A MOTIVIC PAIRING AND THE MELLIN TRANSFORM IN FUNCTION FIELDS

NATHAN GREEN

ABSTRACT. We define two pairings relating the A-motive with the dual A-motive of an abelian Anderson A-module. We show that specializations of these pairings give the exponential and logarithm functions of this Anderson A-module, and we use these specializations to give precise formulas for the coefficients of the exponential and logarithm functions. We then use this pairing to express the exponential and logarithm functions as evaluations of certain infinite products. As an application of these ideas, we prove an analogue of the Mellin tranform formula for the Riemann zeta function in the case of Carlitz zeta values. We also give an example showing how our results apply to Carlitz multiple zeta values.

CONTENTS

1. INTRODUCTION

The Mellin transform is a useful and ubiquitous tool in analytic number theory. It appears in many formulas involving the Riemann zeta function and related special functions. Recall the definition of the Mellin transform for a real-valued function $f(x)$ with suitable decay conditions at $x = 0$ and $x = \infty$,

$$(1.1) \qquad M(f)(s) = \int_0^\infty f(x)x^{s-1}dx,$$

for suitable $s \in \mathbb{C}$ (see [24] for details and convergence discussion). A classical formula relates the Mellin transform of the exponential function to the Riemann zeta function and

the gamma function. Let $f = 1/(e^x - 1)$. Then we have

(1.2) $$M\left(\frac{1}{e^x - 1}\right)(s) = \Gamma(s)\zeta(s).$$

One of the main results in this paper establishes an analogue of the above formula for function fields of curves defined over a finite field. Developing integration techniques for characteristic p function fields has traditionally been very difficult due to the difficulty of defining characteristic p additive measures. In this paper, we sidestep the need for integration by instead defining maps arising from the motivic structure of certain t-modules related to the Carlitz zeta function which take the place of integration in this specific situation.

We now briefly state a special case of the main theorem of this paper and explain why this should be viewed as the correct analogue of the Mellin transform. Let $q = p^r$ be a prime power, and let $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$ (in most sections of the paper we actually prove our results for an arbitrary function field). Let $K_\infty = \mathbb{F}_q((1/\theta))$ and let $\mathbb{C}_\infty$ be a completion of an algebraic closure of K_∞. Set $u = \frac{\tilde{\pi}}{t-\theta} \in \mathbb{C}_\infty(t)$, where $\tilde{\pi}$ is the Carlitz period (see (5.5)) and t is an independent variable. For a specified element $\mathbf{z} \in \mathbb{C}_\infty^n$ (n is the dimension of a particular t-module) we define a map $\delta_{1,\mathbf{z}}^M$ from a t-motive M to $\mathbb{C}_\infty$, which we view as an analogue of integration. We then have the following special case of our main theorem.

Theorem 1.1. *(Special case of Corollary 5.9) Set $n = 1$. Then we have*

$$\delta_{1,\mathbf{z}}^M\left(\frac{u}{\exp_C(u)}\right) = \Gamma_A(n)\zeta_A(n) \in K_\infty,$$

where $\exp_C$ is the Carlitz exponential function, and Γ_A and ζ_A are function field versions of the gamma and Riemann zeta function, respectively (see (5.7) and (5.8)).

Similar formulas also hold for integers $n > 1$, but the notation is more cumbersome, so we refer the reader to §5 for details.

We make a brief comparison between the integral formula for the Mellin transform (1.1) and the definition of our map $\delta_{1,\mathbf{z}}^M$ (see Def. 2.14). The Mellin transform satisfies the functional equation (for suitable $s \in \mathbb{C}$)

$$M(xf(x))(s) = M(f(x))(s + 1),$$

thus it turns multiplication by x against the function into addition in the complex argument. On the other hand, in Proposition 2.15 we show for $\mathbf{z} \in \mathbb{C}_\infty^n$

$$\delta_{1,\mathbf{z}}^M(tf(t)) = \delta_{1,\phi_t(\mathbf{z})}^M(f(t)),$$

so our map $\delta_{1,\mathbf{z}}^M$ turns multiplication by t against the function into the *Anderson A-module action* in the argument. Since we are in characteristic p, it is not natural to expect multiplication to turn into addition (else multiplication by t^p would cause the transform to vanish). The Anderson A-module action is the natural replacement.

On the other hand, we observe that the Mellin transform zeta values formula (1.2) is valid for all complex values $s \neq 1$, whereas our formula Corollary 5.9 is valid only for positive integer values n. Thus we interpret our formulas as giving an algebraic interpolation of the Mellin transform formula at integer values, rather than being a true replacement for the Mellin transform. However, strong similarities remain. For example, we are able to

recover function field gamma values using $\delta_{1,\mathbf{z}}^M$ (see Remark 5.12) which parallels the classical definition of the gamma function as a Mellin transform.

To solidify the analogy between these two Mellin transform formulas, we recall the classical definition of Bernoulli numbers,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

Comparing this with the definition of the Carlitz-Bernoulli numbers (originally due to Carlitz [6]) as found in [23],

$$\frac{z}{\exp_C(z)} = \sum_{n=0}^{\infty} \frac{B_{n,C} z^n}{\Gamma_n},$$

we see that $\frac{z}{\exp_C(z)}$ is the natural function field analogue of the term $\frac{1}{e^x-1}$ (the extra copy of x in the numerator is due to normalization issues). The Bernoulli numbers, both over $\mathbb{C}$ and over function fields, are intimately connect with special values of zeta functions, which strongly supports our assertion that Corollary 5.9 should be viewed as an analogue of the Mellin transform.

The truly exciting part about these new formulas, is that we prove them in a much more general setting than what we have discussed above. We give a "product formula" (Cor. 5.6) for the logarithm and exponential functions of an arbitrary abelian Anderson A-module, which in the particular case of tensor powers of the Carlitz module specifies to Theorem 1.1. We call it a "product formula" because it is only a true product formula for simple cases; in general our formulas gives a finite sum of infinite matrix products. In particular, this formula applies to the case of t-modules associated to function field multiple zeta values (defined in (5.11)). As an example application of our main theorems, we give a formula of the the form

$$\text{"product formula"} = (\theta^2 + \theta)\zeta_A(1,3),$$

where the left hand side is a sum of terms which look like a higher dimensional version of $\frac{u}{\exp_C(u)}$, expressed as an infinite product (see (5.13)). Thus, we are able to realize function field multiple zeta values (MZVs) using a generalization of our Mellin transform formula. This mirrors the situation for complex valued MZVs, which can be expressed as a higher-dimensional Mellin transform (see [11, (2.1)]). In Section 5 we give one specific example of this behavior; we plan to study such formulas in more generality in a future project. In Remark 5.14 we discuss how zeta values potentially appear in the terms of our "product formula" and we are hopeful that the LHS of our formula can be decomposed into terms involving zeta values and MZV, thus giving new relations between MZV. This also is the topic of a future project.

The major innovation in this paper is a new technique we develop, which we call a motivic pairing. This pairing allows us to give explicit formulas for the coefficients of the exponential and logarithm functions of Anderson A-modules (see Def. 2.7; original definition in [1]). Our formulas apply in a very general setting, and give large improvements over previously known formulas for the coefficients of the exponential and logarithm. For example, they subsume the powerful results contained in [3, Prop. 2.2] in the case of A-finite t-modules.

We now briefly discuss the setting of our main theorems and briefly describe the contents of each section. An Anderson A-module ϕ of dimension d is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : A \to \mathrm{Mat}_d(\mathbb{C}_\infty)[\tau],$$

where $\mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$ is a skew polynomial ring where we view τ acting via the q-power Frobenius (see 2.1). Associated to the Anderson A-module ϕ, there is an exponential function $\mathrm{Exp}_\phi : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ which serves a similar role as the exponential map of a Lie group. The inverse function of Exp_ϕ is called the logarithm function, Log_ϕ. As functions from $\mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$, both Exp_ϕ and Log_ϕ are $\mathbb{F}_q$-linear power series with coefficients in $\mathrm{Mat}_d(\mathbb{C}_\infty)$. For each Anderson A-module we functorially define an associated A-motive and dual A-motive. We review this theory in detail in Section 2.

The main results of Sections 3 and 4 give explicit formulas for the coefficients of the exponential and logarithm functions of an arbitrary Anderson A-module subject to a couple technical conditions (see Defs. 2.3 and 2.12). These formulas are contained in Corollaries 3.8 and 4.5. Our main technique is to construct two pairings between the associated A-motive and dual A-motive of ϕ (see Def. 2.3 for the definition of A-motives). We then show that a specialization of these parings returns the exponential and logarithmic functions and this allows us to prove precise formulas for Exp_ϕ and Log_ϕ in Theorems 3.7 and 4.4. We view this A-motivic set up as being a very natural setting to study these coefficients, and our formulas subsume many previous ad hoc approaches (see [20], [14], [9]). As an immediate application of our formulas, we give a sufficient condition for the exponential and logarithm coefficients of t-modules to be invertible in Theorems 4.7 and 4.8.

Using this motivic set up, in Section 5 we develop a pair of "product formulas" for the exponential and logarithm functions in Theorem 5.5. We specialize these product formulas to the case of t-modules related to zeta values and multiple zeta values to give a function field version of the Mellin transform formula for the Riemann zeta function in Theorem 5.9 and Example 5.13. We also give an additional brief application of these new techniques in Section 6 to recover and slightly extend a result of Anderson related to log-algebraicity.

2. Review of Anderson Motives and Modules

In this section we review the definitions and much of the theory involved in A-motives, dual A-motives and Anderson A-modules. Most of these results are originally due to Anderson [1], although here we follow the exposition of Hartl and Juschka [17]. We also prove several new results which will be used in subsequent sections to define our motivic pairings. Let $\mathbb{F}_q$ be the finite field of size $q = p^r$ and let X be a projective, smooth, geometrically connected curve defined over $\mathbb{F}_q$ with fixed point at infinity $\infty \in X(\mathbb{F}_q)$. Let K be the function field of X and let A be the functions on X regular away from ∞. Let K_∞ be the completion of K at the place of K corresponding to ∞, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. Let L be an algebraically closed subfield with $K \subset L \subset \mathbb{C}_\infty$. We comment that in some subsections we will set $L = \mathbb{C}_\infty$, and the reader doesn't lose much generality by simply making this assumption throughout the paper. We view $L \otimes_{\mathbb{F}_q} K$ as the function field of the curve $L \times_{\mathbb{F}_q} X$ and $L \otimes_{\mathbb{F}_q} A$ as its coordinate ring. Let $\Xi \in X(K)$ be a pull back of the point $t \in \mathbb{P}^1(\mathbb{F}_q(t))$ under the projection $X \to \mathbb{P}^1$.

Definition 2.1. Let $(L \otimes_{\mathbb{F}_q} A)[\tau]$ be the skew polynomial ring with commutativity relation

$$\tau(\ell \otimes a) = (\ell^q \otimes a)\tau, \quad \forall \ell \otimes a \in L \otimes_{\mathbb{F}_q} A,$$

where everything else commutes. We similarly define $(L \otimes_{\mathbb{F}_q} A)[\sigma]$, where $\sigma = \tau^{-1}$. When there is no confusion, we will sometimes refer to $L \otimes 1 \subset L \otimes A$ simply as $L \subset L \otimes A$ and similarly for $A \subset L \otimes A$.

Definition 2.2. Let J be the maximal ideal of $L \otimes_{\mathbb{F}_q} A$ generated by $(1 \otimes a) - (a \otimes 1)$ for all $a \in A$ (recall $A \subset L$).

Definition 2.3. An abelian A-motive of rank r and dimension d is an $(L \otimes_{\mathbb{F}_q} A)[\tau]$-module M which is a free $L[\tau]$ module (via $L \otimes 1$) of rank d and a finitely generated, projective $L \otimes A$ module of constant local rank r such that for $n \in \mathbb{Z}$ sufficiently large we have

$$J^n M / \tau M = 0.$$

Definition 2.4. An A-finite dual A-motive of rank r and dimension d is an $(L \otimes_{\mathbb{F}_q} A)[\sigma]$-module N which is a free $L[\sigma]$ module (via $L \otimes 1$) of rank d and a finitely generated, projective $L \otimes A$ module of constant local rank r such that for $n \in \mathbb{Z}$ sufficiently large we have

$$J^n N / \sigma N = 0.$$

Definition 2.5. In this article, we will assume all the A-motives and dual A-motives are Abelian and A-finite, respectively, and will hence shorten this to simply A-motive or dual A-motive.

Definition 2.6. For an element $h = z \otimes f \in L \otimes A$, we define the ith Frobenius twist of h to be

$$h^{(i)} = z^{q^i} \otimes f \in L \otimes A.$$

We extend this definition coordinate-wise to vectors and matrices of elements in $L \otimes A$. We also extend this definition to points on X by twisting each coordinate of X.

Definition 2.7. A d-dimensional Anderson A-module is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathrm{Mat}_d(\mathbb{C}_\infty)[\tau]$, such that for each $a \in A$,

$$\phi_a = d[a] + A_1\tau + \dots, \quad A_i \in \mathrm{Mat}_d(\mathbb{C}_\infty)$$

where $d[a] = aI + N$ for some nilpotent matrix $N \in \mathrm{Mat}_d(\mathbb{C}_\infty)$ (depending on a). A Drinfeld A-module is a dimension $d = 1$ Anderson A-module. A t-module is a d-dimensional Anderson A-module for which $A = \mathbb{F}_q[t]$ (we have $X = \mathbb{P}^1$).

