, Λ_G forms a finitely generated and discrete A-module. We call any non-zero element of Λ_G *a period of G*. Indeed, by [1, Thm.4], when G is the Anderson t-module either in Example 2.2(i) or in 2.2(iii), Λ_G is free of rank r as an A-module. Moreover, if G is the k-th tensor power of the Carlitz module, then Λ_G is free of rank one.

2.2. **Anderson generating functions.** For any $c \in \mathbb{C}_\infty^\times$, we define the Tate algebra

$$\mathbb{T}_c := \left\{ g = \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |ca_i| \to 0 \quad \text{as } i \to \infty \right\}.$$

It is equipped with the multiplicative norm $\|\cdot\|_c$ given by

$$\|g\|_c := \max\{|c^i||a_i| \;\mid\; i \geq 0\}.$$

To ease the notation, we denote $\mathbb{T}_1$ by $\mathbb{T}$ and $\|\cdot\|_1$ by $\|\cdot\|$.

Let ϕ be a Drinfeld module of rank r given as in (1.4). In what follows, we define a certain element in $\mathbb{T}$ which will be later useful to describe a particular property of Anderson t-motives of Drinfeld modules. For any $z \in \mathbb{C}_\infty$, *the Anderson generating function $s_\phi(z;t)$* is given by

$$s_\phi(z;t) := \sum_{i=0}^{\infty} \exp_\phi\left(\frac{z}{\theta^{i+1}}\right) t^i \in \mathbb{T}.$$

Let t be a variable over $\mathbb{C}_\infty$. For any $f = \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[[t]]$ and $j \in \mathbb{Z}$, we set $f^{(j)} := \sum_{i\geq 0} a_i^{q^j} t^i \in \mathbb{C}_\infty[[t]]$. We now state a fundamental property of Anderson generating functions due to Pellarin.

Proposition 2.3 (Pellarin,[28, §4.2])**.** *Let* $\lambda \in \mathrm{Ker}(\exp_\phi)$. *Then*

$$(t-\theta)s_\phi(\lambda;t) = k_1 s_\phi(\lambda;t)^{(1)} + \cdots + k_r s_\phi(\lambda;t)^{(r)}.$$

2.3. **Anderson t-motives.** We define the non-commutative ring $\mathbb{C}_\infty[t,\tau] := \mathbb{C}_\infty[t][\tau]$ with respect to the condition $\tau f = f^{(1)}\tau$ where $f = \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[t]$.

Definition 2.4. (i) *An Anderson t-motive M* is a left $\mathbb{C}_\infty[t,\tau]$-module which is free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\tau]$ (possibly of different ranks) such that there exists a non-negative integer μ satisfying

$$(t-\theta)^\mu M \subset \tau M.$$

(ii) Morphisms of Anderson t-motives are given by morphisms of left $\mathbb{C}_\infty[t,\tau]$-modules.

(iii) Let M_1 and M_2 be two Anderson t-modules. *The tensor product of M_1 and M_2* is the Anderson t-motive $M_1 \otimes_{\mathbb{C}_\infty[t]} M_2$ where τ acts diagonally.

Let $\mathbf{m} \in \mathrm{Mat}_{d\times 1}(M)$ be a $\mathbb{C}_\infty[t]$-basis for M and $\mathfrak{Q} \in \mathrm{GL}_r(\mathbb{T})$ be such that

$$\tau \cdot \mathbf{m} = \mathfrak{Q}\mathbf{m}.$$

We call M *rigid analytically trivial* if there exists $\Upsilon \in \mathrm{GL}_r(\mathbb{T})$ such that

$$\Upsilon^{(1)} = \mathfrak{Q}\Upsilon.$$

We also call Υ *a rigid analytic trivialization of M*.

Due to Anderson [1, Thm. 1], there exists an anti-equivalence of categories of Anderson t-modules and Anderson t-motives. We briefly describe this functor now. Given an Anderson t-module $G = (\mathbb{G}^d_{a/\mathbb{C}_\infty}, \phi)$, there exists a unique Anderson t-motive M_G given by the group of morphisms $\mathbb{G}^d_{a/\mathbb{C}_\infty} \to \mathbb{G}_{a/\mathbb{C}_\infty}$ of $\mathbb{C}_\infty$-algebraic groups. This group of morphisms is naturally a $\mathbb{C}_\infty[\tau]$-module and is isomorphic to $\mathrm{Mat}_{1\times d}(\mathbb{C}_\infty[\tau])$ as $\mathbb{C}_\infty[\tau]$-modules. It is equipped with a $\mathbb{C}_\infty[t,\tau]$-module structure given by

$$ct^i \cdot m := c \circ m \circ \phi_{\theta^i}, \quad m \in M_G.$$

In what follows, we describe the Anderson t-motives corresponding to the Anderson t-modules given in Example 2.2.

2.3.1. *Anderson t-motive of Drinfeld modules.* Let ϕ be the Drinfeld module of rank r given as in (1.4). We define $M_\phi := \mathbb{C}_\infty[\tau]$ and equip it with the $\mathbb{C}_\infty[t]$-module structure given by

$$ct^i \cdot \mathfrak{a}\tau^j := c\mathfrak{a}\tau^j\phi_{\theta^i}, \quad \mathfrak{a}, c \in \mathbb{C}_\infty.$$

One can see that M_ϕ forms a left $\mathbb{C}_\infty[t,\tau]$-module, satisfying $(t-\theta)M_\phi \subset \tau M_\phi$, which is free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\tau]$. We define the matrix

$$\Theta := \begin{pmatrix} & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ \frac{t-\theta}{k_r} & -\frac{k_1}{k_r} & \dots & \dots & -\frac{k_{r-1}}{k_r} \end{pmatrix} \in \mathrm{GL}_r(\mathbb{T}).$$

We choose $m := [m_1, \dots, m_r]^{\mathrm{tr}} \in \mathrm{Mat}_{r\times 1}(M_\phi)$ to be a $\mathbb{C}_\infty[t]$-basis for M_ϕ so that

$$\tau \cdot m = \Theta m.$$

Observe that $\{m_1\}$ forms a $\mathbb{C}_\infty[\tau]$-basis for M_ϕ.

Let $\{\lambda_1, \dots, \lambda_r\}$ be an A-basis for the period lattice Λ_ϕ. For any $i \in \{1, \dots, r\}$, we define the Anderson generating function $f_i := s_\phi(\lambda_i; t)$. Consider the matrix

$$\Upsilon := \begin{pmatrix} f_1 & \cdots & \cdots & f_r \\ f_1^{(1)} & \cdots & \cdots & f_r^{(1)} \\ \vdots & & & \vdots \\ f_1^{(r-1)} & \cdots & \cdots & f_r^{(r-1)} \end{pmatrix} \in \mathrm{Mat}_{r\times r}(\mathbb{T}). \tag{2.3}$$

By [28, §4.2], we know that $\Upsilon \in \mathrm{GL}_r(\mathbb{T})$ and moreover it satisfies

$$\Upsilon^{(1)} = \Theta\Upsilon.$$

Hence M_ϕ is rigid analytically trivial.

For later use, we also consider another $\mathbb{C}_\infty[t]$-basis

$$\mathfrak{c}^\phi := [\mathfrak{c}_1^\phi, \dots, \mathfrak{c}_r^\phi]^{\mathrm{tr}} := [k_1^{(-1)}m_1 + k_2^{(-1)}m_2 + \cdots + k_r^{(-1)}m_r, k_2^{(-2)}m_1 + k_2^{(-2)}m_2 + \cdots + k_r^{(-2)}m_{r-1}$$
$$, \dots, k_{r-1}^{(1-r)}m_1 + k_r^{(1-r)}m_2, k_r^{(-r)}m_1]^{\mathrm{tr}} \in \mathrm{Mat}_{r\times 1}(M_\phi)$$

and note that

$$\tau \cdot \mathfrak{c}^\phi = \Phi^{\mathrm{tr}}\mathfrak{c}^\phi \tag{2.4}$$

where

$$\Phi := \begin{pmatrix} & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ \frac{t-\theta}{k_r^{(-r)}} & -\frac{k_1^{(-1)}}{k_r^{(-r)}} & \cdots & \cdots & -\frac{k_{r-1}^{(-(r-1))}}{k_r^{(-r)}} \end{pmatrix} \in \mathrm{GL}_r(\mathbb{T}). \tag{2.5}$$

2.3.2. *Anderson t-motive of the tensor powers of the Carlitz module.* Let $k \in \mathbb{Z}_{\geq 1}$. We consider the left $\mathbb{C}_\infty[t,\tau]$-module

$$M_{C^{\otimes k}} := M_C \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} M_C = \mathbb{C}_\infty[\tau] \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\tau]$$

so that τ acts diagonally. Let m_1 be a basis for M_C as a $\mathbb{C}_\infty[t]$-module. Then $m := m_1 \otimes \cdots \otimes m_1$ is a $\mathbb{C}_\infty[t]$-basis for $M_{C^{\otimes k}}$ so that

$$\tau m = (t-\theta)^k m.$$

Moreover, the set $\{m, (t-\theta)m, \dots, (t-\theta)^{k-1}m\}$ forms a $\mathbb{C}_\infty[\tau]$-basis for $M_{C^{\otimes k}}$ and hence it is of dimension k over $\mathbb{C}_\infty[\tau]$. In particular, $M_{C^{\otimes k}} \cong \mathrm{Mat}_{1\times k}(\mathbb{C}_\infty)[\tau]$ as $\mathbb{C}_\infty[\tau]$-modules.

We now fix a $(q-1)$-st root of $-\theta$ and define *the Anderson-Thakur element* ω_C by

$$\omega_C := (-\theta)^{1/(q-1)} \prod_{j=0}^{\infty} \left(1 - \frac{t}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}. \tag{2.6}$$

One can observe that $(\omega_C^k)^{(1)} = (t-\theta)^k \omega_C^k$ and hence $M_{C^{\otimes k}}$ is rigid analytically trivial.

2.3.3. *Anderson t-motive of the tensor product of Drinfeld modules with the tensor powers of the Carlitz module.* We consider the left $\mathbb{C}_\infty[t,\tau]$-module

$$M_{\phi\otimes C^{\otimes k}} := M_\phi \otimes_{\mathbb{C}_\infty[t]} \otimes_{\mathbb{C}_\infty[t]} M_{C^{\otimes k}} = \mathbb{C}_\infty[\tau] \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\tau]$$

so that τ acts diagonally. Observe that $(t-\theta)^{k+1} M_{\phi\otimes C^{\otimes k}} \subset \tau M_{\phi\otimes C^{\otimes k}}$. Moreover, $M_{\phi\otimes C^{\otimes k}}$ is free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\tau]$. We consider a $\mathbb{C}_\infty[t]$-basis $\mathfrak{m}$ for $M_{\phi\otimes C^{\otimes k}}$ given by $\mathfrak{m} := [\mathfrak{m}_1, \dots, \mathfrak{m}_r]^{\mathrm{tr}} := [m_1 \otimes m, \dots, m_r \otimes m]^{\mathrm{tr}}$, where m_i are the basis elements from §2.3.1 and m is from §2.3.2. Note that

$$\tau \cdot \mathfrak{m} = (t-\theta)^k \Theta \mathfrak{m}.$$

Let $\tilde{\Upsilon} := \omega_C^k \Upsilon \in \mathrm{GL}_r(\mathbb{T})$. Then it is easy to see that $\tilde{\Upsilon}^{(1)} = (t-\theta)^k \Theta \tilde{\Upsilon}$ and hence $M_{\phi\otimes C^{\otimes k}}$ is rigid analytically trivial.

We further define another $\mathbb{C}_\infty[t]$-basis

$$\mathfrak{c} := [\mathfrak{c}_1, \dots, \mathfrak{c}_r]^{\mathrm{tr}} := [\mathfrak{c}_1^\phi \otimes m, \dots, \mathfrak{c}_r^\phi \otimes m]^{\mathrm{tr}}.$$

Moreover, we note that

$$\tau \cdot \mathfrak{c} = (t-\theta)^k \Phi^{\mathrm{tr}} \mathfrak{c}. \tag{2.7}$$

Lastly, we define a $\mathbb{C}_\infty[\tau]$-basis

$$\begin{aligned}\mathfrak{g} := [g_1, \dots, g_{rk+1}]^{\mathrm{tr}} := [\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r, (t-\theta)\mathfrak{m}_1, (t-\theta)\mathfrak{m}_2, \dots, (t-\theta)\mathfrak{m}_r, \dots,\\ (t-\theta)^{k-1}\mathfrak{m}_1, (t-\theta)^{k-1}\mathfrak{m}_2, \dots, (t-\theta)^{k-1}\mathfrak{m}_r, (t-\theta)^k \mathfrak{m}_1]^{\mathrm{tr}}.\end{aligned}$$

One now sees that

$$t \cdot \mathfrak{g} = \rho_\theta \mathfrak{g}$$

where ρ_θ is given as in Example 2.2(iii).

2.4. **Dual t-motives.** We define $\mathbb{C}_\infty[t,\sigma] := \mathbb{C}_\infty[t][\sigma]$ to be the ring of polynomials of σ with coefficients in $\mathbb{C}_\infty[t]$ subject to the condition

$$\sigma f = f^{(-1)}\sigma, \quad f \in \mathbb{C}_\infty[t].$$

We further define the $*$-operation on elements in $\mathbb{C}_\infty[\tau]$ by

$$g^* := \sum_{i\geq 0} c_i^{(-i)} \sigma^i, \quad g = \sum_{i\geq 0} c_i \tau^i.$$

We extend this operation to elements in $\mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$ by defining $\mathfrak{M}^* := ((m_{\mu\nu}^*))^{\mathrm{tr}}$ for any $\mathfrak{M} = (m_{\mu\nu}) \in \mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$.

Definition 2.5. (i) *A dual t-motive* N is a left $\mathbb{C}_\infty[t,\sigma]$-module which is free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\sigma]$ such that there exists $\ell \in \mathbb{Z}_{\geq 0}$ satisfying

$$(t-\theta)^\ell N \subset \sigma N.$$

(ii) The morphisms of dual t-motives are given by left $\mathbb{C}_\infty[t,\sigma]$-module homomorphisms.
(iii) The tensor product of dual t-motives N_1 and N_2 is defined to be the left $\mathbb{C}_\infty[t,\sigma]$-module $N_1 \otimes N_2 := N_1 \otimes_{\mathbb{C}_\infty[t]} N_2$ where σ acts diagonally.