The map $d : A \to \mathrm{Mat}_n(\overline{K}_\infty)$, where $d[a]$ is the constant term of ϕ_a, is a ring homomorphism, and we will use this notation regularly throughout the paper. For more details on these constructions we refer the reader to [12, §5]

Definition 2.8. We define a map $* : (L \otimes A)[\tau] \to (L \otimes A)[\sigma]$, for $\alpha = \sum_{i=0}^m a_i \tau^i \in (L \otimes A)[\tau]$, by setting

$$\alpha^* = \sum_{i=0}^m a_i^{(-i)} \sigma^i.$$

We extend this map to matrices $B \in \mathrm{Mat}_{m \times n}((L \otimes A)[\tau])$ by setting B^* to be matrix $B^\top$ with the $*$ map applied to each coordinate (here $^\top$ is the transpose). Note that we have $(AB)^* = B^* A^*$.

To each Anderson A-module ϕ, there exists an associated Anderson A-motive and dual A-motive, which we denote by M_ϕ and N_ϕ. This correspondence is given functorially by

$$M_\phi = \mathrm{Hom}_{\mathbb{F}_q}(\phi, \mathbb{G}_a) \cong \mathrm{Mat}_d(L[\tau]), \quad N_\phi = \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_a, \phi) \cong L[\tau]^d,$$

(the underlying algebraic group of ϕ is $\mathbb{G}_a^d$) where $a \in A$, $b \in L$ and τ act on $m \in M_\phi$ by

$$a \cdot m = m \circ \phi_a, \quad b \cdot m = bm, \quad \tau \cdot m = \tau m,$$

(here τ acts as the q-power Frobenius) and $a \in A$, $b \in L$ and σ act on $n \in N_\phi$ by

$$a \cdot n = \phi_a \circ n, \quad b \cdot n = bn, \quad \sigma \cdot m = m\tau.$$

Thus M_ϕ and N_ϕ are $(L \otimes A)[\tau]-$ and $(L \otimes A)[\sigma]-$modules, and it is shown in [17, §2.5.2] that they are A-motives and dual A-motives, respectively. Note that we may identify $N_\phi = \mathrm{Mat}_{1 \times d}(L[\sigma])$, in which case the actions can be written as

$$a \cdot n = n \circ \phi_a^*, \quad b \cdot n = bn, \quad \sigma \cdot m = \sigma m.$$

For details and proofs of the above facts we refer the reader to [17, §2.5] and [5, §4-5]. In a recent paper [18], Maurischat proved an important relationship between M_ϕ and N_ϕ.

Theorem 2.9 (Maurischat). *For an Anderson A-module ϕ, we have that M_ϕ is abelian if and only if N_ϕ is A-finite.*

Definition 2.10. Each Anderson A-module ϕ has an associated exponential and logarithm function. The exponential function is the unique $\mathbb{F}_q$-linear power series, denoted

$$\mathrm{Exp}_\phi(\mathbf{z}) = \sum_{i=0}^\infty Q_i \mathbf{z}^{(i)},$$

for $\mathbf{z} \in \mathbb{C}_\infty^d$ with $Q_i \in \mathrm{Mat}_{d \times d}(\mathbb{C}_\infty)$, such that $Q_0 = I$ and

$$\mathrm{Exp}_\phi(d[a]\mathbf{z}) = \phi_a(\mathrm{Exp}_\phi(\mathbf{z}))$$

for all $a \in A$. The exponential function is entire as a function $\mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$.

The logarithm function is defined as the formal power series inverse of Exp_ϕ and is denoted

$$\mathrm{Log}_\phi(\mathbf{z}) = \sum_{i=0}^\infty P_i \mathbf{z}^{(i)}.$$

It is the unique $\mathbb{F}_q$-linear function with $P_i = I$ satisfying

$$\mathrm{Log}_\phi(\phi_a \mathbf{z}) = d[a](\mathrm{Log}_\phi(\mathbf{z})).$$

We note that the logarithm function has some bounded domain of convergence in $\mathbb{C}_\infty^d$.

Throughout the remainder of this paper we will fix an Anderson A-module ϕ. We will denote the associated A-motive and dual A-motive by M and N respectively, suppressing the ϕ from their notation.

Let $M_K = M \otimes_A K$ be the $(L \otimes_{\mathbb{F}_q} K)[\tau]$-module where the τ-action on K is trivial. Define $N_K = N \otimes_A K$ similarly where the σ action is also trivial on K. Note that M_K and N_K are $L \otimes_{\mathbb{F}_q} K$-free modules of rank r.

Lemma 2.11. *Let ϕ be an Anderson A-module and let M and N be the associated A-motive and dual A-motive, respectively.*

   (1) We may choose $L \otimes_{\mathbb{F}_q} K$-bases for M_K and N_K, denoted $\{c_1, \ldots, c_r\}$ and $\{d_1, \ldots, d_r\}$ respectively, such that if the τ-action on M_K in this basis is given by

$$\tau(c_1, \ldots, c_r)^\top = \Phi(c_1, \ldots, c_r)^\top,$$

for $\Phi \in \mathrm{Mat}_{r \times r}(L \otimes K)$, then the σ-action on N is given by

$$\sigma(d_1, \ldots, d_r)^\top = \Phi^\top (d_1, \ldots, d_r)^\top.$$

Further, we may choose these bases such that Φ is lower triangular (and $\Phi^\top$ is upper triangular).

(2) *We may choose an $L[\tau]$-basis for M and an $L[\sigma]$-basis for N, denoted $\{g_1, \ldots, g_d\}$ and $\{h_1, \ldots, h_d\}$ respectively, such that for $b \in 1 \otimes A$, if the action of b on M is given by the matrix*

$$b(g_1, \ldots, g_d)^\top = \Theta_b(g_1, \ldots, g_d)^\top,$$

for $\Theta_b \in \mathrm{Mat}_{d \times d}(L[\tau])$, then the action of b on N is give by the matrix

$$b(h_1, \ldots, h_d)^\top = \Theta_b^*(h_1, \ldots, h_d)^\top,$$

where Θ_b^* is defined in 2.8.

Proof. The proof of (1) follows from [17, Prop. 2.4.3]. By definition 2.3, it follows that $\det(\Phi) \in J^d$. From here, it is a simple linear algebra exercise to see that we may choose Φ to be lower triangular. The proof of part (2) follows from [18, (5.2)] (see also [5, §4.4]). $\square$

Definition 2.12. We fix the bases of Lemma 2.11 for use throughout the paper. Further, given such bases and the matrix Φ as defined above, we say that the Anderson A-module ϕ is Ξ-regular if $\det(\Phi)$ and each coordinate of Φ is regular at $\Xi^{(i)}$ for all $i \in \mathbb{Z}$. Note that if $A = \mathbb{F}_q[\theta]$, then $\Phi \in \mathrm{Mat}_d(\mathbb{C}_\infty[t])$ and $\det(\Phi) \in (t - \theta)^d \mathbb{C}_\infty$, thus all t-modules are automatically Ξ-regular. Throughout the rest of the paper we will assume that all our Anderson A-modules are Ξ-regular.

Definition 2.13. For an Anderson A-module ϕ, let N be the associated dual A-motive with rank r and dimension d and let $\{h_1, \ldots, h_d\} \subset N$ be an $L[\sigma]$-basis for N as described in Lemma 2.11. We define two maps,

$$\delta_0^N, \delta_1^N : N \to L^d,$$

for $n \in N$ by first expressing n in the basis $\{h_1, \ldots, h_d\}$,

(2.1)
$$
\begin{aligned}
n =& c_{1,0} h_1 + c_{1,1} \sigma(h_1) + \cdots + c_{1,m} \sigma^k(h_1) \\
&+ c_{2,0} h_2 + c_{2,1} \sigma(h_2) + \cdots + c_{2,m} \sigma^k(h_2) \\
&\vdots \\
&+ c_{d,0} h_d + c_{d,1} \sigma(h_d) + \cdots + c_{d,m} \sigma^k(h_d),
\end{aligned}
$$

then writing

(2.2)
$$\delta_0^N(n) = \begin{pmatrix} c_{d,0} \\ c_{d-1,0} \\ \vdots \\ c_{1,0} \end{pmatrix}$$

and

(2.3)
$$\delta_1^N(n) = \begin{pmatrix} c_{d,0} \\ c_{d-1,0} \\ \vdots \\ c_{1,0} \end{pmatrix} + \begin{pmatrix} c_{d,1} \\ c_{d-1,1} \\ \vdots \\ c_{1,1} \end{pmatrix}^{(1)} + \cdots + \begin{pmatrix} c_{d,m} \\ c_{d-1,m} \\ \vdots \\ c_{1,m} \end{pmatrix}^{(m)}.$$

Definition 2.14. For an Anderson A-module ϕ, let M be the associated A-motive with rank r and dimension d and let $\{g_1, \ldots, g_d\} \subset M$ be an $L[\tau]$-basis for M as described in Lemma 2.11. Fix an element $\mathbf{z} \in L^d$. We define two maps,

$$\delta_0^M : M \to L^d, \quad \delta_{1,\mathbf{z}}^M : M \to L$$

for $m \in M$ by first expressing m in the basis $\{g_1, \ldots, g_d\}$,

(2.4)
$$
\begin{aligned}
m = & c_{1,0} g_1 + c_{1,1} \tau(g_1) + \cdots + c_{1,m} \tau^k(g_1) \\
& + c_{2,0} g_2 + c_{2,1} \tau(g_2) + \cdots + c_{2,m} \tau^k(g_2) \\
& \vdots \\
& + c_{d,0} g_n + c_{d,1} \tau(g_d) + \cdots + c_{d,m} \tau^k(g_d),
\end{aligned}
$$

then writing (notice the reversal of order in the vector compared Definition 2.13)

(2.5)
$$\delta_0^M(m) = \begin{pmatrix} c_{1,0} \\ c_{2,0} \\ \vdots \\ c_{d,0} \end{pmatrix}.$$

For the definition of $\delta_{1,\mathbf{z}}^M$, we recall that $M = \operatorname{Hom}_{\mathbb{F}_q}(\phi, \mathbb{G}_a) \cong \operatorname{Mat}_{1 \times d}(L[\tau])$, where $a \in A$ and $b \in L$ act on $m \in M$ via

$$am = m \circ \phi_a, \quad bm = b \cdot m.$$

We then define

(2.6)
$$\delta_{1,\mathbf{z}}^M(m) = m(\mathbf{z}).$$

Note that this definition depends on the element $\mathbf{z} \in L^d$, but in our applications we will often use it for an arbitrary element of L^d, so when convenient we will suppress this dependency in our notation and simply write δ_1^M.

Proposition 2.15. *For ϕ, $m \in M$, and $n \in N$ as given above, we have the following for all $a \in A$.*

(1) $\delta_1^N(an) = \phi_a(\delta_1^N(n))$
(2) $\delta_{1,\mathbf{z}}^M(am) = \delta_{1,\phi_a(\mathbf{z})}^M(m)$
(3) $\delta_0^N(an) = d[a]\delta_1^N(n)$
(4) $\delta_0^M(am) = d[a]^\top \delta_0^M(m)$.

Thus, both δ_1^M and δ_1^N allow us to recover the Anderson A-module ϕ, while both δ_0^M and δ_0^N allow us to recover the Lie action of ϕ.

Proof. These statements all follow from [17, Prop. 2.4.3 and Prop. 2.5.8]. They can also easily be seen by direct calculation following from the definitions. $\square$

Definition 2.16. For $c \in \mathbb{C}_\infty$, we define the Tate algebra in the variable t with radius c

$$\mathbb{T}_c = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \,\middle|\, \left| c^i b_i \right| \to 0 \right\}.$$

We then identify $\mathbb{F}_q[t]$ as a subring of A via a fixed injective ring morphism $\mathbb{F}_q[t] \to A$ (this is not unique) and define the ring

$$(2.7) \qquad \mathbb{Y}_c = (\mathbb{C}_\infty \otimes_{\mathbb{F}_q} A) \otimes_{\mathbb{C}_\infty[t]} \mathbb{T}_c.$$

Note that this ring is a ring of regular functions on an open rigid analytic subspace of the rigid analytic space associated to our curve X. We refer the reader to [17, §2.3.3] for full details on the rigid analytic geometry. We let τ (and σ) act on $\mathbb{Y}_c$ diagonally via Frobenius twisting (inverse twisting). Thus we let $L = \mathbb{C}_\infty$ and set $M_\theta = M \otimes_{(\mathbb{C}_\infty \otimes A)} \mathbb{Y}_\theta$ and $N_\theta = N \otimes_{(\mathbb{C}_\infty \otimes A)} \mathbb{Y}_\theta$ with diagonal τ and σ action, respectively. We set N_Γ to be the submodule of N_θ consisting of all $h \in N_\theta$ such that $\sigma(h) \in N_\theta$.

We note here that whenever we are dealing with a Tate algebra, we will set $L = \mathbb{C}_\infty$, so that our coefficients live in a complete field. However, this will mostly occur in Section 6, so most of the paper will be developed for general L.

Definition 2.17. We define a norm on $\mathbb{T}_c$ for $h = \sum_{i=0}^\infty b_i t^i \in \mathbb{T}_c$, by setting

$$\|h\|_c = \sup |c|_\infty^i \cdot |b_i|_\infty,$$

where $|\cdot|_\infty$ is the norm used for the completion K_∞. We then extend this norm to $\mathbb{Y}_c$ by recognizing that $\mathbb{Y}_c \cong \mathbb{T}_c^\ell$ where ℓ is the degree of $K/\mathbb{F}_q(t)$, and taking the max of the norm in each coordinate. Finally, we extend this norm to matrices $\mathrm{Mat}_{m \times m}(\mathbb{Y}_c)$, again by taking the max of the norm of each coordinate. Note that this matrix norm is sub-multiplicative, so for $A, B \in \mathrm{Mat}_{m \times m}(\mathbb{Y}_c)$ we have

$$\|AB\|_c \le \|A\|_c \cdot \|B\|_c.$$

Finally, we comment that the ring $\mathbb{Y}_c$ is complete in the defined norm. In particular in this paper, we are usually concerned with the norm $\|\cdot\|_\theta$, where $\theta \in \mathbb{C}_\infty$ an image of t under the inclusion $\mathbb{F}_q[t] \to A \to \mathbb{C}_\infty$.