Let $\mathbf{n} \in \mathrm{Mat}_{r\times 1}(N)$ be a $\mathbb{C}_\infty[t]$-basis for N and $\mathfrak{Z} \in \mathrm{GL}_r(\mathbb{T})$ be such that

$$\sigma \cdot \mathbf{n} = \mathfrak{Z}\mathbf{n}.$$

We say that M is *rigid analytically trivial* if there exists $\Psi \in \mathrm{GL}_r(\mathbb{T})$ such that

$$\Psi^{(-1)} = \mathfrak{Z}\Psi.$$

We further call Ψ *a rigid analytic trivialization of* N.

In an unpublished work of Anderson (see also [21, §2.5]), he showed the equivalence between the category of Anderson t-modules and the category of dual t-motives. Similar to the case of Anderson t-motives, this result yields that for any Anderson t-module $G = (\mathbb{G}^d_{a/\mathbb{C}_\infty}, \phi)$, there exists a unique Anderson t-motive $N_G := \mathrm{Mat}_{1\times d}(\mathbb{C}_\infty[\sigma])$ equipped with a $\mathbb{C}_\infty[t,\tau]$-module structure given by

$$ct^i \cdot n := cn\phi^*_{\theta^i}, \quad n \in N_G.$$

In what follows, we describe the dual t-motives corresponding to Anderson t-modules given in Example 2.2.

2.4.1. *Dual t-motive of Drinfeld modules.* Let ϕ be a Drinfeld module given as in (1.4). We define N_ϕ to be the $\mathbb{C}_\infty[\sigma]$-module $\mathbb{C}_\infty[\sigma]$ equipped with the $\mathbb{C}_\infty[t]$-module action given by

$$ct^i \cdot \mathfrak{a}\sigma^j := c\mathfrak{a}\sigma^j\phi^*_{\theta^i}, \quad \mathfrak{a}, c \in \mathbb{C}_\infty.$$

It is free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\sigma]$ satisfying $(t-\theta)N_\phi \subset \sigma N_\phi$. We choose a $\mathbb{C}_\infty[t]$-basis $\mathfrak{d}^\phi := [\mathfrak{d}^\phi_1, \ldots, \mathfrak{d}^\phi_r]^{\mathrm{tr}} \in \mathrm{Mat}_{r\times 1}(N_\phi)$ for N_ϕ satisfying

$$\sigma \cdot \mathfrak{d}^\phi = \Phi\mathfrak{d}^\phi. \tag{2.8}$$

Moreover, $\{\mathfrak{d}^\phi_1\}$ forms a $\mathbb{C}_\infty[\sigma]$-basis for N_ϕ.

Following the notation in [4, §3.3], set

$$V := \begin{pmatrix} k_1 & k_2^{(-1)} & k_3^{(-2)} & \ldots & k_r^{(1-r)} \\ \vdots & \vdots & \vdots & \iddots & \\ \vdots & \vdots & k_r^{(-2)} & & \\ \vdots & k_r^{(-1)} & & & \\ k_r & & & & \end{pmatrix} \in \mathrm{GL}_r(\mathbb{C}_\infty) \tag{2.9}$$

and consider the matrix $\Psi := V^{-1}((\Upsilon^{(1)})^{\mathrm{tr}})^{-1} \in \mathrm{GL}_r(\mathbb{T})$. Then, by Proposition 2.3, we obtain $(\Upsilon^{(1)})^{\mathrm{tr}} = \Upsilon^{\mathrm{tr}}\Theta^{\mathrm{tr}}$. Moreover, one has

$$V^{(-1)}\Phi = \Theta^{\mathrm{tr}}V. \tag{2.10}$$

Thus, we have $\Psi^{(-1)} = \Phi\Psi$ and hence N_ϕ is rigid analytically trivial.

2.4.2. *Dual t-motive of the tensor powers of the Carlitz module.* Set

$$N_{C^{\otimes k}} := N_C \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} N_C = \mathbb{C}_\infty[\sigma] \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\sigma]$$

and equip it with the diagonal σ-action. Thus $N_{C^{\otimes k}}$ is a left $\mathbb{C}_\infty[t,\sigma]$-module. One can choose $\mathbb{C}_\infty[t]$-basis $n := \mathfrak{d}^C_1 \otimes \cdots \mathfrak{d}^C_1$ for $N_{C^{\otimes k}}$ so that

$$\sigma n = (t-\theta)^k n.$$

On the other hand, the set $\{n,(t-\theta)n,\dots,(t-\theta)^{k-1}n\}$ forms a $\mathbb{C}_\infty[\sigma]$-basis for $N_{C^{\otimes k}}$ and hence $N_{C^{\otimes k}}\cong \mathrm{Mat}_{1\times k}(\mathbb{C}_\infty)[\sigma]$. Consider the element $\Omega:=(\omega_C^{(1)})^{-1}$. It can be easily seen that $(\Omega^k)^{(-1)}=(t-\theta)^k\Omega$ and thus implies the rigid analytic triviality of $N_{C^{\otimes k}}$.

2.4.3. *Dual t-motive of the tensor product of Drinfeld modules with the tensor powers of the Carlitz module.* We set

$$N_{\phi\otimes C^{\otimes k}}:=N_\phi\otimes_{\mathbb{C}_\infty[t]}\otimes_{\mathbb{C}_\infty[t]}N_{C^{\otimes k}}=\mathbb{C}_\infty[\sigma]\otimes_{\mathbb{C}_\infty[t]}\cdots\otimes_{\mathbb{C}_\infty[t]}\mathbb{C}_\infty[\sigma]$$

and equip it with the diagonal σ-action. It can be seen that $N_{\phi\otimes C^{\otimes k}}$ forms a left $\mathbb{C}_\infty[t,\sigma]$-module and it is also a free and finitely generated over $\mathbb{C}_\infty[t]$ and $\mathbb{C}_\infty[\sigma]$. Moreover,

$$(t-\theta)^{k+1}N_{\phi\otimes C^{\otimes k}}\subset\sigma N_{\phi\otimes C^{\otimes k}}.$$

We consider the $\mathbb{C}_\infty[t]$-basis for $N_{\phi\otimes C^{\otimes k}}$ given by $\mathfrak{d}:=[\mathfrak{d}_1,\dots,\mathfrak{d}_r]:=[\mathfrak{d}_1^\phi\otimes n,\dots,\mathfrak{d}_r^\phi\otimes n]^{\mathrm{tr}}$ for $N_{\phi\otimes C^{\otimes k}}$. Note that

$$\sigma\cdot\mathfrak{d}=(t-\theta)^k\Phi\mathfrak{d}. \tag{2.11}$$

To see that $N_{\phi\otimes C^{\otimes k}}$ is rigid analytically trivial, we define the matrix $\tilde{\Psi}:=\Omega^k\Psi\in\mathrm{GL}_r(\mathbb{T})$ and observe that $\tilde{\Psi}^{(-1)}=(t-\theta)^k\Phi\tilde{\Psi}$.

We set $\tilde{\mathfrak{h}}_r:=1$ and for each $i\in\{1,\dots,r-1\}$, we let

$$\tilde{\mathfrak{h}}_i:=k_{i+1}^{(-1)}\mathfrak{d}_2^\phi+k_{i+1}^{(-2)}\mathfrak{d}_3^\phi+\cdots+k_r^{(-(r-i))}\mathfrak{d}_{r-i+1}^\phi.$$

Moreover, we consider the $\mathbb{C}_\infty[\sigma]$-basis for $N_{\phi\otimes C^{\otimes k}}$ defined by

$$\begin{aligned}\mathfrak{h}:=\{h_1,\dots,h_{rk+1}\}:=[(t-\theta)^k\tilde{\mathfrak{h}}_r\otimes n,(t-\theta)^{k-1}\tilde{\mathfrak{h}}_1\otimes n,\dots,(t-\theta)^{k-1}\tilde{\mathfrak{h}}_r\otimes n,\\ (t-\theta)\tilde{\mathfrak{h}}_1\otimes n,\dots,(t-\theta)\tilde{\mathfrak{h}}_r\otimes n,\tilde{\mathfrak{h}}_1\otimes n,\dots,\tilde{\mathfrak{h}}_r\otimes n]^{\mathrm{tr}}\end{aligned}$$

and observe that

$$t\cdot\mathfrak{h}=\rho_\theta^*\mathfrak{h}.$$

2.5. **Logarithms of Anderson t-modules.** In this section we review the background and some of the main theorems of [19] which gives a factorization theorem for the logarithm function of a t-module. We state our first lemma which describes a particular choice of bases for Anderson t-motives and dual t-motives.

Lemma 2.6. [19, Lem. 2.10] *Let G be an Anderson t-module and M_G (N_G resp.) be the corresponding Anderson t-motive (dual t-motive resp.).*

(i) There exists a $\mathbb{C}_\infty[t]$-basis $\{c_1,\dots,c_r\}$ ($\{d_1,\dots,d_r\}$ resp.) for M_G (N_G resp.) such that

$$\tau[c_1,\dots,c_r]^{\mathrm{tr}}=\mathfrak{Q}[c_1,\dots,c_r]^{\mathrm{tr}}$$

and

$$\sigma[d_1,\dots,d_r]^{\mathrm{tr}}=\mathfrak{Q}^{\mathrm{tr}}[d_1,\dots,d_r]^{\mathrm{tr}}$$

for some $\mathfrak{Q}\in\mathrm{GL}_r(\mathbb{T})$.

(ii) There exists a $\mathbb{C}_\infty[\tau]$-basis $\mathcal{G}:=[g_1,\dots,g_d]^{\mathrm{tr}}$ for N_G and a $\mathbb{C}_\infty[\sigma]$-basis $\mathcal{H}:=[h_1,\dots,h_d]^{\mathrm{tr}}$ for N_G such that

$$t\cdot\mathcal{G}=\mathfrak{V}\mathcal{G}$$

and

$$t\cdot\mathcal{H}=\mathfrak{V}^*\mathcal{H}$$

for some $\mathfrak{V}\in\mathrm{Mat}_{r\times r}(\mathbb{C}_\infty)[\tau]$.

Let $N \cong \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$ be a dual t-motive for some $r \in \mathbb{Z}_{\geq 1}$ and let $h = \{h_1, \dots, h_d\}$ be a $\mathbb{C}_\infty[\sigma]$-basis. Any $n \in N$ can be written as

$$n = \sum_{i=1}^{d} \left(\sum_{j=0}^{m_i} \alpha_{i,j} \sigma^j \right) h_i$$

for some $\alpha_{i,j} \in \mathbb{C}_\infty$ and $m_i \in \mathbb{Z}_{\geq 0}$. Then we define the map $\delta_0^N : N \to \mathbb{C}_\infty^d$ by

$$\delta_0^N(n) := \begin{pmatrix} \alpha_{0,1} \\ \vdots \\ \alpha_{0,d} \end{pmatrix}.$$

Now let $\{d_1, \dots, d_r\}$ be a $\mathbb{C}_\infty[t]$-basis for N as in Lemma 2.6(i). We consider

$$\tilde{N} := \oplus_{i=1}^{r} \mathbb{C}_\infty(t) d_i \cong \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty(t))$$

and for any $\tilde{n} \in \tilde{N}$, write $\tilde{n} = \sum_{i=1}^{d} a_i d_i$ for some $a_i \in \mathbb{C}_\infty(t)$. We define

$$\sigma^{-1}(\tilde{n}) := (\mathfrak{Q}^{-1})^{(1)} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}^{(1)}. \tag{2.12}$$

Moreover, we consider $N_\theta := N \otimes_{\mathbb{C}_\infty[t]} \mathbb{T}_\theta$. By [19, Prop. 2.8], there exists an extension of δ_0^N-map to $\delta_0^N : N_\theta \to \mathbb{C}_\infty^d$.

Remark 2.7. If $n \in N$, then one can write $n = \sum_{i=0}^{r} a_i c_i$ for some $a_i \in \mathbb{C}_\infty[t]$. Since $\det(\mathfrak{Q}) = (t-\theta)^\ell \mathfrak{a}$ for some $\ell \in \mathbb{Z}_{\geq 1}$ and $\mathfrak{a} \in \mathbb{C}_\infty^\times$, $\sigma^{-1}(n) \in \tilde{N}$ has only a pole at $t = \theta^q$. Thus, one can evaluate $\sigma^{-j}(n)$ at δ_0^N for any integer j. We refer the reader to [19, Prop. 2.18] for details on this extensions of δ_0^N.

We define another crucial map for our purposes. Let $M \cong \mathrm{Mat}_{1\times d}(\mathbb{C}_\infty)[\tau]$ be an Anderson t-motive and fix $\mathbf{z} = (z_1, \dots, z_d)^{\mathrm{tr}} \in \mathbb{C}_\infty^d$. We define $\delta_{1,\mathbf{z}}^M : M \to \mathbb{C}_\infty$ by

$$\delta_{1,\mathbf{z}}^M(m) := m\mathbf{z} := m_1(z_1) + \cdots + m_d(z_d), \quad m = [m_1, \dots, m_d] \in \mathrm{Mat}_{1\times d}(\mathbb{C}_\infty)[\tau], \tag{2.13}$$

where we view τ as acting as the q-power Frobenius. We further define $M_{\mathbf{z}}$ to be the set of elements $(\mathfrak{a}_1, \dots, \mathfrak{a}_d)\mathbf{z}$ where, for each $i \in \{1, \dots, d\}$, $\mathfrak{a}_i = \sum_{j=0}^{\infty} a_{i,j}\tau^j \in \mathbb{C}_\infty[[\tau]]$ satisfies $(a_{1,\mu}\tau^\mu, \dots, a_{d,\mu}\tau^\mu)\,\mathbf{z} \to 0$ as $\mu \to \infty$. Then we extend the map $\delta_{1,\mathbf{z}}^M$ to $M_{\mathbf{z}}$ by defining $\delta_{1,\mathbf{z}}^M : M_{\mathbf{z}} \to \mathbb{C}_\infty^d$

$$\delta_{1,\mathbf{z}}^M(\tilde{m}) := \lim_{\mu\to\infty} \delta_{1,\mathbf{z}}([\mathfrak{a}_1^\mu, \dots, \mathfrak{a}_d^\mu])$$

where $\tilde{m} = \left[\sum_{j=0}^{\infty} a_{1,j}\tau^j, \dots, \sum_{j=0}^{\infty} a_{d,j}\tau^j\right]$ and $\mathfrak{a}_i^\mu := \sum_{j=0}^{\mu} a_{i,j}\tau^j$. Again, we refer the reader to [19, Def. 2.19] for full details on this extension.