Proposition 2.18. *There exists an extension of the map δ_0^N to a $\mathbb{C}_\infty \otimes A$-module homomorphism*

$$\delta_0^N : N_\theta \to \mathbb{C}_\infty^d.$$

There also exists an extension of δ_0^M to elements of $M_K = M \otimes K$ (here M is defined over an arbitrary L) whose coordinates are regular at Ξ when M_K is viewed as $(L \otimes A)^r$.

Proof. The proof of the first extension of δ_0^N is given in [17, Prop 2.5.8] and relies on the fact that the map δ_0^N factors through $N/J^d N$ (J is the maximal ideal generated by $(1 \otimes a) - (a \otimes 1)$). The proof of the second again follows quickly from the fact that δ_0^M also factors through $M/J^d M$. We note that these ideas are originally due to Anderson from unpublished work. $\qquad \square$

Definition 2.19. Note that $N \otimes_{L[\sigma]} L[[\sigma]] \cong L[[\sigma]]^d$, and thus for $n \in N \otimes_{L[\sigma]} L[[\sigma]]$, similarly to (2.1), we may write for $c_{i,j} \in L$

$$
\begin{aligned}
n = & c_{1,0} h_1 + c_{1,1} \sigma(h_1) + \ldots \\
& + c_{2,0} h_2 + c_{2,1} \sigma(h_2) + \ldots \\
& \vdots \\
& + c_{d,0} h_n + c_{d,1} \sigma(h_d) + \ldots.
\end{aligned}
$$

(2.8)

We let $N[[\sigma]]$ denote the subset of $n \in N \otimes_{L[\sigma]} L[[\sigma]]$ such that we have $(c_{1,j}, \ldots, c_{d,j})^{(j)} \to 0$ as $j \to \infty$ in the above expansion for n. Thus there exists an extension of the map $\delta_1^N :$ $N[[\sigma]] \to \mathbb{C}_\infty^d$ defined by applying δ_1^N to partial sums of (2.8), then taking the limit.

Using the $L \otimes_{\mathbb{F}_q} K$-basis $\{c_1, \ldots, c_r\}$ from Lemma 2.11, if we view $M_K \cong (L \otimes_{\mathbb{F}_q} K)^r$, then for $(a_1, \ldots, a_r) \in \mathrm{Mat}_{1 \times r}(L \otimes K)$, we find that

$$\tau(a_1, \ldots, a_r)(c_1, \ldots, c_r)^\top = (a_1, \ldots, a_r)^{(1)}\tau(c_1, \ldots, c_r)^\top = (a_1, \ldots, a_r)^{(1)}\Phi(c_1, \ldots, c_r)^\top.$$

After transposing, we may write the τ action of M viewed as $(L \otimes_{\mathbb{F}_q} K)^r$ as

$$\tau(a_1, \ldots, a_r)^\top = \Phi^\top((a_1, \ldots, a_r)^{(1)})^\top.$$

Similarly we find that $N_K \cong (L \otimes_{\mathbb{F}_q} K)^r$ and

$$\sigma(a_1, \ldots, a_r)^\top = \Phi((a_1, \ldots, a_r)^{(-1)})^\top.$$

Definition 2.20. For $m \in M_K$ we use the isomorphism in the preceding discussion to write $m = (a_1, \ldots, a_r) \in (L \otimes_{\mathbb{F}_q} K)^r$ and define

$$\tau^{-1}(m) = ((\Phi^{-1})^\top)^{(-1)}((a_1, \ldots, a_r)^{(-1)})^\top.$$

Note that the coordinates of Φ^{-1} are regular at $\Xi^{(i)}$ for $i \neq 0$ by Definition 2.12 and by the Cayley-Hamilton inverse formula, thus the coordinates of $(\Phi^{-1})^{(i)}$ are regular at Ξ for all $i < 0$. Thus if $m \in M \subset M_K$, then we may evaluate δ_0^M at $\tau^{-i}(m)$. We similarly define for $n \in N_K$

$$\sigma^{-1}(n) = (\Phi^{-1})^{(1)}((a_1, \ldots, a_r)^{(1)})^\top.$$

Note that if $n \in N \subset N_K$, then any poles of $\sigma^{-1}(n)$ will be at $\Xi^{(i)}$ for $i \geq 1$, and thus we may evaluate δ_0^N at $\sigma^{-1}(n)$. We further extend this σ-action to $n \in N_\theta$ and by similar arguments conclude that $\sigma^{-1}(n)$ is in N_θ and thus may be evaluated under δ_0^N.

We note that $\tau(\tau^{-1}(m)) = \tau^{-1}(\tau(m)) = m$ and $\sigma(\sigma^{-1}(n)) = \sigma^{-1}(\sigma(n)) = n$ for all $m \in M_K$ and $n \in N_K$.

3. The Exponential Motivic Pairing

The goal of this section is to define a pairing between the rings $(L \otimes_{\mathbb{F}_q} A)[\tau]$ and $(L \otimes_{\mathbb{F}_q} A)[\sigma]$, as well as a related pairing between the motives M and N, which specializes to give the exponential function. This will allow us to give extremely explicit formulas for the coefficients of the exponential function for any abelian Ξ-finite Anderson A-module.

Definition 3.1. Let ϕ be a Ξ-regular Anderson A-module. Let M and N be the associated A-motive and dual A-motive. Let $\{g_1, \ldots, g_d\}$ and $\{h_1, \ldots, h_d\}$ be the bases described in Lemma 2.11 above. For $x \in (L \otimes_{\mathbb{F}_q} A)[\tau]$ and $y \in (L \otimes_{\mathbb{F}_q} A)[\sigma]$ and $\mathbf{z} \in L^d$, define the pairing

$$F : (L \otimes_{\mathbb{F}_q} A)[\tau] \times (L \otimes_{\mathbb{F}_q} A)[\sigma] \to \mathbb{C}_\infty^d,$$

(3.1)
$$F(x, y; \mathbf{z}) = \delta_1^N \left(\sum_{i=0}^\infty \sum_{k=1}^d \delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z} \sigma^i(y(h_k))) \right)$$

Remark 3.2. We recall for the reader's convenience that $\delta_0^M(g) \in L^d$ for any $g \in M \otimes K$ whose coordinates are regular at Ξ. Thus, in the above summand, $\delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z} \in L$, and thus it makes sense to multiply this by $\sigma^i(y(h_k)) \in N$. Taking the sum for $i \geq 0$ gives an element of $N \otimes L[[\sigma]]$. We will see below that the sum is actually in the module $N[[\sigma]]$ (see Def. 2.19), and thus it makes sense to evaluate it under δ_1^N.

Lemma 3.3. *The infinite sum*

$$\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z}\sigma^i(y(h_k))) \in N[[\sigma]].$$

Proof. To save notation we temporarily denote $\Theta = (\Phi^{-1})^\top \in \mathrm{Mat}_r(L \otimes K)$. We write $x(g_k) = \mathbf{a}_k \in (L \otimes K)^r$ so that by definition 2.20 we have

$$\tau^{-i}(x(g_k)) = \Theta^{(-1)} \cdots \Theta^{(-i)}(\mathbf{a}_k)^{(-i)}.$$

Recall that M is a free $L[\sigma]$-module and that $\Phi^\top$ is upper triangular. Then, since $\det(\Phi) \in J^d$, it follows that each coordinate of the diagonal of $\Phi^\top$ is in J. Write $\Phi^\top = D + E'$ where $D \in \mathrm{Mat}_r(J)$ is a diagonal matrix and E' is a nilpotent, strictly upper triangular matrix. Thus we can write Θ as $\Theta = D^{-1} + E$ for some other nilpotent, strictly upper triangular matrix E. Thus $\Theta^{(-1)} \cdots \Theta^{(-i)}$ is a sum of terms consisting of $(D^{-1})^{(j)}$ and $E^{(k)}$ for various $-i \leq j, k \leq -1$. In particular, any term with more than r copies of $E^{(k)}$ for any values of k will vanish, since E is strictly upper triangular. Thus each term has at least $i - r$ copies of $(D^{-1})^{(j)}$ for various j. A short calculation shows that $\|(D^{-1})^{(j)}\|_\theta < 1$ for all $j \leq -1$ (recall the definition of the norm from Definition 2.17). Further, for $j \ll 0$, this norm is independent of j, since it is dominated by terms in $1 \otimes K$, which are unaffected by twisting. It follows that

$$\delta_0^M\left(\Theta^{(-1)} \cdots \Theta^{(-i)}(\mathbf{a}_k)^{(-i)}\right)^\top \mathbf{z} \to 0 \in L^d, \quad i \to \infty.$$

We conclude from this that

$$\left(\sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z}\right)^{(i)} \to 0$$

as $i \to \infty$ which finishes the lemma. $\square$

It will be useful to have a finer version of the pairing F. To this end, for any two elements $g \in M$ and $h \in N$ we define

$$(3.2) \qquad F(x, y; \mathbf{z}; g, h) = \delta_1^N\left(\sum_{i=0}^{\infty} \delta_0^M(\tau^{-i}(x(g)))^\top \mathbf{z}\sigma^i(y(h)))\right).$$

The pairing $F(x, y; \mathbf{z}; g, h)$ exists and converges defined based on the same reasoning as Lemma 3.3.

Lemma 3.4. *For any $g \in M$ and $h \in N$, the pairing $F(x, y; \mathbf{z}; g, h)$ satisfies:*

(1) $F(ax, y; \mathbf{z}; g, h) = F(x, ay; \mathbf{z}; g, h)$ *for all $a \in L$.*
(2) $F(\tau x, y; \mathbf{z}; g, h) = F(x, \sigma y; \mathbf{z}; g, h)$.
(3) $F(bx, y; \mathbf{z}; g, h) = F(x, y; d[b]\mathbf{z}; g, h)$ *for all $b \in A$.*
(4) $F(x, by; \mathbf{z}; g, h) = \phi_b \circ F(x, y; \mathbf{z}; g, h)$ *for all $b \in A$.*

Proof. The first equality of part (1) follows by observing that $\tau^{-1}a = a^{1/q}\tau^{-1}$, that $\sigma a = a^{1/q}\sigma$ and recalling that δ_0^M is L-linear. Statement (2) follows from the observation that $\delta_0^M(\tau(g)) = 0$ for any $g \in M$. Parts (3) and (4) follow from the A-linearity of δ_0^M and δ_1^N as described in Proposition 2.15. $\square$

Proposition 3.5. *For the pairing $F(x, y; \mathbf{z})$, we have*

(1) $F(ax, y; \mathbf{z}) = F(x, ay; \mathbf{z})$ *for all $a \in L$.*
(2) $F(ax, y; \mathbf{z}) = F(x, ay; \mathbf{z}) = aF(x, y; \mathbf{z})$ *for all $a \in L$, if $x, y \in L$.*
(3) $F(\tau x, y; \mathbf{z}) = F(x, \sigma y; \mathbf{z})$.
(4) $F(bx, y; \mathbf{z}) = F(x, by; \mathbf{z}) = \phi_b \circ F(x, y; \mathbf{z}) = F(x, y; d[b]\mathbf{z})$ *for all $b \in A$ and $x, y \in A$.*

Remark 3.6. Before giving the proof of Proposition 3.5, we make a brief comment about calling F a pairing, which will apply to the other pairings presented in the paper. The above proposition does not actually show that F is an $(L \otimes A)[\tau]$-bilinear pairing. It would rather be more accurate to describe F as a mapping which is simultaneously A-bilinear, L-bilinear and which is τ- and σ-symmetric. That being said, we will call F a pairing for ease of discussion.