Example 2.8. Since the map $\delta_{1,\mathbf{z}}^M$ is central to the main formulas of our paper, we give a short example showing how one computes the image of this map, at least in the case of Carlitz (compare this calculation with the framework in [13, §6.5]). Using [19, (5.4)], for

$\mathbf{z} \in \mathbb{C}_\infty$ inside the radius of convergence of the Carlitz logarithm, in $(M_C)_{\mathbf{z}}$, we may write

$$\begin{aligned}-\widetilde{\pi}\Omega &= 1 + \frac{1}{(\theta - \theta^q)}(t-\theta) + \frac{1}{(\theta-\theta^q)(\theta-\theta^{q^2})}(t-\theta)(t-\theta^q) + \dots \\ &= 1 + \frac{1}{\ell_1}\tau(1) + \frac{1}{\ell_2}\tau^2(1) + \dots,\end{aligned}$$

where $1/\ell_i$ is the ith coefficient of the Carlitz logarithm (one can also compute this expansion of $\widetilde{\pi}\Omega$ directly using induction without relying on the machinery of [19]). We then have

$$\begin{aligned}\delta^M_{1,\mathbf{z}}(-\widetilde{\pi}\Omega) &= z + \frac{1}{\ell_1}z^q + \frac{1}{\ell_2}z^{q^2} + \dots \\ &= \log_C(\mathbf{z}),\end{aligned}$$

which is consistent with [19, Cor. 5.7].

Let $G = (\mathbb{G}^d_{a/\mathbb{C}_\infty}, \phi)$ be an Anderson t-module given as in (2.1). For each $j \in \{0, \dots, \ell-1\}$, we set

$$\Theta_{\phi,\tau^{\ell-j}} := A^{(-j)}_{j+1}\tau + \dots + A^{(-j)}_{\ell}\tau^{\ell-j}. \tag{2.14}$$

The following was one of the main theorems of [19] and gives an interpretation of the logarithm function of an Anderson t-module in terms of a limit of evaluations of the motivic maps $\delta^\square_i$ given above. After substituting definitions, this formula becomes an infinite product of matrices (or a finite sum of such terms), hence we call it a factorization of the logarithm.

Theorem 2.9. [19, Cor. 5.7(2)] *Let $\mathbf{z}$ be an element in the domain of convergence of Log_G and let M_G (N_G resp.) be the Anderson t-motive (dual t-motive resp.) corresponding to G. Let $\mathcal{G}$ and $\mathcal{H}$ be the $\mathbb{C}_\infty[\tau]$-basis ($\mathbb{C}_\infty[\sigma]$-basis resp.) of Anderson t-motive (dual t-motive resp.) as in Lemma 2.6(ii) and let $\mathfrak{e}_i \in \mathrm{Mat}_{d\times 1}(\mathbb{F}_q)$ be the i-th unit vector. Then*

$$\mathrm{Log}_G(\mathbf{z}) = \lim_{n\to\infty} \delta^{M_G}_{1,\mathbf{z}}\left((t\,\mathrm{Id}_d - d[\theta])^{-1} \sum_{\mu=1}^{d}\sum_{\nu=0}^{\ell-1} \delta^{N_G}_0(\sigma^{\nu-n}(h_\mu))\tau^n(\mathcal{G}^{\mathrm{tr}}\Theta^{\mathrm{tr}}_{\phi,\tau^{\ell-\nu}}\mathfrak{e}_\mu)\right).$$

2.6. **Tensor construction.** Let $G = (\mathbb{G}^d_{a/\mathbb{C}_\infty}, \phi)$ be an Anderson t-module given as in Definition 2.1. In this subsection, we detail a modified construction of the pairing $G(x,y)$ found in [19] which was used in the proof of Theorem 2.9, which will allow us to more easily analyze the convergence of the quantities described in §3. Recall the bases $\mathcal{G}$ and $\mathcal{H}$ given in Lemma 2.6(ii). For $x \in \mathbb{C}_\infty[t,\sigma]$ and $y \in \mathbb{C}_\infty[t,\tau]$, define

$$G^{\otimes}_n(x,y) := \sum_{i=0}^{n}\sum_{k=1}^{d} \sigma^{-i}(x(h_k)) \otimes_{\mathbb{C}_\infty} \tau^i(y(g_k))) \in \tilde{N}_G \otimes M_G. \tag{2.15}$$

We note that if we apply the map δ^N_0 to the first coordinate of each simple tensor in (2.15), then the resulting sum is in $\mathbb{C}^d_\infty \otimes M \cong M^d$ and we recover $G_n(x,y)$ of [19, Def. 5.1]. In fact, since δ^N_0 is $\mathbb{C}_\infty$-linear, this is equivalent to applying $\delta^N_0 \otimes 1$ to the whole sum (2.15).

The pairing $G^{\otimes}_n(x,y)$ has many similar properties to $G_n(x,y)$ (detailed in [19, Prop. 5.4]). We briefly discuss the properties of $G^{\otimes}_n(x,y)$ here. For convenience we recall Definition [19, 5.3]:

Definition 2.10. For each $j \in \{0,\dots,\ell-1\}$, we define

$$\Theta_{\phi,\sigma^{\ell-j}} := A_{j+1}\tau + \dots + A_\ell\tau^{\ell-j},$$

where the A_i are the coefficients of the t-module as in Definition 2.1

Proposition 2.11. *Let $x \in \mathbb{C}_\infty[t,\sigma]$ and $y \in \mathbb{C}_\infty[t,\tau]$.*

(1) For any $c \in \mathbb{C}_\infty$ we have

$$G_n^\otimes(cx, y) = G_n^\otimes(x, cy).$$

(2) We have

$$G_n^\otimes(x, \tau y) - G_n^\otimes(\sigma x, y) = \sum_{k=1}^{d} \sigma^{-n}(xh_k) \otimes \tau^{n+1}(yg_k) - \sigma(xh_k) \otimes yg_k,$$

and more generally for $m < n$ and $c \in \mathbb{C}_\infty$

$$G_n^\otimes(x, c\tau^m y) - G_n^\otimes(c^{(-m)}\sigma^m x, y) = \sum_{k=1}^{d}\sum_{\ell=0}^{m-1} \sigma^{\ell-n}(xh_k) \otimes \tau^n(c^{(-\ell)}\tau^{m-\ell}yg_k) - c^{(\ell-m)}\sigma^{\ell-m}(xh_k) \otimes \tau^\ell(yg_k).$$

(3) We have

$$G_n^\otimes(1, t) - G_n^\otimes(t, 1) = \sum_{k=1}^{d}\sum_{\ell=0}^{r-1} \sigma^{\ell-n}(h_k) \otimes \tau^n\left(\Theta_{\phi,\tau^{r-\ell}} g_k\right) - \Theta^*_{\phi,\sigma^{r-\ell}} h_k \otimes \tau^\ell g_k.$$

Proof. Part (1) is a straightforward calculation. The first part of (2) follows because the two terms being subtracted create a telescoping series, which leaves the highest and lowest degree (in τ) terms after cancellation. The second part follows by using part (1), recalling that $a\tau = \tau a^{(-1)}$, and then repeatedly applying the first part of (2). Part (3) follows by recalling from §2.3.1 and §2.4.1 that t acts as ϕ_θ on M_G and as ϕ^*_θ on N_G, then by applying parts (1) and (2) to the individual terms of ϕ_θ and ϕ^*_θ. □

In what follows, we also obtain a factorization of $G_n^\otimes(1,1)$ similarly to [19, Thm. 5.4(3)]. Let us denote

$$G_n^\otimes := G_n^\otimes(1,1).$$

Proposition 2.12. *We have the following factorization of $G_n^\otimes$:*

$$((1 \otimes t) - (t \otimes 1))G_n^\otimes = \sum_{k=1}^{d}\sum_{\ell=0}^{r-1} \sigma^{\ell-n}(h_k) \otimes \tau^n\left(\mathfrak{e}_k^{\mathrm{tr}}\Theta_{\phi,\tau^{r-\ell}}\mathcal{G}\right) - \mathfrak{e}_k^{\mathrm{tr}}\Theta^*_{\phi,\sigma^{r-\ell}}\mathcal{H} \otimes \tau^\ell g_k.$$

Proof. This proposition follows from Proposition 2.11(3) after noting that $G_n^\otimes(1,t) = (1 \otimes t)G_n^\otimes(1,1)$, and that $G_n^\otimes(t,1) = (t \otimes 1)G_n^\otimes(1,1)$. □

3. Logarithms of Drinfeld modules

Our goal in this section is to interpret the logarithms of Drinfeld modules in terms of formulas investigated in [19]. First, we introduce the notion of fundamental periods. Let ϕ be a Drinfeld module as in (1.4). For any positive integer n, consider *the set $\phi[\theta^n]$ of θ^n-torsion points* which consists of elements $z \in \mathbb{C}_\infty$ such that $\phi_{\theta^n}(z) = 0$. By [24, Thm. 4.4], there exists a positive integer n_ϕ, depending on ϕ, and elements $\xi_1, \dots, \xi_r \in \phi[\theta^{n_\phi}]$ such that $\lambda_i := \theta^{n_\phi}\log_\phi(\xi)$, which we call a *a fundamental period of ϕ* for each $1 \le i \le r$, forms an A-basis for the period lattice Λ_ϕ.

Throughout this section, we fix a Drinfeld module ϕ given by

$$\phi_\theta = \theta + k_1\tau + \cdots + k_r\tau^r \tag{3.1}$$

so that $|k_i| \leq 1$ for each $1 \leq i \leq r-1$ and $k_r \in \mathbb{F}_q^\times$. It this case, by [24, Prop. 3.1], we have $n_\phi = 1$.

Let us further fix a basis $\{\xi_1, \dots, \xi_r\}$ for the $\mathbb{F}_q$-vector space $\phi[\theta]$ ([18, §4.5]).

3.1. The product formula for Υ.

Consider

$$B := \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_r \\ \xi_1^q & \xi_2^q & \dots & \xi_r^q \\ \vdots & \vdots & & \vdots \\ \xi_1^{q^{r-1}} & \xi_2^{q^{r-1}} & \dots & \xi_r^{q^{r-1}} \end{pmatrix} \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty). \tag{3.2}$$

Since $\xi_1, \dots, \xi_r$ are $\mathbb{F}_q$-linearly independent and B is a Moore matrix, the inverse of B exists. Further, we define certain quantities $\mathcal{B}_n(t) \in K(t)$ from [6, (6.4)] and refer the reader to [6, §5,6] for full details. Let

$$\mathcal{B}_n(t) := \sum_{S\in P_r(n)} \prod_{i=1}^r \prod_{j\in S_i} \frac{k_i^{q^j}}{t-\theta^{q^{i+j}}},$$

where the sum is over so-called shadowed partitions (see [6, §5]). We comment that, by [5, (6.5)], if we set $\log_\phi := \sum_{n\geq 0} \beta_n\tau^n$, then we have $\mathcal{B}_n(\theta) = \beta_n$.

Observe, by the Newton polygon method, that each non-zero element in $\phi[\theta]$ has norm $q^{1/(q^r-1)}$. Let $\{\lambda_1, \dots, \lambda_r\}$ be a set of fundamental periods, forming an A-basis for $\mathrm{Ker}(\exp_\phi)$, and hence we define accordingly $\Upsilon \in \mathrm{GL}_r(\mathbb{T})$ given in (2.3). We also set $F := B^{-1}\Theta^{-1}B^{(1)} \in \mathrm{GL}_r(\mathbb{T})$ and $\Pi_n := B\prod_{i=0}^n F^{(n)} \in \mathrm{GL}_r(\mathbb{T})$.

Khaochim and Papanikolas obtained a product formula for Υ as well as a certain expression for the entries of Π_n in terms of $\mathcal{B}_n$ which will later be essential for us to prove our main results.

Theorem 3.1 (Khaochim and Papanikolas, [24, Prop. 4.3, Thm. 4.4])**.** *The following identities hold.*

(i)

$$(\Pi_n)_{ij} = \left(\xi_j - \frac{t}{t-\theta}\sum_{\mu=0}^{n-(i-1)} \mathcal{B}_\mu(t)\xi_j^{q^\mu}\right)^{(i-1)}.$$

In particular, $\lim_{n\to\infty}(\Pi_n)_{ij}^{(1)}$ *exists with respect to the norm* $\|\cdot\|_\theta$ *on* $\mathbb{T}_\theta$.

(ii)

$$\Upsilon = \lim_{n\to\infty}\Pi_n = B\prod_{n=0}^\infty F^{(n)}.$$

Recall the matrices Θ, Φ and V from §2. For each $n \geq 1$, let us set

$$\mathcal{P}_n := ((\Phi^{\mathrm{tr}})^{-1})^{(1)} \cdots ((\Phi^{\mathrm{tr}})^{-1})^{(n)} \in \mathrm{GL}_r(\mathbb{T}).$$

We further define

$$\mathcal{M} := V^{\mathrm{tr}}B^{(1)}. \tag{3.3}$$

By (2.10), we have

(3.4)
$$\begin{aligned}
&\mathcal{P}_{n+1}^{(-1)}\\
&= (\Phi^{\mathrm{tr}})^{-1}((\Phi^{\mathrm{tr}})^{-1})^{(1)}\cdots((\Phi^{\mathrm{tr}})^{-1})^{(n)}\\
&= (V^{(-1)})^{\mathrm{tr}}\Theta^{-1}(\Theta^{-1})^{(1)}\cdots(\Theta^{-1})^{(n)}((V^{-1})^{\mathrm{tr}})^{(n)}\\
&= (V^{(-1)})^{\mathrm{tr}}B(B^{-1}\Theta^{-1}B^{(1)})(B^{-1}\Theta^{-1}B^{(1)})^{(1)}\cdots(B^{-1}\Theta^{-1}B^{(1)})^{(n)}(B^{-1})^{(n+1)}((V^{-1})^{\mathrm{tr}})^{(n)}\\
&= (V^{(-1)})^{\mathrm{tr}}\Pi_n(\mathcal{M}^{-1})^{(n)}.
\end{aligned}$$

Thus, by (3.4), we obtain

$$\mathcal{P}_n = (\mathcal{P}_n^{(-1)})^{(1)} = ((\Phi^{\mathrm{tr}})^{-1}((\Phi^{\mathrm{tr}})^{-1})^{(1)}\cdots((\Phi^{\mathrm{tr}})^{-1})^{(n-1)})^{(1)} = V^{\mathrm{tr}}\Pi_{n-1}^{(1)}(\mathcal{M}^{-1})^{(n)}. \tag{3.5}$$

For each $n \geq 1$, we further set $\Psi_n := V^{-1}((\Pi_n^{(1)})^{\mathrm{tr}})^{-1}$. Recall the invertible matrix Ψ from §2.4.1 and observe, by Theorem 3.1, that

$$\Psi = \lim_{n\to\infty}\Psi_n = V^{-1}((\Upsilon^{(1)})^{\mathrm{tr}})^{-1}. \tag{3.6}$$

By taking the inverse of very left and right hand side of (3.4), we have

$$\mathcal{S}_n := (\Phi^{\mathrm{tr}})^{(n)}(\Phi^{\mathrm{tr}})^{(n-1)}\cdots(\Phi^{\mathrm{tr}}) = \mathcal{M}^{(n)}\Pi_n^{-1}((V^{(-1)})^{-1})^{\mathrm{tr}} = \mathcal{M}^{(n)}(\Psi_n^{(-1)})^{\mathrm{tr}}. \tag{3.7}$$

Thus, for any $\tilde{n} \in \tilde{N}_\phi$ ($\tilde{m} \in M_\phi$ resp.) given by $\tilde{n} = \sum_{i=1}^r a_i\mathfrak{d}_i^\phi$ ($\tilde{m} = \sum_{i=1}^r b_i\mathfrak{c}_i^\phi$ resp.), using (2.4), (2.8) and (2.12), we have

$$\sigma^{-n}(\tilde{n}) = \mathcal{P}_n\begin{bmatrix}a_1\\ \vdots\\ a_r\end{bmatrix}^{(n)} \quad\text{and}\quad \tau^n(\tilde{m}) = [b_1,\dots,b_r]^{(n)}\mathcal{S}_{n-1}. \tag{3.8}$$

3.2. **Tensor construction for Drinfeld modules.** We use the bases described in §2.3.1 and §2.4.1. Let $G = (\mathbb{G}_{a/\mathbb{C}_\infty},\phi)$ where ϕ is as given in (3.1). In this case, the definition of $G_n^\otimes$ from (2.15) reduces to

$$G_n^\otimes = \sum_{i=0}^n \sigma^{-i}(\mathfrak{d}_1^\phi)\otimes\tau^i(\mathfrak{c}_r^\phi) \in \tilde{N}_\phi\otimes M_\phi, \tag{3.9}$$

and Proposition 2.12 reduces to

$$\begin{aligned}
((1\otimes t)-(t\otimes 1))G_n^\otimes &= \sum_{\ell=0}^{r-1}\sigma^{\ell-n}(\mathfrak{d}_1^\phi)\otimes\tau^n\left(\Theta_{\phi,\tau^{r-\ell}}\mathfrak{c}_r^\phi\right) - \Theta^*_{\phi,\sigma^{r-\ell}}(\mathfrak{d}_1^\phi)\otimes\tau^\ell(\mathfrak{c}_r^\phi)\\
&= \sum_{\ell=0}^{r-1}\sigma^{\ell-n}(\mathfrak{d}_1^\phi)\otimes\tau^{n+1}(\mathfrak{c}_{\ell+1}^\phi) - \sum_{\ell=0}^{r-1}\sigma(\mathfrak{d}_\ell^\phi)\otimes\tau^\ell(\mathfrak{c}_r^\phi),
\end{aligned} \tag{3.10}$$

where the equality in the second line follows from the definition of the basis $\mathfrak{c}^\phi$ in §2.3.1 and Definition 2.10. Going forward we will denote $\gamma := \sum_{\ell=0}^{r-1}\sigma(\mathfrak{d}_\ell^\phi)\otimes\tau^\ell(\mathfrak{c}_r^\phi)$. This term has no dependence on n and thus makes no contribution towards the convergence of the left hand side, thus we will minimize the notation of these terms throughout the following discussion.

We now wish to move towards viewing these identities as living in rings of matrices over Tate algebras. To this end, we identify $M_\phi \cong \mathrm{Mat}_{1\times r}\,\mathbb{C}_\infty[t]$ and identify $N_\phi \cong \mathbb{C}_\infty[t]^r$ using the bases described above. Applying the definition of the $\tau-$ and $\sigma-$ action on these bases detailed in (3.8), formula (3.10) becomes

$$((1 \otimes t) - (t \otimes 1))G_n^{\otimes} = \sum_{\ell=0}^{r-1} \mathcal{P}_{n-\ell}\mathfrak{e}_1 \otimes \mathfrak{e}_{\ell+1}^{\mathrm{tr}}\mathcal{S}_n - \gamma. \tag{3.11}$$

A short calculation shows that both of these (finite) sums are in $\mathbb{T}_\theta^r \otimes_{\mathbb{C}_\infty} \mathbb{T}_\theta^r$.

Remark 3.2. We note that we write vectors to the left of the tensor as a column and vectors to the right as a row in order to simplify notation in what comes next, namely, so that we can multiply by $r \times r$ matrices on the left and on the right of such a simple tensor and it is clear what that means. To avoid cumbersome notation, we will denote such elements as living in $\mathbb{T}_\theta^r \otimes_{\mathbb{C}_\infty} \mathbb{T}_\theta^r$ rather than $\mathbb{T}_\theta^r \otimes_{\mathbb{C}_\infty} \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$.

Our immediate goal is to prove that the right hand side of (3.11) converges in some ring of Tate algebras as $n \to \infty$.

Definition 3.3. Let $c \in \mathbb{C}_\infty^\times$. Recall the norm $\|\cdot\|_c$ on $\mathbb{T}_c^r$ from §2.2. We extend this norm to simple tensors $a \otimes b \in \mathbb{T}_c^r \otimes_{\mathbb{C}_\infty} \mathbb{T}_c^r$ by setting

$$\|a \otimes b\|_c = \|a\|_c \cdot \|b\|_c,$$

then extending it to all $\mathbb{T}_c^r \otimes \mathbb{T}_c^r$ by taking the supremum over all sums involving simple tensors. It follows trivially from the definition that this is in fact a non-archimedean (or ultrametric) norm on $\mathbb{T}_c^r \otimes_{\mathbb{C}_\infty} \mathbb{T}_c^r$. In fact, this is an example of a cross norm on the tensor product of two Banach spaces (see [30, §6] for more details on cross norms). We then form the completion of $\mathbb{T}_c^r \otimes \mathbb{T}_c^r$ under this norm, and denote the resulting space $\widehat{\mathbb{T}_c^r \otimes \mathbb{T}_c^r}$.

Lemma 3.4. *For $a_n, b_n \in \mathbb{T}_c^r$, the sum of simple tensors*

$$\sum_{n=0}^{\infty} a_n \otimes b_n$$

converges in $\widehat{\mathbb{T}_c^r \otimes \mathbb{T}_c^r}$ if and only if $\|a_n \otimes b_n\|_c \to 0$ as $n \to \infty$.

Proof. First, note that the sum $\sum_{i=0}^{\infty} a_n \otimes b_n$ trivially diverges if $\|a_n \otimes b_n\|_c$ does not converge to 0. On the other hand, if the individual simple tensors do converge to 0 in norm, then the convergence of the series follows from the ultrametric triangle inequality. □

3.3. **The element α_n.** Our main goal in this subsection is to define an element $\alpha_n \in \mathbb{T}^r \otimes \mathbb{T}^r$ for each $n \in \mathbb{Z}_{\geq 1}$ which will be useful to interpret the RHS of (3.11) in terms of matrices Π_n and Ψ_n in (3.10). Recall the matrix B from (3.2) and set $\mathfrak{D} := \det(B)$ and write $B^{-1} = \frac{1}{\mathfrak{D}}(\mathfrak{c}_{ji})_{ij}$ where $\mathfrak{c}_{ji}$ is the (j,i)-cofactor of B. By the construction of B, for each $1 \leq \ell \leq r$, we obtain

$$\mathfrak{c}_{1\ell} = \begin{cases} -\mathfrak{c}_{r\ell}^q & \text{if } r \text{ is even} \\ \mathfrak{c}_{r\ell}^q & \text{if } r \text{ is odd.} \end{cases} \tag{3.12}$$

Since $\xi_1, \dots, \xi_r$ are elements in $\phi[\theta]$, we have

$$B^{(1)} = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ -\frac{\theta}{k_r} & -\frac{k_1}{k_r} & \dots & \dots & -\frac{k_{r-1}}{k_r} \end{pmatrix} B. \tag{3.13}$$

This relation shows that

$$\mathfrak{D}^q = \begin{cases} \frac{\theta \mathfrak{D}}{k_r} & \text{if } r \text{ is even} \\ -\frac{\theta \mathfrak{D}}{k_r} & \text{if } r \text{ is odd.} \end{cases} \tag{3.14}$$

Hence, for each $m \geq 0$, we have

$$(B^{-1})^{(m)} = \begin{cases} \frac{k_r^{1+q+\dots+q^{m-1}}}{\theta^{1+q+\dots+q^{m-1}}\mathfrak{D}} (\mathfrak{c}_{ji}^{q^m})_{ij} & \text{if } r \text{ is even} \\ (-1)^m \frac{k_r^{1+q+\dots+q^{m-1}}}{\theta^{1+q+\dots+q^{m-1}}\mathfrak{D}} (\mathfrak{c}_{ji}^{q^m})_{ij} & \text{if } r \text{ is odd.} \end{cases} \tag{3.15}$$

For any positive integer n, in what follows, we define α_n, a quantity related to the right-hand side of (3.11) without the γ term and after factoring out $\Pi_{n-1}^{(1)}$ on the left and $(\Psi_n^{(-1)})^{\mathrm{tr}}$ on the right,

$$\begin{aligned} \alpha_n := (\mathcal{M}^{-1})^{(n-(r-1))}\mathfrak{e}_1 \otimes \mathfrak{e}_r^{\mathrm{tr}}\mathcal{M}^{(n)} + F^{(n-(r-2))}(\mathcal{M}^{-1})^{(n-(r-2))}\mathfrak{e}_1 \otimes \mathfrak{e}_{r-1}^{\mathrm{tr}}\mathcal{M}^{(n)} + \\ F^{(n-(r-2))}F^{(n-(r-3))}(\mathcal{M}^{-1})^{(n-(r-3))}\mathfrak{e}_1 \otimes \mathfrak{e}_{r-2}^{\mathrm{tr}}\mathcal{M}^{(n)} + \dots + \\ F^{(n-(r-2))} \cdots F^{(n)}(\mathcal{M}^{-1})^{(n)}\mathfrak{e}_1 \otimes \mathfrak{e}_1^{\mathrm{tr}}\mathcal{M}^{(n)} \in \mathbb{T}^r \otimes \mathbb{T}^r. \end{aligned}$$

The precise relationship between (3.11) and α_n will be given in (3.23). We further define another quantity

$$\begin{aligned} \beta &:= (\mathcal{M}^{-1})^{(-1)}\mathfrak{e}_1 \otimes \mathfrak{e}_r^{\mathrm{tr}}\mathcal{M}^{(r-2)} + \mathcal{M}^{-1}\mathfrak{e}_1 \otimes \mathfrak{e}_{r-1}^{\mathrm{tr}}\mathcal{M}^{(r-2)} + (\mathcal{M}^{-1})^{(1)}\mathfrak{e}_1 \otimes \mathfrak{e}_{r-2}^{\mathrm{tr}}\mathcal{M}^{(r-2)} \\ &\quad + \dots + (\mathcal{M}^{-1})^{(r-2)}\mathfrak{e}_1 \otimes \mathfrak{e}_1^{\mathrm{tr}}\mathcal{M}^{(r-2)} \\ &= \frac{1}{k_r^{(-1)}}B^{-1}\mathfrak{e}_r \otimes k_r^{(-1)}\mathfrak{e}_1^{\mathrm{tr}}B^{(r-1)} + \frac{1}{k_r}(B^{-1})^{(1)}\mathfrak{e}_r \otimes (k_{r-1}, k_r, 0, \dots, 0)B^{(r-1)} + \\ &\quad \frac{1}{k_r^{(1)}}(B^{-1})^{(2)}\mathfrak{e}_r \otimes (k_{r-2}^q, k_{r-1}^q, k_r^q, 0, \dots, 0)B^{(r-1)} + \dots + \\ &\quad \frac{1}{k_r^{(r-2)}}(B^{-1})^{(r-1)}\mathfrak{e}_r \otimes (k_1^{q^{r-2}}, \dots, k_{r-1}^{q^{r-2}}, k_r^{q^{r-2}})B^{(r-1)} \in \mathbb{T}^r \otimes \mathbb{T}^r. \end{aligned} \tag{3.16}$$

Remark 3.5. Important Notational Comment: Since $\mathfrak{c}^\phi$ ($\mathfrak{d}^\phi$ resp.) forms a $\mathbb{C}_\infty[t]$-basis for M_ϕ (for N_ϕ resp.), we conclude that $\{\mathfrak{d}_i^\phi \otimes \mathfrak{c}_j^\phi\}$ for $1 \leq i, j \leq r$ forms a $\mathbb{C}_\infty[t] \otimes \mathbb{C}_\infty[t]$-basis for $M_\phi \otimes_{\mathbb{C}_\infty} N_\phi$. We then tensor this with $\mathbb{T}$ and view $\mathbb{T} \otimes_{\mathbb{C}_\infty} (M_\phi \otimes_{\mathbb{C}_\infty} N_\phi) \cong \mathbb{T}^r \otimes_{\mathbb{C}_\infty} \mathbb{T}^r$ as $\mathrm{Mat}_{r\times r}(\mathbb{T})$ with a $\mathbb{T}$-basis given by $\{\mathfrak{d}_i^\phi \otimes \mathfrak{c}_j^\phi\}$ as above. Thus, there exists a bijection $f : \mathbb{T}^r \otimes \mathbb{T}^r \to \mathrm{Mat}_{r\times r}(\mathbb{T})$ sending each

$$g = \sum_{i,j=1}^{r} b_{ij}\mathfrak{d}_i^\phi \otimes \mathfrak{c}_j^\phi \in \mathbb{T}^r \otimes \mathbb{T}^r$$

to $f(g) := (b_{ij}) \in \mathrm{Mat}_{r\times r}(\mathbb{T})$. However, we are especially interested in using this notation in the case of an element, such as β given above, with $\beta \in \mathbb{C}_\infty^r \otimes \mathbb{C}_\infty^r \in \mathbb{T}^r \otimes \mathbb{T}^r$. In particular we have

$$f\left(\sum_{i=1}^{r} \mathfrak{d}_i^{\phi} \otimes \mathfrak{c}_i^{\phi}\right) = \mathrm{Id}_r\,.$$

Throughout the remainder of this paper we will often omit the map f from our notation when it does not cause confusion. For example, we will view α_n and β as matrices over $\mathbb{T}$ and $\mathbb{C}_\infty$, respectively.

Recall the matrix $\mathcal{M}$ defined in (3.3). The next lemma will be crucial to determine the limiting behavior of α_n.