Proof. The proof of parts (1) and (3) follow directly from Lemma 3.4. The proof of part (2) follows from (1) together with the fact that $\delta_1^N(a\sigma^i(h_j)) = a^{q^i}\delta_1^N(\sigma^i(h_j))$ for the basis elements h_j. For the proof of part (4), we fundamentally use the fact that the action of A on M and on N is related via the $*$ operator of Definition 2.8. Recall from Lemma 2.11 that there exists a matrix $\Theta_b \in \mathrm{Mat}_{d \times d}(L[\tau])$ such that for if we label $\mathbf{g} = (g_1, \dots, g_d)^\top$ and $\mathbf{h} = (h_1, \dots, h_d)^\top$

$$b\mathbf{g} = \Theta_b \mathbf{g}, \quad \text{and} \quad b\mathbf{h} = \Theta_b^* \mathbf{h}$$

where the elements of $\mathbf{g}$ form an $L[\tau]$-basis for M and $\mathbf{h}$ an $L[\sigma]$-basis for N. Let us label the jth row of Θ_b as $\Theta_{b,j}$. Then, using the fact that $x \in A$ and thus commutes with b and g_k we get,

$$F(bx, y; \mathbf{z}) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(bx(g_k)))^\top \mathbf{z}\sigma^i(y(h_k)) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(bg_k)))^\top \mathbf{z}\sigma^i(y(h_k)) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(\Theta_{b,k}\mathbf{g})))^\top \mathbf{z}\sigma^i(y(h_k)) \right).$$

Then, using Lemma 3.4 parts (1) and (2) on the individual terms of $\Theta_{b,k}\mathbf{g}$, we get

$$\delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(\Theta_{b,k}\mathbf{g})))^\top \mathbf{z}\sigma^i(y(h_k)) \right) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z}\sigma^i(y(\Theta_{b,k}^*\mathbf{h})) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(x(g_k)))^\top \mathbf{z}\sigma^i(by(h_k)) \right)$$

$$= F(x, by; \mathbf{z}).$$

The final two equalities of part (4) follow from Lemma 3.4 parts (3) and (4). $\square$

Theorem 3.7. *For $\mathbf{z} \in L^d$, we have*

$$F(1, 1; z) = \mathrm{Exp}_\phi(\mathbf{z}).$$

Proof. Observe first that $F(1, 1; \mathbf{z})$ may be expressed as an $\mathbb{F}_q$-linear power series in $\mathbf{z}$. Further, its first nonzero term is $\mathrm{Id}_d \mathbf{z}$. It remains to show that $F(1, 1; \mathbf{z})$ satisfies the correct functional equation given in Def. 2.10. Using Proposition 3.5(3), we see that for all $a \in A$

$$F(1, 1; d[a]\mathbf{z}) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(g_k))^\top d[a]\mathbf{z}\sigma^i(h_k) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} (d[a]^\top \delta_0^M (\tau^{-i}(g_k)))^\top \mathbf{z}\sigma^i(h_k) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(ag_k))^\top \mathbf{z}\sigma^i(h_k) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(g_k))^\top \mathbf{z}\sigma^i(ah_k) \right)$$

$$= \delta_1^N \left(a \sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(g_k))^\top \mathbf{z}\sigma^i(h_k) \right)$$

$$= \phi_a \left(\delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(g_k))^\top \mathbf{z}\sigma^i(h_k) \right) \right)$$

$$= \phi_a(F(1, 1; \mathbf{z})).$$

Thus $F(1, 1; \mathbf{z})$ equals the exponential function for ϕ. $\qquad\square$

Corollary 3.8. *Let ϕ be a Ξ-regular Anderson A-module (2.12). If we write*

$$\mathrm{Exp}_\phi(\mathbf{z}) = \sum_{i=0}^{\infty} Q_i \mathbf{z}^{(i)},$$

then

$$Q_i = (\delta_0^M (\tau^{-i}(g_1))^{(i)}, \dots, \delta_0^M (\tau^{-i}(g_d))^{(i)})^\top.$$

Proof. The follows immediately from Theorem 3.7 after noting that $\delta_1^N (\sigma^i(h_k)) = \mathbf{e}_k$, the kth standard basis vector. $\qquad\square$

Remark 3.9. We note that the pairing $F : (L \otimes A)[\tau] \times (L \otimes A)[\sigma] \to \mathbb{C}_\infty^d$ is not perfect. Indeed, M and N are torsion $(L \otimes A)[\tau]-$ and $(L \otimes A)[\sigma]-$ modules, respectively, and one can use this torsion to show that the induced map

$$(L \otimes A)[\sigma] \to \mathrm{Hom}_{(L \otimes A)[\tau]}((L \otimes A)[\tau], \mathbb{C}_\infty^d)$$

is not injective. However, if we restrict the pairing to $F : L \times L \to \mathbb{C}_\infty^d$, then the induced map is injective as long as $\mathrm{Exp}_\phi(\mathbf{z}) \neq 0$. Further, if one restricts the pairing to $A \times A \to \mathbb{C}_\infty^d$, then a short calculation shows that the induced map is actually the Anderson A-module morphism ϕ.

Example 3.10 (Tensor Powers of Carlitz)**.** In this section we give a detailed example of many of the above constructions in the case of the nth tensor power of the Carlitz module for fixed $n > 0$. We have $K = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$ and we set $L = \mathbb{C}_\infty$. Let us identify $1 \otimes A = \mathbb{F}_q[t]$ and $A \otimes 1 = \mathbb{F}_q[\theta]$ for independent variables t and θ, thus obviating the need for the tensor product in this situation (thus $\mathbb{C}_\infty$ is written using the θ variable). We may realize the t-motive M and the dual t-motive N as $M = N = \mathbb{C}_\infty[t]$ (here $L \otimes_{\mathbb{F}_q} A = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A = \mathbb{C}_\infty[t]$) with

$$\tau(m) = (t - \theta)^n m^{(1)}, \quad \sigma(s) = (t - \theta)^n s^{(-1)},$$

for $m \in M$ and $s \in N$. The associated Anderson A-module (in this case called a t-module) is denoted $C^{\otimes n}$ and is given by

(3.3)
$$C_t^{\otimes n} = \begin{pmatrix} \theta & 1 & 0 & \cdots & 0 \\ 0 & \theta & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \theta \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \tau & \cdots & 0 \end{pmatrix}$$

(we refer the reader to [5, §3.3] for details). In this case, M and N both have the same $\mathbb{C}_\infty[\tau]-$ and $\mathbb{C}_\infty[\sigma]-$ bases, namely

$$\{1, (t - \theta), \dots, (t - \theta)^{n-1}\},$$

and in order to satisfy the conditions of Lemma 2.11 we set $g_k = (t-\theta)^{k-1}$ and $h_k = (t-\theta)^{n-k}$ for $1 \le k \le n$. Further, we have $M \otimes K = N \otimes K = \mathbb{C}_\infty(t)$ and

$$\tau^{-1}(m) = \frac{1}{(t - \theta^{1/q})^n} m^{(-1)} \in \mathbb{C}_\infty(t)$$

$$\sigma^{-1}(m) = \frac{1}{(t - \theta^q)^n} m^{(1)} \in \mathbb{C}_\infty(t).$$

Let us denote $D_i(t) = (t - \theta) \dots (t - \theta^{q^{i-1}})$, so that

$$(\tau^{-i}(1)\big|_{t=\theta})^{q^i} = \frac{1}{D_i(\theta^{q^i})^n}.$$

The maps δ_0^M and δ_0^N may each be given by hyperderivatives. Namely, for $m \in M$, we write

$$m = m(\theta) + \partial_t(m)\big|_{t=\theta}(t - \theta) + \partial_t^2(m)\big|_{t=\theta}(t - \theta)^2 + \dots,$$

where $\partial_t^k(m)$ denotes the kth partial hyperderivative of m with respect to the variable t. Because δ_0^M factors through $M/(t - \theta)^n M$, we find

$$\delta_0^M(m) = (m(\theta), \partial_t(m))\big|_{t=\theta}, \dots, \partial_t^{n-1}(m)\big|_{t=\theta})^\top.$$

Similarly, we calculate for $s \in N$

$$\delta_0^N(s) = (\partial_t^{n-1}(s)\big|_{t=\theta}, \dots, \partial_t(s))\big|_{t=\theta}, s(\theta))^\top.$$

Thus, using the product rule for hyperderivatives, we find that

(3.4)
$$\delta_0^M(m)^\top \delta_0^N(n) = \partial_t^n(mn)\big|_{t=\theta}.$$

We refer the reader to $[7, \S 3.2\text{-}3.4]$ and $[20, \S 2.3\text{-}2.4]$ for full details on hyperderivatives. Our Corollary 3.8 then shows that the coefficients of the exponential are given by

$$
Q_i = \left. \begin{pmatrix}
\frac{1}{D_i(t)^n} & \partial_t\left(\frac{1}{D_i(t)^n}\right) & \cdots & \partial_t^{n-1}\left(\frac{1}{D_i(t)^n}\right) \\
\frac{t-\theta}{D_i(t)^n} & \partial_t\left(\frac{t-\theta}{D_i(t)^n}\right) & \cdots & \partial_t^{n-1}\left(\frac{t-\theta}{D_i(t)^n}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(t-\theta)^{n-1}}{D_i(t)^n} & \partial_t\left(\frac{(t-\theta)^{n-1}}{D_i(t)^n}\right) & \cdots & \partial_t^{n-1}\left(\frac{(t-\theta)^{n-1}}{D_i(t)^n}\right)
\end{pmatrix} \right|_{t=\theta^{q^i}}.
$$

This recovers Proposition 4.3.6(b) of $[20]$.

4. The Logarithmic Motivic Pairing

In this section we will define a pairing similar to F in the previous section which will give the logarithm function upon specialization. This will again give us an extremely explicit formula for the coefficients of the logarithm function. Many of the proofs in this section follow nearly identically to those in Section 3, so we leave many of the details to the reader. Our Theorem 4.4 and Corollary 4.5 generalizes formulas for the logarithm function of certain t-modules given by Anglés, Ngo Dac and Ribeiro in $[3, \text{Prop. } 2.2]$. In this section, we also give a sufficient criteria for the coefficients of the exponential and logarithm function to be invertible.

Definition 4.1. Recall that Log_ϕ has some bounded domain of convergence $D \subset \mathbb{C}_\infty^d$. Let $\mathbf{z} \in D \cap L^d$ be fixed. Also recall that $\{g_i\}$ and $\{h_i\}$ are the bases of Lemma 2.11. We define a second pairing

$$
G : (L \otimes A)[\tau] \times (L \otimes A)[\sigma] \to \mathbb{C}_\infty^d,
$$

for $x \in (L \otimes A)[\tau]$ and $y \in (L \otimes A)[\sigma]$ by setting (recall that the map δ_1^M implicitly depends on the element $\mathbf{z}$)

$$
G(x, y; \mathbf{z}) = \sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^N \left(\sigma^{-i}(x(h_k)) \delta_1^M (\tau^i(y(g_k))) \right).
$$

For now we view $G(x, y; \mathbf{z})$ as a formal $\mathbb{F}_q$-linear power series in the variable $\mathbf{z}$ (this follows quickly from the definition of δ_1^M (2.14)). However, we will see in Theorem 4.4 that for the given $\mathbf{z}$ the pairing coincides with the logarithm function, and thus it actually converges.

As in the previous section, we also define a finer pairing which will be useful in our proofs. For any $g \in M$ and $h \in N$ we set

$$
(4.1) \qquad G(x, y; \mathbf{z}; g, h) = \sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^N \left(\sigma^{-i}(x(h)) \delta_1^M (\tau^i(y(g))) \right).
$$

Lemma 4.2. *For any $g \in M$ and $h \in N$, the pairing $G(x, y; \mathbf{z}; g, h)$ satisfies:*

(1) For all $a \in L$, we have $G(ax, y; \mathbf{z}; g, h) = G(x, ay; \mathbf{z}; g, h)$.
(2) We have $G(\tau x, y; \mathbf{z}; g, h) = G(x, \sigma y; \mathbf{z}; g, h)$.

Proof. The proof is similar to Lemma 3.4, and we leave the details to the reader. $\square$

Proposition 4.3. *For the pairing $G(x, y; \mathbf{z})$, we have*

(1) $G(ax, y; \mathbf{z}) = G(x, ay; \mathbf{z})$ for all $a \in L$.

(2) $G(\tau x, y; \mathbf{z}) = G(\sigma x, y; \mathbf{z})$

(3) $G(bx, y; \mathbf{z}) = G(x, by; \mathbf{z})$ for all $b \in A$ and $x, y \in A$.

Proof. This proposition follows nearly identically to Proposition 3.5. Again, we leave the details to the reader. $\qquad\square$

Theorem 4.4. *Recall that* Log_ϕ *has some bounded domain of convergence* $D \subset \mathbb{C}_\infty^d$. *Let* $\mathbf{z} \in D \cap L^d$ *be fixed. We have*

$$G(1, 1; \mathbf{z}) = \mathrm{Log}_\phi(\mathbf{z}).$$

Proof. As in the proof of Theorem 3.7, we observe first that $G(1, 1; \mathbf{z})$ is a formal $\mathbb{F}_q$-linear power series in $\mathbf{z}$ with linear term $\mathbf{z}$. It remains to show that $d[a]G(1, 1; \mathbf{z}) = G(1, 1; \phi_z(\mathbf{z}))$. We do this using Proposition 4.3(3) and give an abbreviated version of the calculation here,

$$G(1, 1; \phi_a(\mathbf{z})) = \delta_0^N \left(\sum_{i=0}^\infty \sum_{k=1}^d \sigma^{-i}(h_k) \delta_{1,\mathbf{z}}^M(\tau^i(g_k(\phi_a))) \right)$$

$$= \delta_0^N \left(\sum_{i=0}^\infty \sum_{k=1}^d \sigma^{-i}(h_k) \delta_{1,\mathbf{z}}^M(\tau^i(tg_k)) \right)$$

$$= \delta_0^N \left(\sum_{i=0}^\infty \sum_{k=1}^d \sigma^{-i}(th_k) \delta_{1,\mathbf{z}}^M(\tau^i(g_k)) \right)$$

$$= d[a] \delta_0^N \left(\sum_{i=0}^\infty \sum_{k=1}^d \sigma^{-i}(h_k) \delta_{1,\mathbf{z}}^M(\tau^i(g_k)) \right)$$

$$= d[a] G(1, 1; \mathbf{z}).$$

$\qquad\square$

Corollary 4.5. *Let ϕ be a Ξ-regular Anderson A-module (2.12). If we write*

$$\mathrm{Log}_\phi(\mathbf{z}) = \sum_{i=0}^\infty P_i \mathbf{z}^{(i)},$$

then

$$P_i = \left(\delta_0^N(\sigma^{-i}(h_1)), \dots, \delta_0^M(\sigma^{-i}(h_d)) \right).$$

Proof. This follows immediately from Theorem 4.4 by noting that if we write $\mathbf{z} = (z_1, \dots, z_d)^\top$, then $\delta_{1,\mathbf{z}}^M(\tau^i(g_k)) = z_k^{q^i}$. $\qquad\square$

Remark 4.6. We comment that in the case of the nth tensor power of the Carlitz module, the above formula recovers Proposition 4.3.6(a) of [20].

Our formulas in Corollaries 3.8 and 4.5 also allow us to give a sufficient condition for the coefficients of the logarithm and exponential functions to be invertible in the case of t-modules (when $A = \mathbb{F}_q[t]$) for $L = \mathbb{C}_\infty$. Our starting point is [19, Prop. 3.5.7], which gives an explicit formula for the map δ_0^N in terms of hyperderivatives. This proposition assumes that the dual t-motive N (and corresponding t-module) has a certain structure. However, in [19] the authors prove that after a suitable change of basis, any dual t-motive can be made to satisfy these condition, thus it is really not a restriction. The conditions on the dual t-motive N are given as follows.

(1) There exists $C \in \mathrm{GL}_r(\mathbb{C}_\infty[t])$ such that the σ-action on N is represented by a matrix

$$\Phi = C \begin{pmatrix} (t-\theta)^{\ell_1} & & \\ & \ddots & \\ & & (t-\theta)^{\ell_r} \end{pmatrix},$$

such that $\ell_k \geq 0$ with $\sum \ell_k = d$, the $\mathbb{C}_\infty[\sigma]$-dimension of N.

(2) If e_i are basis elements for N as a $\mathbb{C}_\infty[t]$-module and s_i for N as a $\mathbb{C}_\infty[\sigma]$-module, then we have for $1 \leq i \leq m$ and $1 \leq j \leq \ell_i$

$$(t-\theta)^{j-1} e_i = s_{\ell_1 + \cdots + \ell_i - j + 1}.$$

If N satisfies these two conditions, then for $\alpha = (\alpha_1, \ldots, \alpha_r)^\top \in \mathbb{T}_\theta^r$ we have

$$(4.2) \qquad \delta_0^N(\alpha) = \begin{pmatrix} \partial_t^{\ell_1-1}(\alpha_1) \\ \vdots \\ \partial_t(\alpha_1) \\ \alpha_1 \\ \vdots \\ \partial_t^{\ell_r-1}(\alpha_r) \\ \vdots \\ \partial_t(\alpha_r) \\ \alpha_r \end{pmatrix} \Bigg|_{t=\theta}.$$

Similarly, by [19, Remark 3.5.13] we can give a similar construction for our t-motive M so that the map δ_0^M can be expressed using hyperderivatives (except with the order of the vector reversed).

We comment that the above constructions are commensurate with our Lemma 2.11, since if one changes the $\mathbb{C}_\infty[\sigma]$-basis for N using a matrix $\alpha \in \mathrm{Mat}_d(\mathbb{C}_\infty[\sigma])$, then one can simply change the $\mathbb{C}_\infty[\tau]$-basis of M by $(\alpha^*)^{-1}$ and our lemma still holds. A similar calculation also holds for a change of $\mathbb{C}_\infty[t]$-bases for M and N.

We now state a sufficient condition for the matrices Q_i and P_i of Corollaries 3.8 and 4.5 to be invertible.

Theorem 4.7. *We maintain the conditions (1) and (2) given above for the dual t-motive N. We further require that the matrix C be upper triangular. In this situation, the coefficients P_i of the logarithm function of the associated t-module ϕ are invertible.*

Proof. Our starting point is Corollary 4.5. By definition 2.4 we see that $\det(\Phi) = (t-\theta)^k$ for some k. Then, since N is finitely generated as a $\mathbb{C}_\infty[\sigma]$-module, we may assume without loss of generality that each of the diagonal elements of Φ is a positive power of $(t-\theta)$. Thus

$$\Phi^{-1} = \begin{pmatrix} \frac{1}{(t-\theta)^{k_1}} & * & \cdots & * \\ 0 & \frac{1}{(t-\theta)^{k_2}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{(t-\theta)^{k_r}} \end{pmatrix},$$

for positive integers k_m. For each $1 \leq f \leq d$, the basis element $h_f = (t-\theta)^k e_j$, for some non-negative k and some j with $1 \leq j \leq r$. Viewing $N \cong \mathbb{C}_\infty[t]^r$, we may assume e_j is the

jth standard basis vector. Thus the fth column of P_i is

$$\delta_0^N\left((\Phi^{-1})^{(1)}\cdots(\Phi^{-1})^{(i)}(t-\theta^{q^i})^k e_j\right).$$

We note that the jth entry of $(\Phi^{-1})^{(1)}\cdots(\Phi^{-1})^{(i)}(t-\theta^{q^i})^k e_j$ equals

$$(4.3)\qquad\qquad \frac{(t-\theta^{q^i})^k}{((t-\theta^q)\cdots(t-\theta^{q^i}))^{k_j}}.$$

By (4.2), $\delta_0^N(\sigma^{-i}(h_f))$ is a block vector, whose blocks consist of successive t-hyperderivatives of functions, which are then evaluated at $t=\theta$. Combining this fact with our above formulas, we see that the $\ell_1+\cdots+\ell_{j-1}+1$ through $\ell_1+\cdots+\ell_{j-1}+\ell_j$ entries of $\delta_0^N(\sigma^{-i}(h_f))$ are the $s=0$ through $s=\ell_j-1$ hyperderivatives of (4.3) which are then evaluated at $t=\theta$. The final ingredients for our proof are the hyperderivative product formula [20, Prop. 2.3.12] and the formula given in [20, Cor. 2.4.7], which states

$$(4.4)\qquad\qquad \partial_t^j\left(\frac{1}{(t-\theta^{q^i})^m}\right)=(-1)^j\binom{m+j-1}{j}\frac{1}{(t-\theta^{q^i})^{m+j}}.$$

Taken together, these formulas allow us to compute the hyperderivatives of the terms appearing in (4.3). In conclusion, we see that P_i is a block-upper-diagonal matrix, whose diagonal blocks all consist of successive hyperderivatives of terms of the form (4.3), which we can calculate using (4.4). Finally, a short calculation involving symmetric Pascal matrices (to account for the binomial coefficients appearing in (4.4)) then shows that each of the blocks along the diagonal is invertible. We conclude that the matrix P_i is invertible. Alternatively, we may conclude the proof by observing that the diagonal blocks of P_i are each equal to the matrix coefficients the exponential function of the ℓ_jth tensor power of the Carlitz module, which we can prove are invertible using formulas in [20]. $\qquad\square$

On the other hand, if our t-motive M satisfies a condition similar to that given for N, then we can give a sufficient condition for the exponential coefficients Q_i to be invertible. Namely:

(1) There exists $C\in\mathrm{GL}_r(\mathbb{C}_\infty[t])$ such that the τ-action on N is represented by a matrix

$$\Phi=C\begin{pmatrix}(t-\theta)^{\ell_1} & & \\ & \ddots & \\ & & (t-\theta)^{\ell_r}\end{pmatrix},$$

such that $\ell_k\geq 0$ with $\sum\ell_k=d$, the $\mathbb{C}_\infty[\tau]$-dimension of N.

(2) If e_i are basis elements for N as a $\mathbb{C}_\infty[t]$-module and s_i for N as a $\mathbb{C}_\infty[\sigma]$-module, then we have for $1\leq i\leq m$ and $1\leq j\leq\ell_i$

$$(t-\theta)^{j-1}e_i=s_{\ell_1+\cdots+\ell_{i-1}+j}.$$

Theorem 4.8. *Let the t-motive N satisfy conditions (1) and (2) given above. We further require that the matrix C be upper triangular. In this situation, the coefficients Q_i of the exponential function of the associated t-module ϕ are invertible.*

Proof. The proof follows nearly identically to that of Theorem 4.7, using Corollary 3.8 rather than Corollary 4.5. $\qquad\square$

5. Product Formulas for Exp and Log

In this section we prove formulas which express the exponential and logarithm functions as sums of infinite matrix products evaluated under δ_1^N and δ_1^M. Pellarin proves somewhat similar looking formulas in the case of the Carlitz module in [21, Thm. 1] and there are some hints towards the formulas of this section contained in the theorems of [16, §6] on trivial MZVs, but in general our formulas are completely original. As discussed in the introduction, we view a special case of these formulas (Corollary 5.9) as giving a function field analogue of the Mellin transform formula for the Riemann zeta function. Thus, we are led to view the map δ_1^M as an analogue of integration. This analogy is explored in more detail in Remarks 5.11 and 5.12.

Definition 5.1. For $n \in \mathbb{Z}_+$ we define a modified version of the nth partial sum of the pairings F and G from sections §3 and §4. For $x \in (L \otimes_{\mathbb{F}_q} A)[\tau]$ and $y \in (L \otimes_{\mathbb{F}_q} A)[\sigma]$ and $\mathbf{z} \in L^d$ we define

$$(5.1) \qquad F_n(x, y; \mathbf{z}) = \sum_{i=0}^{n} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(x(g_k)))^\top \mathbf{z} \sigma^i(y(h_k))) \in N$$

$$(5.2) \qquad G_n(x, y) = \sum_{i=0}^{n} \sum_{k=1}^{d} \delta_0^N \left(\sigma^{-i}(x(h_k)) \, \tau^i(y(g_k)) \right) \in M^d.$$

Definition 5.2. For a fixed $\mathbf{z} \in L^d$, we extend the definition of $\delta_{1,\mathbf{z}}^M$ to act coordinatewise on M^d. Then we have

$$F(x, y; \mathbf{z}) = \lim_{n \to \infty} \delta_1^N(F_n(x, y; \mathbf{z})), \text{ and } \quad G(x, y; \mathbf{z}) = \lim_{n \to \infty} \delta_{1,\mathbf{z}}^M(G_n(x, y)).$$

Definition 5.3. For fixed $b \in A$, recall the definition of Θ_b from Lemma 2.11 and write

$$\Theta_b = \Theta_0 + \Theta_1 \tau + \cdots + \Theta_k \tau^k, \quad \Theta_i \in \mathrm{Mat}_d(L).$$

We then define

$$\Theta_{b,\tau} = (\Theta_1 \tau + \cdots + \Theta_k \tau^k) + (\Theta_2 \tau + \cdots + \Theta_k \tau^{k-1}) + \cdots + (\Theta_k \tau).$$

Proposition 5.4. *For the partial sums F_n and G_n we have*

(1) For all $a \in L$ we have

$$F_n(ax, y; \mathbf{z}) = F_n(x, ay; \mathbf{z}), \quad G_n(ax, y) = G_n(x, ay)$$

(2) We have

$$F_n(x, \sigma y; \mathbf{z}) - F_n(\tau x, y; \mathbf{z}) = \sum_{k=1}^{d} \delta_0^M (\tau^{-n}(x(g_k)))^\top \mathbf{z} \sigma^{n+1}(y(h_k)))$$

$$G_n(x, \tau y) - G_n(\sigma x, y) = \sum_{k=1}^{d} \delta_0^N \left(\sigma^{-n}(x(h_k)) \, \tau^{n+1}(y(g_k)) \right)$$

(3) For $b \in A$, recall the definition of $\Theta_{b,\tau} \in \mathrm{Mat}_d(L[\tau])$ from Definition 5.3 and let $\mathbf{e}_k \in L^r$ be the kth standard basis vector. For all $x, y \in A$ we have

$$F_n(x, by; \mathbf{z}) - F_n(bx, y; \mathbf{z}) = \sum_{k=1}^{d} \delta_0^M (\tau^{-n}(x(g_k)))^\top \mathbf{z} \sigma^n(y(\mathbf{e}_k^\top \Theta_{b,\tau}^* \mathbf{h}))),$$

$$G_n(x, by) - G_n(bx, y) = \sum_{k=1}^{d} \delta_0^N (\sigma^{-n}(x(h_k))) \tau^n(y(\mathbf{e}_k^\top \Theta_{b,\tau} \mathbf{g}))),$$

Proof. The proofs of parts (1) and (2) follow as in the proofs of Propositions 3.5 and 4.3, and we leave these details to the reader. Part (3) follows similarly to the proof of Propositions 3.5(4) and 4.3(3), so we merely sketch the argument here. We begin by observing that

$$F_n(bx, y; \mathbf{z}) = \sum_{i=0}^{n} \sum_{k=1}^{d} \delta_0^M (\tau^{-i}(x(\Theta_{b,k}\mathbf{g})))^\top \mathbf{z} \sigma^i(y(h_k)).$$

We then apply parts (1) and (2) of this proposition to the individual terms of $\Theta_{b,k}\mathbf{g}$. Note that contrary to the proof of Proposition 3.5, F_n is not τ- and σ-symmetric; we get an extra term as observed in part (2) every time we transform a τ from the first coordinate into a σ in the second coordinate. After a short calculation, we find that all the extra terms give

$$F_n(x, by; \mathbf{z}) - F_n(bx, y; \mathbf{z}) = \sum_{k=1}^{d} \delta_0^M (\tau^{-n}(x(g_k)))^\top \mathbf{z} \sigma^n(y(\mathbf{e}_k^\top \Theta_{b,\tau}^* \mathbf{h}))).$$

The proof of the formula for G_n follows nearly identically. $\qquad\square$

The next proposition gives a product formula for our pairings F and G. We term this a product formula because powers of τ and σ can be expressed as products of twists of matrices related to Φ and so as we take the limit as $n \to \infty$ (after proper normalization) we get a finite sum of infinite matrix products. We shall give explicit examples of such product formulas as they relate to special values of zeta and multiple zeta functions shortly.