Lemma 3.6. *For each $2 \le j \le r$ and a matrix $\mathfrak{J} \in \mathrm{Mat}_{r\times r}(\mathbb{T})$ whose each entry has norm less than 1, we have*

$$\lim_{n\to\infty} \mathfrak{J}(\mathcal{M}^{-1})^{(n-(r-j))}\mathfrak{e}_1 \otimes \mathfrak{e}_{r-(j-1)}^{\mathrm{tr}} \mathcal{M}^{(n)} = 0.$$

Proof. For each $1 \le i \le r-1$, let

$$\mathcal{F}_i := (B^{-1})^{(i)}\mathfrak{e}_r \otimes (k_{r-i}^{q^{i-1}}, \dots, k_{r-1}^{q^{i-1}}, k_r, 0, \dots, 0)B^{(r-1)}.$$

Since, by assumption, $|k_i| \le 1$ and $k_r \in \mathbb{F}_q^\times$, after a simple calculation and using Lemma 3.4, it suffices to show that $\log_q(\|\mathcal{F}_i\|) \le 0$ for each i. Note, from (3.14), that $|\mathfrak{D}| = q^{1/(q-1)}$. Finally, for any $1 \le \mu \le r$, since ξ_μ is a θ-torsion point, one obtains

$$k_{r-i}^{q^{i-1}}\xi_\mu^{q^{r-1}} + \dots + k_{r-1}^{q^{i-1}}\xi_\mu^{q^{r+i-2}} + k_r\xi_\mu^{q^{r+i-1}} = -\theta^{q^{i-1}}\xi_\mu^{q^{i-1}} - k_1^{q^{i-1}}\xi_\mu^{q^i} - \dots - k_{r-i-1}^{q^{i-1}}\xi_\mu^{q^{r-2}}.$$

Since $|k_i| \le 1$, we see that

$$\log_q(|k_{r-i}^{q^{i-1}}\xi_\mu^{q^{r-1}} + \dots + k_{r-1}^{q^{i-1}}\xi_\mu^{q^{r+i-2}} + k_r\xi_\mu^{q^{r+i-1}}|) \le q^{i-1} + \frac{q^{i-1}}{q^r-1}.$$

Similarly, a direct calculation implies that, for each $1 \le \nu \le r$, $|\mathfrak{c}_{r\nu}|$ is bounded by $q^{1+q+\dots+q^{r-2}}/(q^r-1)$. Combining all these facts above, we obtain

$$\begin{aligned}\log_q(\|\mathcal{F}_i\|) &\le -\frac{q^i}{q-1} + \frac{q^i+\dots+q^{i+r-2}}{q^r-1} + q^{i-1} + \frac{q^{i-1}}{q^r-1}\\ &= -\left(1+q+\dots+q^{i-1}+\frac{1}{q-1}\right) + 1+q+\dots+q^{i-2}\\ &\qquad + \frac{1+q+\dots+q^{i-2}+q^i+\dots+q^{r-1}}{q^r-1} + q^{i-1} + \frac{q^{i-1}}{q^r-1}\\ &= 0\end{aligned}$$

as desired. □

By [24, Thm. 3.29], we have $F = \mathrm{Id}_r + \tilde{F}$ where each entry of $\tilde{F} \in \mathrm{Mat}_{r\times r}(\mathbb{T})$ has norm less than 1. Thus, by choosing $\mathfrak{J} = \tilde{F}$ in Lemma 3.6 and using the definition of α_n and β, we have

$$\alpha := \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta^{(n-(r-2))} \tag{3.17}$$

if and only if $\lim_{n\to\infty}\beta^{(n-(r-2))}$ exists. The main goal of this section is to prove the existence of this limit and hence, as a consequence, our next theorem.

Theorem 3.7. *In $\mathbb{T}^r \otimes \mathbb{T}^r$, we have*

$$\alpha = \sum_{i=1}^{r} \mathfrak{d}_i^{\phi} \otimes \mathfrak{c}_i^{\phi}.$$

In other words, via the identification in Remark 3.5, $\alpha = \mathrm{Id}_r$.

3.4. **The proof of Theorem 3.7.** The proof of Theorem 3.7 occupies §3.4.1 and §3.4.2. Our main strategy, as it will be elaborated in what follows, is to show that $\beta = \mathrm{Id}_r$. Note, as it is used in the proof of Lemma 3.6, that since $\xi_1, \dots, \xi_r$ are elements in $\phi[\theta]$, we have

$$k_i \xi_j^{q^i} + \cdots + k_r \xi_j^{q^r} = -\theta \xi_j - k_1 \xi_j^{q} - \cdots - k_{i-1} \xi_j^{q^{i-1}} \tag{3.18}$$

for any $2 \le i \le r$ and $1 \le j \le r$.

3.4.1. *Even rank case.* Let us set $r = 2n$ for some positive integer n. By using (3.14), (3.15), (3.18) as well as the definition of β given in (3.16), we obtain

$$\beta = \frac{1}{\mathfrak{D}} \begin{pmatrix} \frac{\mathfrak{c}_{(2n)1}}{k_{2n}^{(-1)}} & \mathfrak{c}_{(2n)1}^{q} & \cdots & \mathfrak{c}_{(2n)1}^{q^{2n-1}} \\ \frac{\mathfrak{c}_{(2n)2}}{k_{2n}^{(-1)}} & \mathfrak{c}_{(2n)2}^{q} & \cdots & \mathfrak{c}_{(2n)2}^{q^{2n-1}} \\ \vdots & \vdots & & \vdots \\ \frac{\mathfrak{c}_{(2n)(2n)}}{k_{2n}^{(-1)}} & \mathfrak{c}_{(2n)(2n)}^{q} & \cdots & \mathfrak{c}_{(2n)(2n)}^{q^{2n-1}} \end{pmatrix} \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & k_{2n}^{(-1)} \\ -1 & -\frac{k_1}{\theta} & -\frac{k_2}{\theta} & \cdots & -\frac{k_{2n-2}}{\theta} & 0 \\ & -\frac{k_{2n}}{\theta} & -\frac{k_{2n}k_1^q}{\theta^{1+q}} & & -\frac{k_{2n}k_{2n-3}^q}{\theta} & \vdots \\ & & -\frac{k_{2n}^{1+q}}{\theta^{1+q}} & \ddots & \vdots & \vdots \\ & & & \ddots & -\frac{k_{2n}^{1+\cdots+q^{2n-4}} k_1^{q^{2n-3}}}{\theta^{1+\cdots+q^{2n-3}}} & \vdots \\ & & & & -\frac{k_{2n}^{1+\cdots+q^{2n-3}}}{\theta^{1+\cdots+q^{2n-3}}} & 0 \end{pmatrix} B.$$

Let us set $\mathfrak{B} := \beta B^{-1}$. Our goal is to show that $\mathfrak{B} = B^{-1}$. Firstly, by (3.12) and a simple calculation, the first and last column of $\mathfrak{B}$ and B^{-1} are equal. Hence $\mathfrak{B} = B^{-1}$ when $n = 1$. Now assume that $n > 1$. Note that (3.13) also implies

(3.19)

$$(B^{(1)})^{-1} \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ -\frac{\theta}{k_{2n}} & -\frac{k_1}{k_{2n}} & \cdots & \cdots & -\frac{k_{2n-1}}{k_{2n}} \end{pmatrix} = B^{-1} = \frac{1}{\mathfrak{D}} \begin{pmatrix} -\mathfrak{c}_{(2n)1}^{q} & \mathfrak{c}_{21} & \cdots & \mathfrak{c}_{(2n-1)1} & \mathfrak{c}_{(2n)1} \\ -\mathfrak{c}_{(2n)2}^{q} & \mathfrak{c}_{22} & \cdots & \mathfrak{c}_{(2n-1)2} & \mathfrak{c}_{(2n)2} \\ \vdots & \vdots & & \vdots & \vdots \\ -\mathfrak{c}_{(2n)(2n)}^{q} & \mathfrak{c}_{2(2n)} & \cdots & \mathfrak{c}_{(2n-1)(2n)} & \mathfrak{c}_{(2n)(2n)} \end{pmatrix}.$$

For each $2 \le m \le 2n-1$ and $1 \le i \le 2n$, we claim that

(3.20)

$$\mathfrak{c}_{mi} = -\Bigg(\frac{\mathfrak{c}_{(2n)i}^{q^m} k_{2n}^{1+q+\cdots+q^{m-2}}}{\theta^{1+q+\cdots+q^{m-2}}} + \frac{k_1^{q^{m-2}} \mathfrak{c}_{(2n)i}^{q^{m-1}} k_{2n}^{1+q+\cdots+q^{m-3}}}{\theta^{1+q+\cdots+q^{m-2}}} + \frac{k_2^{q^{m-3}} \mathfrak{c}_{(2n)i}^{q^{m-2}} k_{2n}^{1+q+\cdots+q^{m-4}}}{\theta^{1+q+\cdots+q^{m-3}}} + \cdots + \frac{k_{m-2}^{q} \mathfrak{c}_{(2n)i}^{q^2} k_{2n}}{\theta^{1+q}} + \frac{k_{m-1} \mathfrak{c}_{(2n)i}^{q}}{\theta} \Bigg).$$

When $m=2$, we have $\mathfrak{c}_{2i} = -\frac{\mathfrak{c}_{(2n)i}^{q^2}k_{2n}}{\theta} - \frac{k_1\mathfrak{c}_{(2n)i}^{q}}{\theta}$. Assume that it holds for m. Note, by (3.14) and (3.19), we have

$$\mathfrak{c}_{(m+1)i} = \frac{\mathfrak{c}_{mi}^q k_{2n}}{\theta} - \frac{k_m \mathfrak{c}_{(2n+1)i}^q}{\theta}.$$

Using the induction hypothesis, we obtain

$$\begin{aligned}
\mathfrak{c}_{(m+1)i} &= \frac{\mathfrak{c}_{mi}k_{2n}}{\theta} - \frac{k_m\mathfrak{c}_{(2n+1)i}^q}{\theta} \\
&= -\Bigg(\frac{\mathfrak{c}_{(2n)i}^{q^{m+1}}k_{2n}^{1+q+\cdots+q^{m-1}}}{\theta^{1+q+\cdots+q^{m-1}}} + \frac{k_1^{q^{m-1}}\mathfrak{c}_{(2n)i}^{q^m}k_{2n}^{1+q+\cdots+q^{m-2}}}{\theta^{1+q+\cdots+q^{m-1}}} + \frac{k_2^{q^{m-2}}\mathfrak{c}_{(2n)i}^{q^{m-1}}k_{2n}^{1+q+\cdots+q^{m-3}}}{\theta^{1+q+\cdots+q^{m-2}}} + \cdots + \\
&\qquad \frac{k_{m-2}^{q^2}\mathfrak{c}_{(2n)i}^{q^3}k_{2n}^{1+q}}{\theta^{1+q+q^2}} + \frac{k_{m-1}^{q}\mathfrak{c}_{(2n)i}^{q^2}k_{2n}}{\theta^{1+q}} + \frac{k_m\mathfrak{c}_{(2n+1)i}^{q}}{\theta}\Bigg)
\end{aligned}$$

which proves our claim. Note that the right hand side of (3.20) is the (i,m)-entry of $\mathfrak{B}$. This immediately implies that $\mathfrak{B} = B^{-1}$ and hence we have $\beta = \mathrm{Id}_{2n}$. Furthermore, by (3.17), we obtain $\alpha = \sum_{i=1}^{2n} \mathfrak{d}_i^{\phi} \otimes \mathfrak{c}_i^{\phi}$.

3.4.2. *Odd rank case.* Let us set $r = 2n+1$ for some positive integer n. Using (3.14), (3.15), (3.18) and the definition of β given in (3.16), we see that

$$\beta = \frac{1}{\mathfrak{D}}\begin{pmatrix}
\frac{\mathfrak{c}_{(2n+1)1}}{k_{2n+1}^{(-1)}} & \mathfrak{c}_{(2n+1)1}^{q} & \cdots & \mathfrak{c}_{(2n+1)1}^{q^{2n}} \\
\frac{\mathfrak{c}_{(2n+1)2}}{k_{2n+1}^{(-1)}} & \mathfrak{c}_{(2n+1)2}^{q} & \cdots & \mathfrak{c}_{(2n+1)2}^{q^{2n}} \\
\vdots & \vdots & & \vdots \\
\frac{\mathfrak{c}_{(2n+1)(2n+1)}}{k_{2n+1}^{(-1)}} & \mathfrak{c}_{(2n+1)(2n+1)}^{q} & \cdots & \mathfrak{c}_{(2n+1)(2n+1)}^{q^{2n}}
\end{pmatrix}$$

$$\times \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & k_{2n+1}^{(-1)} \\
1 & \frac{k_1}{\theta} & \frac{k_2}{\theta} & \cdots & \frac{k_{2n-1}}{\theta} & 0 \\
 & -\frac{k_{2n+1}}{\theta} & -\frac{k_{2n+1}k_1^q}{\theta^{1+q}} & & -\frac{k_{2n+1}k_{2n-2}}{\theta^{1+q}} & \vdots \\
 & & \ddots & \ddots & \vdots & \vdots \\
 & & & \frac{k_{2n+1}^{1+\cdots+q^{2n-3}}}{\theta^{1+\cdots+q^{2n-3}}} & \frac{k_{2n+1}^{1+\cdots+q^{2n-3}}k_1^{q^{2n-3}}}{\theta^{1+\cdots+q^{2n-2}}} & \vdots \\
 & & & & -\frac{k_{2n+1}^{1+\cdots+q^{2n-2}}}{\theta^{1+\cdots+q^{2n-2}}} & 0
\end{pmatrix} B.$$

Consider $\mathfrak{C} := \beta B^{-1}$. Our goal is to show that $\mathfrak{C} = B^{-1}$. Firstly, by (3.12) and a simple calculation, the first and last column of $\mathfrak{C}$ and B^{-1} are equal. On the other hand, similar to