Theorem 5.5 (Product Formulas). *Let $b, x, y \in A$ and let $\Theta_{b,\tau}$ be as in Definition 5.3. For the pairings F and G from sections §3 and §4 we have*

$$F(x, y; \mathbf{z}) = \lim_{n \to \infty} \delta_1^N \left((bI - d[b]^\top)^{-1} \sum_{k=1}^{d} \delta_0^M (\tau^{-n}(x(g_k)))^\top \mathbf{z} \sigma^n(y(\mathbf{e}_k^\top \Theta_{b,\tau}^* \mathbf{h}))) \right),$$

$$G(x, y; \mathbf{z}) = \lim_{n \to \infty} \delta_{1,\mathbf{z}}^M \left((bI - d[b])^{-1} \sum_{k=1}^{d} \delta_0^N (\sigma^{-n}(x(h_k))) \tau^n(y(\mathbf{e}_k^\top \Theta_{b,\tau} \mathbf{g}))) \right).$$

Proof. From the definition of $F_n(x, y; \mathbf{z})$ we see that for any $b \in A$

$$F_n(x, by; \mathbf{z}) = b F_n(x, y; \mathbf{z}).$$

Then, by the A-linearity of δ_0^M described in Proposition 2.15, we find

$$F_n(bx, y; \mathbf{z}) = d[b]^\top F_n(x, y; \mathbf{z}).$$

Finally, by subtracting these, factoring and using Proposition 5.4(3) we deduce the first formula. The second formula follows identically. $\qquad\square$

Corollary 5.6. *Let ϕ be a Ξ-regular Anderson A-module.*

(1) *For all $\mathbf{z} \in \mathbb{C}_\infty^d$ we have*

$$\mathrm{Exp}_\phi(\mathbf{z}) = \lim_{n \to \infty} \delta_1^N \left((bI - d[b]^\top)^{-1} \sum_{k=1}^{d} \delta_0^M (\tau^{-n}(g_k))^\top \mathbf{z} \sigma^n(\mathbf{e}_k^\top \Theta_{b,\tau}^* \mathbf{h})) \right).$$

(2) For all $\mathbf{z}$ *in the domain of convergence of* Log_ϕ *we have*

$$\mathrm{Log}_\phi(\mathbf{z}) = \lim_{n\to\infty} \delta_{1,\mathbf{z}}^M \left((bI - d[b])^{-1} \sum_{k=1}^d \delta_0^N(\sigma^{-n}(h_k))\tau^n(\mathbf{e}_k^\top \Theta_{b,\tau}\mathbf{g})) \right).$$

Proof. This follows immediately from Theorems 3.7, 4.4 and 5.5. □

Example 5.7. We return to the notation of Example 3.10 for the nth tensor power of the Carlitz module. In this situation, the formula given in Theorem 5.5 is especially clean and interesting. In this setting we have

$$G_m(1,1) = \sum_{i=0}^m \sum_{k=1}^n \delta_0^N(\sigma^{-i}((t-\theta)^{n-k}))\tau^i((t-\theta)^{k-1})$$

(5.3)
$$= \sum_{i=0}^m \sum_{k=1}^n \delta_0^N \left(\frac{(t-\theta^{q^i})^{n-k}}{(t-\theta^q)^n \cdots (t-\theta^{q^i})^n} \right) (t-\theta)^n \cdots (t-\theta^{q^{i-1}})^n (t-\theta^{q^i})^{k-1}$$

From equation (3.3) we see that $\Theta_{t,\tau}$ on the basis $\{g_1, \ldots, g_n\}$ is given by

$$\Theta_{t,\tau} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \tau & \cdots & 0 \end{pmatrix}.$$

From this, we see that most of the terms in Theorem 5.5 vanish and we are left with

(5.4)
$$G(1,1;\mathbf{z}) = \delta_{1,\mathbf{z}}^M \left(\lim_{m\to\infty} (t-d[t])^{-1}\delta_0^N(\sigma^{-m}(1))\tau^{m+1}(1) \right)$$
$$= \delta_{1,\mathbf{z}}^M \left(\lim_{m\to\infty} (t-d[t])^{-1}\delta_0^N \left(\frac{1}{(t-\theta^q)^n \cdots (t-\theta^{q^m})^n} \right) (t-\theta)^n \cdots (t-\theta^{q^m})^n \right).$$

Recall the formula (see [5, 2.2.1]) for the fundamental period of the Carlitz module $\tilde{\pi}$

(5.5)
$$\tilde{\pi} = (-\theta)^{q/(q-1)} \prod_{i=1}^\infty \left(1 - \theta^{1-q^i} \right)^{-1} \in \mathbb{C}_\infty$$

and the definition of the Anderson-Thakur function

(5.6)
$$\omega_C = (-\theta)^{1/(q-1)} \prod_{i=0}^\infty \left(1 - \frac{t}{\theta^{q^i}} \right)^{-1} \in \mathbb{T}$$

from [2]. Finally, recall that the bottom coordinate of the map δ_0^N in this situation is merely evaluation at θ and that $(t-d[t])$ is upper triangular with bottom-right coordinate $(t-\theta)$. Putting this all together, and examining the bottom coordinate of equation (5.4) we conclude the following theorem.

Theorem 5.8. *Let* $\mathrm{Log}_C^{\otimes n}$ *denote the logarithm function for the nth tensor power of the Carlitz module and let* $\delta_{1,\mathbf{z}}^M$ *be the map from Definition 2.14 for some* $\mathbf{z} \in \mathbb{C}_\infty^n$ *inside the domain of convergence for* $\mathrm{Log}_C^{\otimes n}$. *Then, if we let* p_n *denote the projection onto the nth coordinate, we have the formula*

$$p_n(\mathrm{Log}_C^{\otimes n}(\mathbf{z})) = \delta_{1,\mathbf{z}}^M \left(\frac{\tilde{\pi}^n}{(t-\theta)\omega_C^n} \right).$$

We denote the Carlitz zeta value by

$$(5.7) \qquad\qquad \zeta_A(n) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ a \text{ monic}}} \frac{1}{a^n}, \quad n \in \mathbb{N}.$$

Denote $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j}) \in \mathbb{C}_\infty$ for $i \in \mathbb{N}$. For any non-negative integer n, recall that the Carlitz gamma value is defined by

$$(5.8) \qquad\qquad \Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A$$

where the $n_i \in \mathbb{Z}_{\geq 0}$ are given by writing the base q-expansion $n = \sum_{i=0}^{\infty} n_i q^i$ for $0 \leq n_i \leq q-1$.

We now make an application of Theorem 5.8 to Carlitz zeta values using the formulas for $\zeta_A(n)$ from [2, Thm. 3.8.3]. Let $\mathbf{z} = \in K^n$ be the special point of [2, 3.8.2] (in our notation $\mathbf{z} = \delta_1^N(H_n(t))$ where $H_n(t)$ is the nth Anderson-Thakur polynomial; see [15, §4.8]) and suppose that n and q are such that $\mathbf{z}$ is within the domain of convergence of $\mathrm{Log}_{C^{\otimes n}}$. Further, we apply [13, Thm. 6.7] to the case of Anderson generating functions (see [10] for details) for the nth tensor power of the Carlitz module to find

$$(5.9) \qquad \mathrm{Exp}_{C^{\otimes n}}\left((d[t] - tI)^{-1}\Pi_n\right) = ((t-\theta)^{n-1}\omega_C^n, \ldots, (t-\theta)\omega_C^n, \omega_C^n)^\top,$$

where $\Pi_n \in \mathbb{C}_\infty^n$ is a fundamental period for $\mathrm{Exp}_{C^{\otimes n}}$. We note that in the above formula, we are viewing t as a central variable which is unaffected by Frobenius twisting, in the sense of the formula for $\omega(t)$ in [4, §2.1] Let us denote $\mathbf{u} = (d[t] - tI)^{-1}\Pi_n$, so that we have

$$(5.10) \qquad\qquad p_n(\mathrm{Exp}_{C^{\otimes n}}(\mathbf{u})) = \omega_C^n,$$

and also let $u = \frac{\tilde{\pi}^n}{\theta - t}$ (by [13, Thm. 6.7] u is the bottom coordinate of $\mathbf{u}$).

Corollary 5.9. *Let $\mathbf{z}$ be the special point of* [2, 3.8.2] *subject to the notation and restrictions given above. Then we have*

$$\delta_{1,\mathbf{z}}^M\left(\frac{\tilde{\pi}^n}{(t-\theta)\omega_C^n}\right) = \delta_{1,\mathbf{z}}^M\left(\frac{u}{p_n(\mathrm{Exp}_{C^{\otimes n}}(\mathbf{u}))}\right) = \Gamma(n)\zeta_A(n),$$

where A_+ denotes the monic elements of A.

Proof. The first equality follows from the discussion above. The second equality follows immediately from Theorem 5.8 and by Theorem 3.8.3 of [2]. □

Remark 5.10. It is possible to give Corollary 5.9 for all values of $n \geq 1$ without constraints on the size of $\mathbf{z}$. However, in general the statement becomes much more cumbersome due to the fact that $\mathbf{z}$ is no longer in the domain of convergence of the logarithm. One would then need to break the LHS of Corollary 5.9 into a $C^{\otimes n}$-linear combination of logarithms, as is done in [15, Def. 3.3].

Remark 5.11. As discussed in the introduction, we view Corollary 5.9 as a function field analogue of the Mellin transform formula for the Riemann zeta function

$$\int_0^\infty \frac{1}{e^x - 1}x^{s-1}dx = \Gamma(s)\zeta(s),$$

where the map $\delta_{1,\mathbf{z}}^M$ serves as a replacement of integration. It is likely that the more natural way to state Theorem 5.8 and Corollary 5.9 is using n-dimensional vectors. However, we have chosen to state the theorem after projecting onto the last coordinate to highlight the analogy between our formulas and the Mellin transform formula.

Remark 5.12. Here we briefly describe how we can recover function field gamma values using $\delta_{1,\mathbf{z}}^M$. Set $\mathbf{z} = \delta_0^N(H_n)$ (H_n is an Anderson-Thakur polynomial [15, §4.8]) which is a modified version of Anderson and Thakur's special point. Then a short calculation gives

$$p_n(\delta_{1,\mathbf{z}}^M(\delta_0^N(1) \cdot (t - \theta)^{n-1})) = \Gamma(n).$$

Note that this is the first term of the expansion given in (5.3).

Example 5.13. In this example we sketch an application of our theorems to a specific t-module whose logarithm evaluates to give Carlitz multiple zeta values (MZVs). There are substantial details which go into this construction which are explained elsewhere, thus here we merely provide a sketch and describe how our theorems relate to it (see particularly [8, Ex. 5.4.2] and [15, Ex. 6.2]). We emphasize that this is a specific example of a general phenomenon; it should be possible to use this approach for a general MZV and we plan to explore this theme in more detail in a future project.

For any r-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, define the MZV for the tuple $\mathfrak{s}$ by

$$\text{(5.11)} \qquad \zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the sum is over all r-tuples of monic polynomials $a_1, \ldots, a_r$ in A with the restriction $|a_1|_\infty > |a_2|_\infty > \cdots > |a_r|_\infty$.

We follow the construction in [15, §5.1] of star dual t-motives. We maintain the notation of the previous example for $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, an so on and fix an r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$. We define a dual t-motive with σ-action given on a $\mathbb{C}_\infty[t]$-basis $\{h_1, \ldots, h_r\}$ by

$$\Phi^\star := \begin{pmatrix} \Phi_{1,1}^\star & & \\ \vdots & \ddots & \\ \Phi_{r,1}^\star & \cdots & \Phi_{r,r}^\star \end{pmatrix} \in \text{Mat}_r(\overline{K}[t])$$

where for $1 \le \ell \le j \le r$,

$$\text{(5.12)} \qquad \Phi_{j,\ell}^\star = (-1)^{j-\ell} \prod_{\ell \le k < j} H_k^{(-1)} (t - \theta)^{s_\ell + \cdots + s_r},$$

and where $H_k \in K[t]$ is a particular Anderson-Thakur polynomial coming from the r-tuple $\mathfrak{s}$ (see [15, §4.8]). As a particular example, if we set $q = 2$ and set $\mathfrak{s} = (1, 3)$, then

$$\Phi^\star = \begin{pmatrix} (t - \theta)^4 & 0 \\ -(t - \theta)^5 & (t - \theta) \end{pmatrix}$$

and the associated t-module is then given by

$$\phi_t = \left(\begin{array}{cccc|c} \theta & 1 & & & \\ & \theta & 1 & & \\ & & \theta & 1 & \tau \\ \tau & & & \theta & (\theta^2 + \theta)\tau \\ \hline & & & & \theta + \tau \end{array}\right).$$

Let e_i for $1 \leq i \leq 5$ be the standard basis vectors, so that $\{e_i\}$ is a $\mathbb{C}_\infty[\tau]$- and $\mathbb{C}_\infty[\sigma]$-module basis for M and N, respectively. A short calculation shows that $m_1 = e_1$ and $m_2 = e_5$ forms a $\mathbb{C}_\infty[t]$-basis for M and $n_1 = e_4$ and $n_2 = e_5$ forms a $\mathbb{C}_\infty[t]$-basis for N. A quick check shows that these bases satisfy the conditions of Lemma 2.11. Further, we calculate that we can express

$$\begin{aligned} e_1 &= m_1, & e_1 &= (t - \theta)^3 n_1 \\ e_2 &= (t - \theta)m_1, & e_2 &= (t - \theta)^2 n_1 \\ e_3 &= (t - \theta)^2 m_1, & e_3 &= (t - \theta)n_1 \\ e_4 &= (t - \theta)^3 m_1 + (t - \theta)m_2, & e_4 &= n_1 \\ e_5 &= m_2, & e_5 &= n_2. \end{aligned}$$

So, the first few terms $G(1,1)$ are given by

$$G(1,1) = \delta_1^M(\delta_0^N((t - \theta)^3 n_1)m_1 + \delta_0^N((t - \theta)^2 n_1)(t - \theta)m_1 + \delta_0^N((t - \theta)n_1)(t - \theta)^2 m_1 + \cdots$$

In order to apply Theorem 5.5 we calculate that $\Theta_t = \phi_t$, and hence

$$\Theta_{t,\tau} = \left(\begin{array}{ccc|c} & & & \\ & & & \tau \\ \tau & & & (\theta^2 + \theta)\tau \\ \hline & & & \tau \end{array}\right).$$

Thus, Theorem 5.5 gives

$$G(1,1) = \delta_1^M\left(\lim_{i \to \infty}(t - d[t])^{-1}\left[\delta_0^N(\sigma^{-i}((t - \theta)n_1))\tau^{i+1}(m_2)\right.\right.$$

$$\left.\left. + \delta_0^N(\sigma^{-i}(n_1))[\tau^{i+1}(m_1) + (\theta^2 + \theta)^{q^i}\tau^{i+1}(m_2)] + \delta_0^N(\sigma^{-i}(n_2))\tau^{i+1}(m_2)\right]\right).$$

Collecting like terms and writing $m_1, n_1 = (1,0)^\top$ and $m_2, n_2 = (0,1)^\top$ gives

$$G(1,1) = \delta_1^M\left(\lim_{i \to \infty}(t - d[t])^{-1}\left[\delta_0^N\left(\sigma^{-i}\begin{pmatrix} t - \theta^2 \\ 1 \end{pmatrix}\right)\tau^{i+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \delta_0^N\left(\sigma^{-i}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\tau^{i+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right).$$

Finally, if we set $\mathbf{z} = (0,0,0,0,-1)^\top$ and if we let p_4 denote the projection on to the 4th coordinate, then by [15, Ex. 6.2] we find

$$(5.13) \qquad p_4(G(1,1,\mathbf{z})) = (\theta^2 + \theta)\zeta_A(1,3),$$

thus giving us a "product formula" for $\zeta_A(1,3)$. The above strategy applies equally well to $\zeta_A(\mathfrak{s})$ for any r-tuple $\mathfrak{s}$, but it becomes increasingly difficult to calculate the individual terms involved in the left hand side of (5.13).

Remark 5.14. We observe that certain zeta values seem to appear in the terms of our product formula above. For example, the term

$$\delta_0^N \left(\sigma^{-i} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \tau^{i+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_0^N \left((\Phi^{\star\top(-1)} \cdots \Phi^{\star\top(-i)})^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) (\Phi^\star \cdots \Phi^{\star(i)}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Examining the formula for $\Phi^\star$ given in Example 5.13, we see that the 4th coordinate of the above expression equals

$$\frac{1}{(\theta - \theta^q)^4 \cdots (\theta - \theta^{q^i})^4} \begin{pmatrix} (t-\theta)^4 \cdots (t - \theta^{q^{i-1}})^4 \\ * \end{pmatrix},$$

where the $*$ coordinate contains a complicated sum coming from the product $\Phi^\star \cdots \Phi^{\star(i)}$. The top coordinate, however, matches the formula we obtained in Example 5.7 for $n = 4$, which hints that a term consisting of $\zeta_A(4)$ appears in the expression for $p_4(G(1,1,\mathbf{z}))$. We are hopeful that studying formulas of this type could lead to new expressions for $\mathbb{F}_q(\theta)$-linear relations between multiple zeta values. However, the $*$ coordinate complicates the situation. This theme will be explored in a future project.

6. Log-Algebraic Applications

In this section we make a short application of the motivic pairings of Sections 3 and 4. The main theorem of this section is Theorem 6.12, which gives a generalization of an unpublished theorem originally due to Anderson (see [17, Cor. 2.5.23]). Although the main result of this section merely recovers (and slightly extends) a known theorem, the methods of proof we use here are new. The pairings H and I which we define in this section appear in a concurrent project which develops a non-commutative factorization of the exponential function, and thus we are inspired to include them here.

We now develop two new pairings, H and I, which relate the F and G pairings. The I pairing will prove especially crucial in this section. Throughout this section, we set $L = \mathbb{C}_\infty$.

Definition 6.1. Fix an integer $\ell \geq 0$, and let $x \in (L \otimes A)[\tau]$ and $y \in (L \otimes A)[\sigma]$. Recall that $\{g_i\}$ and $\{h_i\}$ are the bases of Lemma 2.11. Then define the pairing

$$H_\ell : (L \otimes A)[\tau] \times (L \otimes A)[\sigma] \to \mathrm{Mat}_{d \times d}(L),$$

by setting

$$H_\ell(x,y) = \sum_{j=0}^{\ell} \sum_{k=1}^{d} \delta_0^N(\sigma^{-j}(x(h_k))) \left(\delta_0^M(\tau^{j-\ell}(y(g_k)))^\top \right)^{(\ell)}.$$

We also define a finer version of the pairing.

Definition 6.2. For, $\ell \geq 0$, for $x \in (L \otimes A)[\tau]$ and $y \in (L \otimes A)[\sigma]$, and for any $g \in M$ and $h \in N$, we define

$$H_\ell(x,y;g,h) = \sum_{j=0}^{\ell} \delta_0^N(\sigma^{-j}(x(h))) \left(\delta_0^M(\tau^{j-\ell}(y(g)))^\top \right)^{(\ell)}.$$

Lemma 6.3. *The pairing $H_\ell(x,y;g,h)$ satisfies*

(1) $H_\ell(\tau x, y; g, h) = H_\ell(x, \sigma y; g, h)$
(2) $H_\ell(ax, y; g, h) = H_\ell(x, ay; g, h) = a^{q^\ell} H_\ell(x, y; g, h)$ *for all $a \in L$*

Proof. The proof for (1) follows by recalling that $\delta_0(\tau(g)) = 0 = \delta_0(\sigma(h))$ for all $g \in M$ and $h \in N$. The proof for (2) follows using the L-commutativity relations for τ and σ and the fact that both maps δ_0^M, δ_0^N are L-linear. $\square$

Proposition 6.4. *The pairing $H_\ell(x, y)$ satisfies*

(1) $H_\ell(\tau x, y) = H_\ell(x, \sigma y; g, h)$
(2) $H_\ell(ax, y) = H_\ell(x, ay) = a^{q^\ell} H_\ell(x, y)$ *for all* $a \in L$
(3) $H_\ell(bx, y) = H_\ell(x, by)$ *for all* $b \in A$ *and for* $x, y \in A$

In particular, $H_\ell(x, y) = 0$ whenever $\ell > 0$ and $x, y \in A$ and $H_0(1, 1) = I_d$.

Proof. Parts (1) and (2) follow directly from Lemma 6.3. The proof of part (3), follows similarly to the proof of Proposition 3.5 part (4); we leave the details to the reader. To prove the last statement of the proposition, we note that for $b \in A$ we have

$$H_\ell(bx, y) = \sum_{j=0}^{\ell} \sum_{k=1}^{d} \delta_0^N (\sigma^{-j}(bx(g_k))) \left(\delta_0^M (\tau^{j-\ell}(y(h_k)))^\top \right)^{(\ell)}$$

$$= \sum_{j=0}^{\ell} \sum_{k=1}^{d} d[b] \delta_0^N (\sigma^{-j}(x(g_k))) \left(\delta_0^M (\tau^{j-\ell}(y(h_k)))^\top \right)^{(\ell)}$$

$$= d[b] H_\ell(x, y)$$

whereas

$$H_\ell(x, by) = \sum_{j=0}^{\ell} \sum_{k=1}^{d} \delta_0^N (\sigma^{-j}(x(g_k))) \left(\delta_0^M (\tau^{j-\ell}(by(h_k)))^\top \right)^{(\ell)}$$

$$= \sum_{j=0}^{\ell} \sum_{k=1}^{d} \delta_0^N (\sigma^{-j}(x(g_k))) \left((d[b]^\top \delta_0^M (\tau^{j-\ell}(y(h_k))))^\top \right)^{(\ell)}$$

$$= \sum_{j=0}^{\ell} \sum_{k=1}^{d} \delta_0^N (\sigma^{-j}(x(g_k))) \left(\delta_0^M (\tau^{j-\ell}(y(h_k)))^\top \right) d[b])^{(\ell)}$$

$$= H_\ell(x, y) d[b]^{(\ell)}.$$

By part (3), $H_\ell(bx, y) = H_\ell(x, by)$ for $x, y \in A$, then a short linear algebra argument shows that

$$d[b] H_\ell(x, y) = H_\ell(x, y) d[b]^{(\ell)}$$

can never happen if $\ell \geq 1$, unless $H_\ell(x, y) = 0$. On the other hand, if $x = y = 1$ and $\ell = 0$, then $H_0(1, 1) = I_d$. $\square$

We also define a pairing which reverses the role of M and N from the previous pairing H_ℓ. We define

$$I_\ell : (L \otimes A)[\tau] \times (L \otimes A)[\sigma] \to L,$$

by setting

$$I_\ell(x,y) = \sum_{j=0}^{\ell} \sum_{k,m=1}^{d} \left(\delta_0^M(\tau^{-j}(x(g_k)))^\top\right)^{(j)} \delta_0^N(\sigma^{j-\ell}(y(h_m)))^{(j)}.$$

We also define a finer version of this pairing for $g \in M$ and $h \in N$ by setting

$$I_\ell(x,y,g,h) = \sum_{j=0}^{\ell} \left(\delta_0^M(\tau^{-j}(x(g)))^\top\right)^{(j)} \delta_0^N(j - \sigma^\ell(y(h)))^{(j)}.$$

Proposition 6.5. *For $I_\ell(x,y,g,h)$ (and consequently also for $I_\ell(x,y)$) we have*

(1) $I(bx,y) = I(x,by)$ *for all* $b \in A$
(2) $I(\tau x, y) = I(x, \sigma y)$
(3) $I(x, ay) = I(a^{q^\ell} x, y)$ *for all* $a \in L$

Proof. The proof follows very similarly to the proof of Proposition 6.4. We leave the details to the reader. $\square$

Proposition 6.6. *We have $I_\ell(1,1,g_k,h_m) = 1$ when $k = m$ and $\ell = 0$ and we have $I_\ell(1,1,g_k,h_m) = 0$ otherwise.*

Proof. Let $\mathbf{z} = (z_1, \ldots, z_d)^\top$ be in the domain of convergence of Log_ϕ. We first observe that by Theorems 3.7 and 4.4 we have

$$\mathrm{Exp}_\phi(\mathrm{Log}_\phi(\mathbf{z})) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(g_k))^\top \left[\mathrm{Log}_\phi(\mathbf{z})\right] \sigma^i(h_k)) \right)$$

$$= \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(g_k))^\top \left[\delta_0^N \left(\sum_{j=0}^{\infty} \sum_{m=1}^{d} \sigma^{-j}(h_m)\delta_{1,\mathbf{z}}^M(\tau^j(g_m)) \right) \right] \sigma^i(h_k)) \right)$$

$$= \sum_{i=0}^{\infty} \sum_{k=1}^{d} \sum_{j=0}^{\infty} \sum_{m=1}^{d} (\delta_0^M(\tau^{-i}(g_k))^{(i)})^\top \delta_0^N \left(\sigma^{-j}(h_m)\right)^{(i)} \delta_{1,\mathbf{z}}^M(\tau^j(g_m))^{q^i} \delta_1^N \left(\sigma^i(h_k)\right))$$

$$= \sum_{i,j=0}^{\infty} \sum_{k,m=1}^{d} (\delta_0^M(\tau^{-i}(g_k))^{(i)})^\top \delta_0^N \left(\sigma^{-j}(h_m)\right)^{(i)} z_m^{q^{i+j}} \mathbf{e}_k$$

where $\mathbf{e}_k$ is the kth standard basis vector. Then, since $\mathrm{Exp}_\phi(\mathrm{Log}_\phi(\mathbf{z})) = \mathbf{z}$, we compare this coordinate-wise with the above equality to conclude the proof. $\square$

We observe that $N_\theta \cong \mathbb{T}_\theta^{\ell r}$ where ℓ is the degree of $K/\mathbb{F}_q(t)$ and r is the $L \otimes A$-rank of N. For any $h \in N_\theta$ we identify

$$h = \sum_{i=0}^{\infty} \mathbf{c}_i t^i \in \mathbb{T}_\theta^{\ell r}, \quad \mathbf{c}_i \in \mathbb{C}_\infty^{\ell r}.$$

Since the σ-action on N_K is represented by a matrix $\Phi \in \mathrm{GL}_r(L \otimes A)$, for any $m \geq 1$ we can find an element $\alpha \in \mathbb{T}_\theta^{\ell r}$ such that

$$h - \sigma^m(\alpha) = \beta_h \in \mathbb{C}_\infty[t]^{\ell r}$$

and $\delta_0^N(\sigma(\alpha)) = 0$. We then write β_h in terms of the $\mathbb{C}_\infty[\sigma]$-basis of Lemma 2.11 to get

$$(6.1) \qquad h = \beta_h + \sigma^m(\alpha) = c_{1,0}h_1 + c_{2,0}(h_2) + \cdots + c_{d,k}\sigma^k(h_d) + \sigma^m(\alpha)$$

for $c_{i,j} \in \mathbb{C}_\infty$ and for some $k \in \mathbb{Z}$. By definition, for any $h \in N_\Gamma$ we have that $\sigma(h) \in N_\Gamma$ (see Def. 2.7 and following discussion). Thus, for each $w \geq 0$, we can express $\sigma^w(h)$ in the form of 6.1 for some $\alpha \in \mathbb{T}_\theta^{wr}$

$$(6.2) \qquad \sigma^w(h) = c_{1,0}^w h_1 + c_{2,0}^w(h_2) + \cdots + c_{d,k}^w \sigma^k(h_d) + \sigma^{w+1}(\alpha)$$

for $c_{i,j}^w \in L$. In general, the coefficients $c_{i,j}^w \in L$ depend of α, but a short linear algebra argument shows that the coefficients $c_{i,j}^w$ for $j \leq w$ are independent of α. Then, for each $w \geq 0$, we obtain an expression for h by taking σ^{-w} of both sides of (6.2) using Definition 2.20, which we label as

$$(6.3) \qquad h^w = d_{1,-w}^w \sigma^{-w}(h_1) + d_{2,-w}^w \sigma^{-w}(h_2) + \cdots + d_{d,k-w}^w \sigma^{k-w}(h_n) + \sigma(\alpha),$$