(3.19), observe that, by (3.13), (3.12) and (3.14), we have

$$\frac{k_{2n+1}}{\theta\mathfrak{D}}\begin{pmatrix} -\mathfrak{c}_{(2n+1)1}^{q^2} & -\mathfrak{c}_{21}^{q} & \cdots & -\mathfrak{c}_{(2n)1}^{q} & -\mathfrak{c}_{(2n+1)1}^{q} \\ -\mathfrak{c}_{(2n+1)2}^{q^2} & -\mathfrak{c}_{22} & \cdots & -\mathfrak{c}_{(2n)2}^{q} & -\mathfrak{c}_{(2n+1)2}^{q} \\ \vdots & \vdots & & \vdots & \vdots \\ -\mathfrak{c}_{(2n+1)(2n+1)}^{q^2} & -\mathfrak{c}_{2(2n+1)}^{q} & \cdots & -\mathfrak{c}_{(2n)(2n+1)}^{q} & -\mathfrak{c}_{(2n+1)(2n+1)}^{q} \end{pmatrix}\times \tag{3.21}$$

$$\begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ -\frac{\theta}{k_{2n+1}} & -\frac{k_1}{k_{2n+1}} & \cdots & \cdots & -\frac{k_{2n}}{k_{2n+1}} \end{pmatrix} = \frac{1}{\mathfrak{D}}\begin{pmatrix} \mathfrak{c}_{(2n+1)1}^{q} & \mathfrak{c}_{21} & \cdots & \mathfrak{c}_{(2n)1} & \mathfrak{c}_{(2n+1)1} \\ \mathfrak{c}_{(2n+1)2}^{q} & \mathfrak{c}_{22} & \cdots & \mathfrak{c}_{(2n)2} & \mathfrak{c}_{(2n+1)2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathfrak{c}_{(2n+1)(2n)}^{q} & \mathfrak{c}_{2(2n+1)} & \cdots & \mathfrak{c}_{(2n)(2n+1)} & \mathfrak{c}_{(2n+1)(2n+1)} \end{pmatrix}.$$

For each $2 \leq m \leq 2n$ and $1 \leq i \leq 2n+1$, we claim that

$$\begin{aligned}\mathfrak{c}_{mi} = {} & \frac{(-1)^{m-1}\mathfrak{c}_{(2n+1)i}^{q^m}k_{2n+1}^{1+q+\cdots+q^{m-2}}}{\theta^{1+q+\cdots+q^{m-2}}} + \frac{(-1)^m k_1^{q^{m-2}}\mathfrak{c}_{(2n+1)i}^{q^{m-1}}k_{2n+1}^{1+q+\cdots+q^{m-3}}}{\theta^{1+q+\cdots+q^{m-2}}} \\ & + \frac{(-1)^{m+1}k_2^{q^{m-3}}\mathfrak{c}_{(2n+1)i}^{q^{m-2}}k_{2n+1}^{1+q+\cdots+q^{m-4}}}{\theta^{1+q+\cdots+q^{m-3}}} + \cdots + \frac{(-1)^{2m-4}k_{m-3}^{q^2}\mathfrak{c}_{(2n+1)i}^{q^3}k_{2n+1}^{1+q}}{\theta^{1+q+q^2}} \\ & + \frac{(-1)^{2m-3}k_{m-2}^{q}\mathfrak{c}_{(2n+1)i}^{q^2}k_{2n+1}}{\theta^{1+q}} + \frac{(-1)^{2m-2}k_{m-1}\mathfrak{c}_{(2n+1)i}^{q}}{\theta}.\end{aligned}\tag{3.22}$$

When $m = 2$, we have

$$\mathfrak{c}_{2i} = -\frac{\mathfrak{c}_{(2n+1)i}^{q^2}k_{2n+1}}{\theta} + \frac{k_1\mathfrak{c}_{(2n+1)i}^{q}}{\theta}.$$

Assume that it holds for m. Note, by (3.21), we have

$$\mathfrak{c}_{(m+1)i} = -\frac{\mathfrak{c}_{mi}^{q}k_{2n+1}}{\theta} + \frac{k_m\mathfrak{c}_{(2n+1)i}^{q}}{\theta}.$$

By the induction hypothesis, we have

$$\begin{aligned}\mathfrak{c}_{(m+1)i} &= -\frac{\mathfrak{c}_{mi}}{\theta} + \frac{k_m\mathfrak{c}_{(2n+1)i}^{q}}{\theta} \\ &= \frac{(-1)^m\mathfrak{c}_{(2n+1)i}^{q^{m+1}}k_{2n+1}^{1+q+\cdots+q^{m-1}}}{\theta^{1+q+\cdots+q^{m-1}}} + \frac{(-1)^{m+1}k_1^{q^{m-1}}\mathfrak{c}_{(2n+1)i}^{q^m}k_{2n+1}^{1+q+\cdots+q^{m-2}}}{\theta^{1+q+\cdots+q^{m-1}}} \\ &\quad + \frac{(-1)^{m+2}k_2^{q^{m-2}}\mathfrak{c}_{(2n+1)i}^{q^{m-1}}k_{2n+1}^{1+q+\cdots+q^{m-3}}}{\theta^{1+q+\cdots+q^{m-2}}} + \cdots + \frac{(-1)^{2m-3}k_{m-3}^{q^3}\mathfrak{c}_{(2n+1)i}^{q^4}k_{2n+1}^{1+q+q^2}}{\theta^{1+q+q^3}} \\ &\quad + \frac{(-1)^{2m-2}k_{m-2}^{q^2}\mathfrak{c}_{(2n+1)i}^{q^3}k_{2n+1}^{1+q}}{\theta^{1+q+q^2}} + \frac{(-1)^{2m-1}k_{m-1}^{q}\mathfrak{c}_{(2n+1)i}^{q^2}k_{2n+1}}{\theta^{1+q}} + \frac{(-1)^{2m}k_m\mathfrak{c}_{(2n+1)i}^{q}}{\theta}\end{aligned}$$

which proves our claim. It is easy to see that the right hand side of (3.22) is the (i,m)-entry of $\mathfrak{C}$. Therefore $\mathfrak{C} = B^{-1}$ and thus we have $\beta = \mathrm{Id}_{2n+1}$. Furthermore (3.17) now yields $\alpha = \sum_{i=1}^{2n+1}\mathfrak{c}_i^{\phi}\otimes\mathfrak{d}_i^{\phi}$ as desired.

3.5. **Formulas for the logarithms.** Recall the identities given in (3.5) and (3.7). Observe, by (3.8) and (3.11), that

(3.23)
$$\begin{aligned}((1\otimes t)-(t\otimes 1))G_n^{\otimes}+\gamma &= \sum_{\ell=0}^{r-1}\mathcal{P}_{n-\ell}\mathfrak{e}_1\otimes\mathfrak{e}_{\ell+1}^{\mathrm{tr}}\mathcal{S}_n\\ &= V^{\mathrm{tr}}\Pi_{n-r}^{(1)}(\mathcal{M}^{-1})^{(n-(r-1))}\mathfrak{e}_1\otimes\mathfrak{e}_r^{\mathrm{tr}}\mathcal{M}^{(n)}(\Psi_n^{(-1)})^{\mathrm{tr}}\\ &\quad + V^{\mathrm{tr}}\Pi_{n-(r-1)}^{(1)}(\mathcal{M}^{-1})^{(n-(r-2))}\mathfrak{e}_1\otimes\mathfrak{e}_{r-1}^{\mathrm{tr}}\mathcal{M}^{(n)}(\Psi_n^{(-1)})^{\mathrm{tr}}+\cdots\\ &\quad + V^{\mathrm{tr}}\Pi_{n-1}^{(1)}(\mathcal{M}^{-1})^{(n)}\mathfrak{e}_1\otimes\mathfrak{e}_1^{\mathrm{tr}}\mathcal{M}^{(n)}(\Psi_n^{(-1)})^{\mathrm{tr}}\\ &= V^{\mathrm{tr}}\Pi_{n-r}^{(1)}\alpha_n(\Psi_n^{(-1)})^{\mathrm{tr}}\end{aligned}$$

where the last equality follows from $\Pi_{n-\ell}^{(1)}=\Pi_{n-r}^{(1)}F^{(n-(r-2))}\cdots F^{(n-(\ell-1))}$. Hence, using Theorem 3.1, Lemma 3.4 and Theorem 3.7 yields the following result.

Theorem 3.8. *In $\mathbb{T}_\theta^r\otimes\mathbb{T}^r$, we have*

$$\lim_{n\to\infty}\big(((1\otimes t)-(t\otimes 1))G_n^{\otimes}+\gamma\big)=V^{\mathrm{tr}}\Upsilon^{(1)}\alpha(\Psi^{\mathrm{tr}})^{(-1)}.$$

We are now ready to prove the main result of this section.

Theorem 3.9. *Let $\mathbf{z}\in\mathbb{C}_\infty$ be an element in the domain of convergence of $\log_\phi$. Then, we have*

$$\log_\phi(\mathbf{z})=\delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{t-\theta}\mathfrak{e}_1^{\mathrm{tr}}V^{\mathrm{tr}}\Upsilon^{(1)}_{|t=\theta}(\Psi^{\mathrm{tr}})^{(-1)}\right).$$

Proof. Let

$$\begin{aligned}((1\otimes t)-(t\otimes 1))G_n^{\otimes}+\gamma &= \sum_{\ell=0}^{r-1}\sigma^{\ell-n}(\mathfrak{d}_1^\phi)\otimes\tau^n\left(\Theta_{\phi,\tau^{r-\ell}}\mathfrak{c}_r^\phi\right)\\ &= \begin{bmatrix}\zeta_{11}\\ \vdots\\ \zeta_{1r}\end{bmatrix}\otimes[\eta_{11},\dots,\eta_{1r}]+\cdots+\begin{bmatrix}\zeta_{r1}\\ \vdots\\ \zeta_{rr}\end{bmatrix}\otimes[\eta_{r1},\dots,\eta_{rr}]\in\mathbb{T}^r\otimes\mathbb{T}^r.\end{aligned}$$

By Theorem 3.1(i) and (3.8), we see that for each $1\le i,j\le r$, ζ_{ij} is well-defined at $t=\theta$. Moreover, by the definition of $\delta_0^{N_\phi}$-map, we have

(3.24)
$$\begin{aligned}&\sum_{\ell=0}^{r-1}\delta_0^{N_\phi}(\sigma^{\ell-n}(\mathfrak{d}_1^\phi))\tau^n(\Theta_{\phi,\tau^{r-\ell}}\mathfrak{c}_r^\phi)\\ &\quad=\left((\zeta_{11})|_{t=\theta}\eta_{11}+\cdots+(\zeta_{r1})|_{t=\theta}\eta_{r1}\right)\mathfrak{c}_1^\phi+\cdots+\left((\zeta_{11})|_{t=\theta}\eta_{1r}+\cdots+(\zeta_{r1})|_{t=\theta}\eta_{rr}\right)\mathfrak{c}_r^\phi.\end{aligned}$$

Finally, by Theorem 2.9, Theorem 3.8 and (3.24), we obtain

$$\log_\phi(\mathbf{z}) = \lim_{n\to\infty} \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{t-\theta}\sum_{\nu=0}^{r-1}\delta_0^{N_\phi}(\sigma^{-n}(\mathfrak{d}_1^\phi))\tau^n(\Theta_{\phi,\tau^{\ell-\nu}}\mathfrak{c}_r^\phi)\right)$$

$$= \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{t-\theta}\lim_{n\to\infty}\left(\sum_{\nu=0}^{r-1}\delta_0^{N_\phi}(\sigma^{-n}(\mathfrak{d}_1^\phi))\tau^n(\Theta_{\phi,\tau^{\ell-\nu}}\mathfrak{c}_r^\phi)\right)\right)$$

$$= \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{t-\theta}\mathfrak{e}_1^{\mathrm{tr}}V^{\mathrm{tr}}\Upsilon^{(1)}_{|t=\theta}\alpha(\Psi^{\mathrm{tr}})^{(-1)}\right)$$

and the result follows from the identification of $\alpha = \mathrm{Id}_r$ given in Theorem 3.7. □

By specializing the value $\mathbf{z}$ at certain prescribed points, we may conclude that the left-hand side of Theorem 3.9 evaluates to a Taelman L-value (see [31] for more details). We further evaluate terms on the right-hand side to show that it includes periods and exponential functions which also indicates that our next result may be interpreted as a Mellin transform formula for Taelman L-values.

Recall the fundamental periods $\lambda_1, \dots, \lambda_r \in \mathbb{C}_\infty^\times$ of ϕ defined at beginning of the present section and $f_i = s_\phi(\lambda_i; t)$ introduced in §2.3.1.

Corollary 3.10. *Let ϕ be a Drinfeld module as in Theorem 1.1 so that each $k_i \in \mathbb{F}_q^\times$ and let $\overline{\pi} = (\lambda_1, \dots, \lambda_r)$ be a vector of fundamental periods of ϕ. Then we have*

$$L(\phi^\vee, 0) = \delta_{1,\mathbf{z}}^{M_\phi}\left(\frac{1}{\theta - t}\overline{\pi}(\Psi^{\mathrm{tr}})^{(-1)}\right).$$

Proof. Let us set $\mathfrak{m} := \{x \in K_\infty \mid |x| < 1\}$. Then we have $K_\infty = \mathfrak{m} \oplus A$. Note that, by [24, Cor. 4.5], the radius of convergence of $\log_\phi$ is $q^{\frac{q^r}{q^r-1}}$. Hence, $\log_\phi(1)$ is well-defined and $\mathfrak{m}$ is in the domain of convergence of $\log_\phi$. Moreover, by [5, Thm. 3.3], one can calculate the logarithm coefficients of ϕ, which yields the fact that $\log_\phi(\mathfrak{m}) \subseteq \mathfrak{m}$.

To proceed, we define the A-module $H(\phi/A)$ given by the quotient

$$H(\phi/A) := \frac{\phi(K_\infty)}{\exp_\phi(K_\infty) + \phi(A)}.$$

Here, by $\phi(K_\infty)$ and $\phi(A)$, we mean the A-modules K_∞ and A equipped with the A-module structure induced from ϕ. Since $\exp_\phi$ is the formal inverse of $\log_\phi$, we now see that $\exp_\phi(K_\infty) \supseteq \mathfrak{m}$. Thus, $\exp_\phi(K_\infty) + \phi(A) \supseteq \phi(K_\infty)$, implying that $H(\phi/A)$ is trivial. On the other hand, if we set $U(\phi/A) := \{u \in K_\infty \mid \exp_\phi(u) \in A\}$, by [9, Thm. 1.10], we know that $U(\phi/A)$ is an A-module of rank one. Indeed, since the norm of $\log_\phi(1)$, being equal to 1, is minimal among the elements of $U(\phi/A)$, we obtain that $U(\phi/A) = A\log_\phi(1)$. Thus, by [31, Rem. 5, Thm. 1] (see also [7, §3]), we obtain $L(\phi^\vee, 0) = \log_\phi(1)$.

On the other hand, observe that $\mathfrak{e}_1^{\mathrm{tr}}V^{\mathrm{tr}} = (k_1, \dots, k_r)$. Then, we find that for $1 \le i \le r$, the i-th entry of $\mathfrak{e}_1^{\mathrm{tr}}V^{\mathrm{tr}}\Upsilon^{(1)}$ is given by

$$k_1 f_i^{(1)} + k_2 f_i^{(2)} + \cdots + k_r f_i^{(r)} = (t-\theta)f_i,$$

by Proposition 2.3. Therefore evaluating this at $t = \theta$ gives $\mathrm{Res}_\theta f_i$ which equals $-\lambda_i$ by [4, (3.4.3)]. Putting this all together gives

$$\mathfrak{e}_1^{\mathrm{tr}} V^{\mathrm{tr}} \Upsilon^{(1)}_{t=\theta} = -\widetilde{\pi}.$$

The result then follows from Theorem 3.9. □

Remark 3.11. At the present, we do not know if our formulas provide a connection between Drinfeld modular forms and L-series. However, there are some hints in this direction provided by the case of the Carlitz module. In this setting, our formulas give

$$\delta^M_{1,\mathbf{z}}(-\widetilde{\pi}\Omega) = \zeta_A(1).$$

In seeking to connect the LHS of this formula with a Drinfeld modular form, we are inspired to write Ω in terms of the commonly used Drinfeld modular form uniformizer, $u(z) := 1/\exp_C(\widetilde{\pi}z)$. We then write

$$1/\Omega^{(-1)} = \omega_C = \exp_C\left(\frac{\widetilde{\pi}}{\theta - t}\right) = \sum_{i=0}^{\infty} \exp_C\left(\frac{\widetilde{\pi}}{\theta^{i+1}}\right) t^i.$$

Thus the reciprocal of $\Omega^{(-1)}$ can be written as a sum of $u(z)$ evaluated at certain powers of θ. This construction is somewhat forced, and seems unlikely to lead to a meaningful connection with Drinfeld modular forms in our opinion. More natural is to do the following. Recall the adjoint of the Carlitz module, $C^*_\theta(z) = \theta z + z^{1/q}$ (see [18, §3.7]). It comes equipped with an exponential function $\exp^*_C(z)$ which satisfies

$$\theta \exp^*_C(z) = \exp^*_C(C^*_\theta(z)).$$

Formally, C^* also has a logarithm series $\log^*_C$, which is the formal (fractional) power series inverse of $\exp^*_C$, which satisfies

$$C^*_t(\log^*_C(z)) = \log^*_C(\theta z).$$

However, this construction produces a power series with 0 radius of convergence! If we had a way to rigorously construct the function $\log^*_C$, it should produce a function with a free rank 1 period generated by an element π^*, and we would use this to define

$$g(t) = \log^*_C\left(\frac{\pi^*}{\theta - t}\right),$$

and we would have that both $g(t)$ and Ω satisfy

$$tg(t) = C^*_\theta(g(t)), \quad t\Omega = C^*_\theta(\Omega).$$

Thus the two functions are equal up to normalization. Finally, we use this identification to rewrite our main theorem

$$\delta^M_{1,\mathbf{z}}(-\widetilde{\pi}\Omega) = \delta^M_{1,\mathbf{z}}(-\widetilde{\pi}\sum_{i=0}^{\infty} \log^*_C(\pi^*\theta^{-i-1})t^i).$$

We anticipate that there seems to be a more natural connection between the logarithm function of the adjoint Carlitz module $\log^*_C$ (see [18, §3.7]) and Drinfeld modular forms. However, we are unsure how to make this connection rigorous, so this is a topic for future study.

4. Logarithms of tensor product of Drinfeld modules with the tensor powers of the Carlitz module

Throughout this section, we fix a positive integer $k \geq 1$ and let ϕ be a Drinfeld module given by

$$\phi_\theta = \theta + k_1\tau + \cdots + k_r\tau^r.$$

We further emphasize that unlike §3, we do not require any conditions on ϕ. Hence, in this case, unlike the situation in §3, n_ϕ may not be equal to one.

We examine the case where our Anderson t-module is chosen to be $\phi \otimes C^{\otimes k} = (\mathbb{G}^{rk+1}_{a/\mathbb{C}_\infty}, \rho)$ detailed in Example 2.2(iii). We remind the reader of the bases described in §2.3.3 and §2.4.3.

Consider the matrix

$$\mathfrak{T} := \rho_\theta^{\mathrm{tr}} = \begin{bmatrix} \theta & \overbrace{0 \quad \dots \quad 0}^{r(k-1)} & \tau & 0 & \dots & 0 & \tau k_1^{(-1)} \\ & \ddots & & & \ddots & & \vdots \\ & & \ddots & & & \ddots & \tau k_{r-1}^{(-1)} \\ & & & \ddots & & \tau & \tau k_r^{(-1)} \\ 1 & & & & \ddots & & 0 \\ & \ddots & & & \ddots & & \vdots \\ & & \ddots & & & \ddots & \vdots \\ & & & \ddots & & \ddots & 0 \\ & & & & 1 & & \theta \end{bmatrix}.$$

Thus, we have $t \cdot \mathfrak{g} = \mathfrak{T}^{\mathrm{tr}}\mathfrak{g}$ and $t \cdot \mathfrak{h} = (\mathfrak{T}^*)^{\mathrm{tr}}\mathfrak{h}$. In this case, we write

$$\Theta_{\rho,\tau} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & \\ 0 & \dots & \dots & \dots & \dots & 0 \\ \tau & & & & & \\ & \ddots & & & & \\ & & \tau & & & \\ k_1\tau & \dots & k_{r-1}\tau & k_r\tau & 0 \quad \dots & 0 \end{bmatrix}$$

where we note that the first $r(k-1)+1$-rows of $\Theta_{\rho,\tau}$ are zero. Furthermore, the formula given in (2.15) for the t-module ρ, which we denote as $G^{\otimes}_{k,n}$ reduces to

$$G^{\otimes}_{k,n} = \sum_{i=0}^{n} \sum_{j=1}^{rk+1} \sigma^{-i}(h_j) \otimes \tau^i(g_j).$$

By Proposition 2.12, we obtain the following.

Proposition 4.1. *We have*

$$(1\otimes t-t\otimes 1)G_{k,n}^{\otimes}=\sigma^{-n}(h_{r(k-1)+2})\otimes\tau^{n+1}(g_1)+\cdots+\sigma^{-n}(h_{rk})\otimes\tau^{n+1}(g_{r-1}) + \sigma^{-n}(h_{rk+1})\otimes\tau^{n+1}(\mathfrak{c}_1)-\sum_{j=2}^{r+1}\sigma(h_{r(k-1)+j})\otimes g_{j-1}-\sum_{j=1}^{r}k_j^{(-1)}\sigma(h_{rk+1})\otimes g_j. \tag{4.1}$$

4.1. **The element η_n.** Our goal in this subsection is similar to what we aim in §3.3. More precisely, we define an element $\eta_n\in\mathbb{T}^r\otimes\mathbb{T}^r$ for each $n\in\mathbb{Z}_{\geq 1}$ which we use to interpret $((1\otimes t)-(t\otimes 1))G_{k,n}^{\otimes}\in\mathbb{T}^r\otimes\mathbb{T}^r$ in terms of matrices Π_n and Ψ_n in (4.5). Let us set

$$\tilde{V}:=\begin{pmatrix} 0 & k_2^{(-1)} & k_3^{(-2)} & \dots & k_r^{(1-r)} \\ \vdots & \vdots & \vdots & \iddots & \\ \vdots & \vdots & k_r^{(-2)} & & \\ 0 & k_r^{(-1)} & & & \\ 1 & & & & \end{pmatrix}\in \mathrm{GL}_r(\mathbb{C}_\infty).$$

Then we have

$$\tilde{V}\begin{bmatrix}\mathfrak{d}_1\\ \vdots \\ \vdots \\ \mathfrak{d}_r\end{bmatrix}=\begin{bmatrix}h_{r(k-1)+2}\\ \vdots\\ \vdots\\ h_{rk+1}\end{bmatrix}\text{ and } [g_1,\dots,g_{r-1},\mathfrak{c}_1]=[\mathfrak{c}_1,\dots,\mathfrak{c}_r](\tilde{V}^{(-1)})^{-1}. \tag{4.2}$$

Let $\mathfrak{e}_i\in\mathrm{Mat}_{r\times 1}(\mathbb{F}_q)$ be the i-th unit vector. For any positive integer n, we now consider

$$\eta_n:=(\mathcal{M}^{-1})^{(n)}(\tilde{V}^{\mathrm{tr}})^{(n)}\mathfrak{e}_1\otimes\mathfrak{e}_1^{\mathrm{tr}}(((\tilde{V}^{(-1)})^{-1})^{\mathrm{tr}})^{(n+1)}\mathcal{M}^{(n)}+\cdots+ (\mathcal{M}^{-1})^{(n)}(\tilde{V}^{\mathrm{tr}})^{(n)}\mathfrak{e}_r\otimes\mathfrak{e}_r^{\mathrm{tr}}(((\tilde{V}^{(-1)})^{-1})^{\mathrm{tr}})^{(n+1)}\mathcal{M}^{(n)}\in\mathbb{T}^r\otimes\mathbb{T}^r.$$

Observe that

$$\eta_n=(\mathcal{M}^{-1})^{(n)}(\tilde{V}^{\mathrm{tr}})^{(n)}(\mathfrak{e}_1\otimes\mathfrak{e}_1^{\mathrm{tr}}+\cdots+\mathfrak{e}_r\otimes\mathfrak{e}_r^{\mathrm{tr}})((\tilde{V}^{-1})^{\mathrm{tr}})^{(n)}\mathcal{M}^{(n)}.$$

Thus, since $\mathfrak{e}_1\otimes\mathfrak{e}_1^{\mathrm{tr}}+\cdots+\mathfrak{e}_r\otimes\mathfrak{e}_r^{\mathrm{tr}}=\eta_n=\sum_{i=1}^r\mathfrak{d}_i\otimes\mathfrak{c}_i$, we finally obtain our next theorem.

Theorem 4.2. *We have*

$$\eta:=\lim_{n\to\infty}\eta_n=\sum_{i=1}^r\mathfrak{d}_i\otimes\mathfrak{c}_i.$$

In particular, via the identification in Remark 3.5, $\eta=\mathrm{Id}_r$.

To simplify the notation, from now on, we set $N_\rho:=N_{\phi\otimes C^{\otimes k}}$ and $M_\rho:=M_{\phi\otimes C^{\otimes k}}$.

4.2. **The structure of $\delta_0^{N_\rho}$-map.** In what follows, we analyze the behavior of $\delta_0^{N_\rho}$ which is necessary to prove our main result. In particular, we define an explicit isomorphism of $\mathbb{C}_\infty[t,\sigma]$-modules which allows us to compute the values of the map $\delta_0^{N_\rho}$. For more details on such construction, we refer the reader to [12, §A.2].

Consider the $\mathbb{C}_\infty[t,\sigma]$-module $\mathcal{N}:=\mathrm{Mat}_{1\times(rk+1)}(\mathbb{C}_\infty[\sigma])$ whose $\mathbb{C}_\infty[t]$-module structure is given by

$$ct^i\cdot\mathfrak{n}:=c\mathfrak{n}\rho^*_{\theta^i},\quad c\in\mathbb{C}_\infty,\quad \mathfrak{n}\in\mathcal{N}.$$

For any $1 \leq i \leq rk+1$, let $\mathfrak{f}_i \in \mathrm{Mat}_{1\times(rk+1)}(\mathbb{F}_q)$ be the i-th unit vector. For any $1 \leq i \leq r$, we set $\mathfrak{n}_i := \mathfrak{f}_{r(k-1)+i+1} \in \mathcal{N}$. Note, as it is already observed in [12, (83)], that we have $(t-\theta)^k \mathfrak{n}_r = \mathfrak{f}_1$ and for $1 \leq \mu \leq k$, one obtains $(t-\theta)^{k-\mu}\mathfrak{n}_i = \mathfrak{f}_{r(\mu-1)+i+1}$. Furthermore, a direct calculation implies that the set $\{\mathfrak{n}_1, \dots, \mathfrak{n}_r\}$ forms a $\mathbb{C}_\infty[t]$-basis for $\mathcal{N}$.

There exists a $\mathbb{C}_\infty[t,\sigma]$-module isomorphism $\iota : N_\rho \to \mathcal{N}$ given by

$$\iota\left(\sum_{j=1}^{r} \mathfrak{r}_j h_{r(k-1)+j+1}\right) := \mathfrak{r}_1 \cdot \mathfrak{n}_1 + \cdots + \mathfrak{r}_r \cdot \mathfrak{n}_r, \quad \mathfrak{r}_1, \dots, \mathfrak{r}_r \in \mathbb{C}_\infty[t].$$

We further define certain elements $v_{ij} \in \mathbb{C}_\infty$ so that

$$\tilde{V}^{-1} = \begin{pmatrix} & & & & v_{1r} \\ & & & v_{2(r-1)} & 0 \\ & & \iddots & & \vdots \\ & \iddots & & & \vdots \\ v_{r1} & \cdots & \cdots & v_{r(r-1)} & 0 \end{pmatrix}.$$

This implies, by (4.2), that $\iota(\mathfrak{d}_1) = v_{1r}\mathfrak{n}_r$ and for $2 \leq \ell \leq r$, we have

$$\iota(\mathfrak{d}_\ell) = v_{\ell(r-\ell+1)}\mathfrak{n}_{r-\ell+1} + \cdots + v_{\ell(r-1)}\mathfrak{n}_{r-1}.$$

Thus, by the definition of $\delta_0^{N_\rho}$, if $n = \sum_{j=1}^r \left(\sum_{\ell=0}^{m_j} a_{j,\ell}(t-\theta)^\ell\right)\mathfrak{d}_j \in N_\rho$, then

$$\delta_0^{N_\rho}(n) = \begin{pmatrix} * \\ \vdots \\ * \\ a_{r0}v_{r1} \\ \sum_{j=r-1}^r a_{j0}v_{j2} \\ \vdots \\ \sum_{j=2}^r a_{j0}v_{j(r-1)} \\ a_{10}v_{1r} \end{pmatrix}. \tag{4.3}$$

Since $(t-\theta)^{k+1}N_\rho \subset \sigma N_\rho$, the map $\delta_0^{N_\rho}$ may be calculated similarly at $\sigma^{-\ell}(n)$ for any non-negative integer ℓ.

4.3. **Formulas for the logarithms.** Recall the matrices $\mathcal{P}_n$ and $\mathcal{S}_n$ from §3. Since, the Drinfeld module ϕ is arbitrary, the matrix B chosen in [24, Thm. 4] is different than we have defined in §3.1 (see [24, Thm. 3.29]). However, since our calculations in (3.5) and (3.7) are not affected by this change, we still have

$$\mathcal{P}_n = (\mathcal{P}_n^{(-1)})^{(1)} = ((\Phi^{\mathrm{tr}})^{-1}((\Phi^{\mathrm{tr}})^{-1})^{(1)} \cdots ((\Phi^{\mathrm{tr}})^{-1})^{(n-1)})^{(1)} = V^{\mathrm{tr}}\Pi_{n-1}^{(1)}(\mathcal{M}^{-1})^{(n)}$$

and

$$\mathcal{S}_n := (\Phi^{\mathrm{tr}})^{(n)}(\Phi^{\mathrm{tr}})^{(n-1)} \cdots (\Phi^{\mathrm{tr}}) = \mathcal{M}^{(n)}\Pi_n^{-1}((V^{(-1)})^{-1})^{\mathrm{tr}} = \mathcal{M}^{(n)}(\Psi_n^{(-1)})^{\mathrm{tr}}.$$

For each $k \geq 1$, we further set

$$\widetilde{\mathcal{P}}_n^k := (t-\theta^q)^{-k} \cdots (t-\theta^{q^n})^{-k}\mathcal{P}_n$$

and

$$\widetilde{\mathcal{S}}_n^k := (t-\theta)^k(t-\theta^q)^k \cdots (t-\theta^{q^n})^k\mathcal{S}_n.$$

Thus, for any $\tilde{n}\in\tilde{N}_\rho$ ($\tilde{m}\in M_\rho$ resp.) given by $\tilde{n}=\sum_{i=1}^r a_i\mathfrak{d}_i$ ($\tilde{m}=\sum_{i=1}^r b_i\mathfrak{c}_i$ resp.), using (2.7), (2.11) and (2.12), we have

$$\sigma^{-n}(\tilde{n})=\widetilde{\mathcal{P}}_n^k\begin{bmatrix}a_1\\ \vdots\\ a_r\end{bmatrix}^{(n)} \quad\text{and}\quad \tau^n(\tilde{m})=[b_1,\dots,b_r]^{(n)}\widetilde{\mathcal{S}}_{n-1}^k. \tag{4.4}$$

Put $\tilde{\gamma}:=\sum_{j=2}^{r+1}\sigma(h_{r(k-1)+j})\otimes g_{j-1}+\sum_{j=1}^r k_j^{(-1)}\sigma(h_{rk+1})\otimes g_j$. Then, by using (4.1) as well as the definition of $\sigma-$ and $\tau-$ action on $\tilde{N}_\rho$ and M_ρ respectively, we have

$$\begin{aligned}
&(1\otimes t-t\otimes 1)G_{k,n}^{\otimes}+\tilde{\gamma}\\
&=\sigma^{-n}(h_{r(k-1)+2})\otimes\tau^{n+1}(g_1)+\cdots+\sigma^{-n}(h_{rk})\otimes\tau^{n+1}(g_{r-1})+\sigma^{-n}(h_{rk+1})\otimes\tau^{n+1}(\mathfrak{c}_1)\\
&=\sum_{\mu=1}^{rn+1}\sigma^{-n}(h_\mu)\tau^n(\mathfrak{g}^{\mathrm{tr}}\Theta_{\rho,\tau}^{\mathrm{tr}}\mathfrak{f}_\mu)\\
&=\widetilde{\mathcal{P}}_n^k(\tilde{V}^{\mathrm{tr}})^{(n)}\mathfrak{e}_1\otimes\mathfrak{e}_1^{\mathrm{tr}}(((\tilde{V}^{(-1)})^{-1})^{\mathrm{tr}})^{(n+1)}\widetilde{\mathcal{S}}_n^k+\cdots+\widetilde{\mathcal{P}}_n^k(\tilde{V}^{\mathrm{tr}})^{(n)}\mathfrak{e}_r\otimes\mathfrak{e}_r^{\mathrm{tr}}(((\tilde{V}^{(-1)})^{-1})^{\mathrm{tr}})^{(n+1)}\widetilde{\mathcal{S}}_n^k\\
&=(t-\theta^q)^{-k}\cdots(t-\theta^{q^n})^{-k}V^{\mathrm{tr}}\Pi_{n-1}^{(1)}\eta_n(\Psi_n^{(-1)})^{\mathrm{tr}}(t-\theta)^k(t-\theta^q)^k\cdots(t-\theta^{q^n})^k\\
&=(-1)^k\left((-\theta)^{q/(q-1)}\prod_{i=1}^n\left(1-\frac{t}{\theta^{q^i}}\right)^{-1}\right)^kV^{\mathrm{tr}}\Pi_{n-1}^{(1)}\eta_n(\Psi_n^{(-1)})^{\mathrm{tr}}\left((-\theta)^{-1/(q-1)}\prod_{i=0}^n\left(1-\frac{t}{\theta^{q^i}}\right)\right)^k.
\end{aligned}\tag{4.5}$$

Using Theorem 3.1, (3.6) and Theorem 4.2, we can now easily obtain the following result.

Theorem 4.3. *We have*

$$\lim_{n\to\infty}\left((1\otimes t-t\otimes 1)G_{k,n}^{\otimes}+\tilde{\gamma}\right)=\lim_{n\to\infty}\left(\sum_{\mu=1}^{rn+1}\sigma^{-n}(h_\mu)\tau^n(\mathfrak{g}^{\mathrm{tr}}\Theta_{\rho,\tau}^{\mathrm{tr}}\mathfrak{f}_\mu)\right)=(-1)^kV^{\mathrm{tr}}\tilde{\Upsilon}^{(1)}(\tilde{\Psi}^{\mathrm{tr}})^{(-1)}.$$

Before we state the main result of this section, we define $\tilde{\mathfrak{e}}_1:=v_{1r}\mathfrak{e}_1^{\mathrm{tr}}$ and for $2\le j\le r$, we set

$$\tilde{\mathfrak{e}}_j:=\sum_{i=j}^r v_{i(r-(j-1))}\mathfrak{e}_i^{\mathrm{tr}}.$$

Recall the projection $p_i:\mathbb{C}_\infty^{rk+1}\to\mathbb{C}_\infty$ onto the i-th coordinate as well as the entire functions $F_{\tau^i}:\mathbb{C}_\infty\to\mathbb{C}_\infty$ for each $1\le i\le r-1$ defined in §3. Recall also the fundamental periods $\lambda_1,\dots,\lambda_r$ of ϕ defined in §3.

Theorem 4.4. *Let $\mathbf{z}\in\mathbb{C}_\infty^{rk+1}$ be an element in the domain of convergence of Log_ρ. Then, for any $1\le j\le r$, we have*

$$\begin{aligned}
p_{rk+1-(j-1)}(\operatorname{Log}_\rho(\mathbf{z}))&=\delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{(-1)^k}{t-\theta}\tilde{\mathfrak{e}}_jV^{\mathrm{tr}}(\tilde{\Upsilon}^{(1)})_{|t=\theta}(\tilde{\Psi}^{\mathrm{tr}})^{(-1)}\right)\\
&=\delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}\tilde{\mathfrak{e}}_jV^{\mathrm{tr}}(\Upsilon^{(1)})_{|t=\theta}(\Psi^{\mathrm{tr}})^{(-1)}\right).
\end{aligned}$$

In particular, we have

$$p_{rk+1-(j-1)}(\mathrm{Log}_\rho(\mathbf{z})) = \begin{cases} \delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{\tilde{\pi}^k}{\omega_C^k(\theta-t)}(\lambda_1,\dots,\lambda_r)(\Psi^{\mathrm{tr}})^{(-1)}\right) & \text{if } j=1 \\ \delta_{1,\mathbf{z}}^{M_\rho}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(F_{\tau^{r-(j-1)}}(\lambda_1),\dots,F_{\tau^{r-(j-1)}}(\lambda_r))(\Psi^{\mathrm{tr}})^{(-1)}\right) & \text{if } 2\leq j\leq r \end{cases}.$$

Proof. Let

$$\begin{aligned}(1\otimes t - t\otimes 1)G_{k,n}^{\otimes}+\tilde{\gamma} &= \sum_{\ell=1}^{r-1}\sigma^{-n}(h_{r(k-1)+\ell+1})\otimes\tau^{n+1}(g_\ell)+\sigma^{-n}(h_{rk+1})\otimes\tau^{n+1}(\mathfrak{c}_1)\\ &= \sum_{\mu=1}^{rn+1}\sigma^{-n}(h_\mu)\tau^n(\mathfrak{g}^{\mathrm{tr}}\Theta_{\rho,\tau}^{\mathrm{tr}}\mathfrak{f}_\mu)\\ &= \begin{bmatrix}\xi_{11}\\ \vdots\\ \xi_{1r}\end{bmatrix}\otimes[\psi_{11},\dots,\psi_{1r}]+\cdots+\begin{bmatrix}\xi_{r1}\\ \vdots\\ \xi_{rr}\end{bmatrix}\otimes[\psi_{r1},\dots,\psi_{rr}]\in\mathbb{T}^r\otimes\mathbb{T}^r.\end{aligned}$$

Again by Theorem 3.1(i) and (3.8), we see that for each $1\leq i,j\leq r$, ξ_{ij} is well-defined at $t=\theta$. Moreover, by the definition of $\delta_0^{N_\rho}$-map given in (4.3), we have

$$\sum_{\mu=1}^{rn+1}\delta_0^{N_\rho}(\sigma^{-n}(h_\mu))\tau^n(\mathfrak{g}^{\mathrm{tr}}\Theta_{\rho,\tau}^{\mathrm{tr}}\mathfrak{f}_\mu) = \tag{4.6}$$

$$\begin{pmatrix} * \\ \vdots \\ * \\ (\xi_{1j})|_{t=\theta}v_{j1} \\ \sum_{j=r-1}^r(\xi_{1j})|_{t=\theta}v_{j2} \\ \vdots \\ \sum_{j=2}^r(\xi_{1j})|_{t=\theta}v_{j(r-1)} \\ (\xi_{11})|_{t=\theta}v_{1r}\end{pmatrix}(\psi_{11}\mathfrak{c}_1+\cdots+\psi_{1r}\mathfrak{c}_r)+\cdots+\begin{pmatrix} * \\ \vdots \\ * \\ (\xi_{rj})|_{t=\theta}v_{j1} \\ \sum_{j=r-1}^r(\xi_{rj})|_{t=\theta}v_{j2} \\ \vdots \\ \sum_{j=2}^r(\xi_{rj})|_{t=\theta}v_{j(r-1)} \\ (\xi_{r1})|_{t=\theta}v_{1r}\end{pmatrix}(\psi_{r1}\mathfrak{c}_1+\cdots+\psi_{rr}\mathfrak{c}_r).$$

Thus, similar to the situation in Theorem 3.9, the first assertion follows from Theorem 2.9, Theorem 4.3 and (4.6). The second assertion mainly follows from the definition of $\tilde{\Upsilon}$ and $\tilde{\Psi}$. For the last assertion, observe that the identity $(\tilde{V}^{-1})^{\mathrm{tr}}\tilde{V}^{\mathrm{tr}}=\mathrm{Id}_r$ implies

$$\tilde{\mathfrak{e}}_jV^{\mathrm{tr}}=\begin{cases}(k_1,\dots,k_r) & \text{if } j=1\\ \mathfrak{e}_{r-(j-1)}^{\mathrm{tr}} & \text{if } 2\leq j\leq r.\end{cases}$$

Thus, the desired result follows from the first assertion and [4, (3.4.3), (3.4.5)]. □

Let ϕ be a Drinfeld module of rank 2 given as in (1.4) such that $k_1,k_2\in\mathbb{F}_q$. In what follows, we briefly explain how our formulas in Theorem 1.5 are related to special values of Goss L-function of ϕ defined in (1.6). Let us consider the Drinfeld module $\tilde{\phi}$ given by

$$\tilde{\phi}_\theta := \theta - k_1k_2^{-1}\tau+k_2^{-1}\tau^2.$$

By [11, Rem. 5.6], we know that $L(\phi,1)=L(\tilde{\phi}^\vee,0)$. Now, for $k\geq 1$, let $\mathbf{z}_i\in \mathrm{Mat}_{(2k+1)\times 1}(\mathbb{F}_q)$ be the i-th unit vector. Using [11, Thm. 5.9], we have

$$L(\phi,k+1)=\det\begin{bmatrix} p_{2k}(\mathrm{Log}_\rho(\mathbf{z}_{2k})) & p_{2k}(\mathrm{Log}_\rho(\mathbf{z}_{2k+1})) \\ p_{2k+1}(\mathrm{Log}_\rho(\mathbf{z}_{2k})) & p_{2k+1}(\mathrm{Log}_\rho(\mathbf{z}_{2k+1}))\end{bmatrix}.$$

Set $\tilde{\rho}:=\tilde{\phi}\otimes C^{\otimes k}$. Thus, one can obtain the following corollary of Theorem 4.4.

Corollary 4.5. *Let k and $\mathbf{z}_i$ for $i=1,2$ be as above. We have*

$$L(\phi,k+1)=$$

$$\det\begin{bmatrix} \delta^{M_{\tilde{\rho}}}_{1,\mathbf{z}_{2k}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(\tilde{F}_\tau(\lambda_1),\tilde{F}_\tau(\lambda_2))(\Psi^{\mathrm{tr}}_{\tilde{\phi}})^{(-1)}\right) & \delta^{M_{\tilde{\rho}}}_{1,\mathbf{z}_{2k+1}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(\tilde{F}_\tau(\lambda_1),\tilde{F}_\tau(\lambda_2))(\Psi^{\mathrm{tr}}_{\tilde{\phi}})^{(-1)}\right) \\ \delta^{M_{\tilde{\rho}}}_{1,\mathbf{z}_{2k}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(-\lambda_1,-\lambda_2)(\Psi^{\mathrm{tr}}_{\tilde{\phi}})^{(-1)}\right) & \delta^{M_{\tilde{\rho}}}_{1,\mathbf{z}_{2k+1}}\left(\frac{\tilde{\pi}^k}{\omega_C^k(t-\theta)}(-\lambda_1,-\lambda_2)(\Psi^{\mathrm{tr}}_{\tilde{\phi}})^{(-1)}\right)\end{bmatrix}.$$

where $\Psi_{\tilde{\phi}}$ is the matrix defined as in (1.5) with respect to $\tilde{\phi}$ and $\tilde{F}:\mathbb{C}_\infty\to\mathbb{C}_\infty$ is the unique entire function satisfying

$$\tilde{F}_\tau(\theta z)-\theta\tilde{F}_\tau(z)=\exp_{\tilde{\phi}}(z)^q$$

for all $z\in\mathbb{C}_\infty$.

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