We comment that if the coordinates of h are all regular at $\Xi^{(i)}$ for $i \geq 1$, then each $d_{i,j} = 0$ for $j < 0$, since the coordinates of $(\Phi^{-1})^{(i)}$ necessarily have poles at $\Xi^{(i)}$. Therefore, if $h \in N \subset N_\Gamma$, then the expressions h_w are all identical for $w \geq 0$. Finally, for such an expression, we denote

$$\mathbf{d}_w := \begin{pmatrix} d_{n,0} \\ d_{n-1,0} \\ \vdots \\ d_{1,0} \end{pmatrix}.$$

Definition 6.7. For a function $h \in N_\Gamma$, if $\lim_{w\to\infty} \mathbf{d}_w$ exists, then we denote this limit by L_h. Then let $N_\Gamma^* \subset N_\Gamma$ be the set of functions h such that L_h exists and is finite. We define a new extension of δ_0^N to N_Γ^*, which we call $\delta_*^N : N_\Gamma^* \to L^d$, by setting

$$\delta_*^N(h) = \lim_{w\to\infty} \mathbf{d}_w.$$

Note that if each coordinate of h is regular at $\Xi^{(i)}$ for $i \geq 1$, then $\delta_*^N(h) = \delta_0^N(h)$.

Proposition 6.8. *The set N_Γ^* is nonempty. In particular we have*

(1) $N \subset N_\Gamma^*$
(2) If $h \in N_\Gamma$ and satisfies $h - \sigma(h) = \sigma(g)$, for some $g \in N$, then $h \in N_\Gamma^*$.

Proof. By the discussion above, if $h \in N$, then $\delta_*^N(h) = \delta_0(h)$ and the limit in Definition 6.7 exists. If $h \in N_\Gamma$ satisfies $h - \sigma(h) = \sigma(g)$, then we may write $\sigma^i(h) = h - \sigma(g) - \sigma^2(g) - \cdots - \sigma^i(g)$. Thus we may select the same $\alpha \in \mathbb{T}_\theta^{\ell r}$ for expression (6.2) for h and for $\sigma^i(h)$, and these two expressions will differ by $-\sigma(g) - \sigma^2(g) - \cdots - \sigma^i(g)$. Thus, if we express g in terms of the basis of Lemma 2.11, a short calculation shows that the limit in Definition 6.7 exists and that $\delta_*^N(h) = \delta_1^N(g)$. $\square$

Definition 6.9. For $1 \leq k \leq d$, we define a function $J_k : N_\Gamma \to L$ similar to our pairings F, G, H and I of the previous sections. For $h \in N_\Gamma$ and $1 \leq k \leq d$ we set

$$J_k(h) = \sum_{i=0}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)} \in L.$$

We then define $J : N_\Gamma \to L^d$ by

$$J(h) = (J_1, \ldots, J_d)^\top.$$

Lemma 6.10. *For $h \in N_\Gamma$, the series $J(h)$ converges in $\mathbb{C}_\infty^d$.*

Proof. As explained in the proof of Proposition 2.18, the map δ_0^N factors through $N/J^d N$, thus we have for $h \in N_\Gamma$, that $|\delta_0^N(h)| \leq |h|_\theta$. It follows that

$$|\delta_0^N(\sigma^i(h))| \leq |\sigma^i(h)|_\theta = |\Phi \ldots \Phi^{(1-i)} h^{(-i)}|_\theta \leq |\Phi|_\theta^{(1-1/q^i)/(1-1/q)} |h|_\theta^{1/q^i}.$$

The last quantity in the above inequality is bounded independent from i, and thus the convergence of the series $J(h)$ follows from the fact that the exponential function is entire and from the fact that $\delta_0^M(\tau^{-i}(g_k))$ appears in the coefficients of Exp_ϕ via Corollary 3.8. $\quad\square$

Proposition 6.11. *For $h \in N_\Gamma^*$, we have $J(h) = \delta_*^N(h)$.*

Proof. For $w > 0$, we express h as in (6.3),

$$h^w = d_{1,-w}^w \sigma^{-w}(h_1) + d_{2,-w}^w \sigma^{-w}(h_2) + \cdots + d_{d,k-w}^w \sigma^{k-w}(h_n) + \sigma(\alpha).$$

Then we compute (recall that $\delta_0^N(\sigma^i(h_j)) = 0$ for all $i \geq 1$ and all $1 \leq j \leq d$ and that $\delta_0(\sigma(\alpha)) = 0$)

$$J_k(h) = \sum_{i=0}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}$$

$$= \sum_{i=0}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N \left(\sigma^i \left(\sum_{m=-w}^{k-w} \sum_{j=1}^d d_{j,m}^w \sigma^m(h_j) + \sigma(\alpha)\right)\right)^{(i)}$$

$$= \sum_{i=0}^w \sum_{m=-w}^{-i} \sum_{j=1}^d \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} d_{j,m}^w \delta_0^N(\sigma^{i+m}(h_j))^{(i)} + \sum_{i=w+1}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}$$

$$= \sum_{m=-w}^0 \sum_{j=1}^d d_{j,m}^w \left[\sum_{i=0}^m \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^{i-m}(h_j))^{(i)}\right] + \sum_{i=w+1}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}$$

$$= \sum_{m=-w}^0 \sum_{j=1}^d d_{j,m}^w I_m(1,1,g_k,h_j) + \sum_{i=w+1}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}$$

$$= d_{k,0}^w + \sum_{i=w+1}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}.$$

Note that in the last line in the above calculation, we have applied Proposition 6.6. We recognize

$$\sum_{i=w+1}^\infty \left(\delta_0^M(\tau^{-i}(g_k))^\top\right)^{(i)} \delta_0^N(\sigma^i(h))^{(i)}$$

as the tail of $J_k(h)$, which is a convergent series. Thus taking the limit as $w \to \infty$ of both sides of the above equality causes the series tail to vanish and proves the theorem by Definition 6.7. $\quad\square$

Theorem 6.12. *Let $h \in N_\Gamma$ be such that*

$$h - \sigma(h) = \sigma(g),$$

for some $g \in N$. Then

$$\mathrm{Exp}_\phi(\delta_0^N(h)) = \delta_*^N(h).$$

In particular, if $\delta_*^N(h)$ is in $\overline{K}$, then h is log-algebraic.

Proof. Note that the functional equation for h gives

$$h = \sigma^i(h) + \sum_{j=1}^{i} \sigma^j(g).$$

We then begin with Theorem 3.7. We recall that $\delta_1^N(\sigma(w)) = \delta_1^N(w)$ for all $w \in N$, that $\delta_1^N(h_k) = \mathbf{e}_k$ (the kth standard basis element), and that $\delta_0^N(\sigma^i(g)) = 0$ for $g \in N$. Using this, we find

$$\mathrm{Exp}_\phi(\delta_0^N(h)) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(g_k))^\top \delta_0^N(h)\sigma^i(h_k) \right)$$

$$= \sum_{i=0}^{\infty} \sum_{k=1}^{d} (\delta_0^M(\tau^{-i}(g_k))^\top)^{(i)} \delta_0^N(h)^{(i)} \delta_1^N\left(\sigma^i(h_k)\right)$$

$$= \sum_{i=0}^{\infty} \sum_{k=1}^{d} (\delta_0^M(\tau^{-i}(g_k))^\top)^{(i)} \delta_0^N\left(\sigma^i(h) + \sum_{j=1}^{i} \sigma^j(g) \right)^{(i)} \delta_1^N\left(h_k\right)$$

$$= \sum_{i=0}^{\infty} \sum_{k=1}^{d} (\delta_0^M(\tau^{-i}(g_k))^\top)^{(i)} \delta_0^N(\sigma^i(h))^{(i)} \mathbf{e}_k$$

$$= \sum_{k=1}^{d} J_k(h)\mathbf{e}_k$$

$$= \delta_*^N(h),$$

where $\mathbf{e}_k$ is the kth standard basis vector. $\square$

6.1. **Example: Carlitz Tensor Powers.** In this brief subsection, we illustrate Theorem 6.12 in the particular case of tensor powers of the Carlitz. In this simple case, we are able to achieve more specific formulas which leads to a pair of open questions at the end of this section. We continue with the notation and example of Example 3.10. Let $h \in N_\Gamma$ such that

$$h - \sigma(h) = \sigma(g),$$

for some $g \in \mathbb{C}_\infty[t]$. This functional equation shows that h has a meromorphic continuation to $\mathbb{C}_\infty$ with possible poles at θ^{q^i} for $i \geq 1$ of order at most n, and thus we are able to take residues of hdt at the values θ^{q^i}. Our Theorem 6.12 then gives

$$\mathrm{Exp}_C^{\otimes n}(\delta_0^N(h)) = \delta_1^N \left(\sum_{i=0}^{\infty} \sum_{k=1}^{d} \delta_0^M(\tau^{-i}(g_k))^\top \delta_0^N(h)\sigma^i(h_k) \right)$$

$$= \delta_*^N(h)$$

However, in this very concrete, basic case of tensor powers of Carlitz, we get more specialized formulas. Combining this all together, and applying (3.4) we get

$$
\mathrm{Exp}_C^{\otimes n}(\delta_0^N(h)) = \delta_1^N\left(\sum_{i=0}^{\infty}\sum_{k=1}^{d}\delta_0^M(\tau^{-i}(g_k))^{\top}\delta_0^N(h)\sigma^i(h_k)\right)
$$

$$
= \delta_1^N\left(\sum_{i=0}^{\infty}\sum_{k=1}^{d}\delta_0^M(\tau^{-i}(g_k))^{\top}\delta_0^N(\sigma^i(h))\sigma^i(h_k)\right)
$$

$$
= \delta_1^N\left(\sum_{i=0}^{\infty}\sum_{k=1}^{d}\delta_0^M(\tau^{-i}(g_k))^{\top}\delta_0^N((D_i(t)^{(1-i)})^n h^{(-i)})\sigma^i(h_k)\right)
$$

$$
= \sum_{i\geq 0}\left(\partial_t^{n-1}\frac{(t-\theta)^{n-1}(D_i(t)^{(1)})^n h}{D_i(t)^n},\ldots,\partial_t^{n-1}\frac{(D_i(t)^{(1)})^n h}{D_i(t)^n}\right)^{\top}\Bigg|_{t=\theta^{q^i}}
$$

$$
= \sum_{i\geq 0}\left(\partial_t^{n-1}\left((t-\theta)^{n-1}(t-\theta^{q^i})^n h\right),\ldots,\partial_t^{n-1}\left((t-\theta^{q^i})^n h\right)\right)^{\top}\Bigg|_{t=\theta^{q^i}}
$$

$$
= \sum_{i\geq 0}\begin{pmatrix}\mathrm{Res}_{t=\theta^{q^i}}hdt\\ \mathrm{Res}_{t=\theta^{q^i}}(t-\theta)hdt\\ \vdots\\ \mathrm{Res}_{t=\theta^{q^i}}(t-\theta)^{n-1}hdt\end{pmatrix}.
$$

Finally, using the fact that h is regular away from θ^{q^i} for $i \geq 1$ and the fact that the sum of residues over all points of a curve must be zero, we conclude that the above sum equals

$$
\mathrm{Exp}_C^{\otimes n}(\delta_0^N(h)) = \begin{pmatrix}-\mathrm{Res}_{\infty}hdt\\ -\mathrm{Res}_{\infty}(t-\theta)hdt\\ \vdots\\ -\mathrm{Res}_{\infty}(t-\theta)^{n-1}hdt\end{pmatrix}.
$$

Thus in this case we see that $\delta_*^N(h)$ is related to the negative of the residue of h at ∞. There are some hints towards a theory of this sort in Sinha's Main Diagram [22, §4.2.3]. This invites the following pair of question related to residues.

Question 1. *For a general Anderson A-module ϕ, and $h \in N_\Gamma$ as above, can we relate $\mathrm{Exp}_\phi(\delta_0^N(h))$ to the residue of h at ∞?*

Question 2. *In [17, Thm. 2.5.13] the authors describe an isomorphism Ξ from the algebraic dual of M to N constructed by taking a finite sum of residues at ∞. They pose an open question about a pairing derived from this map. How do the pairings F or G from our paper compare the map Ξ from [17] and can we use them to construct this pairing explicitly?*

References

[1] G. Anderson. *t-motives. Duke Math. J.*, 53(2):457–502, 1986.

[2] G. Anderson and D. Thakur. Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)*, 132(1):159–191, 1990.

[3] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. On special *L*-values of *t*-modules. *Adv. Math.*, 372 (2020), 107313, 33 pp.

[4] B. Anglès and F. Pellarin. Universal Gauss-Thakur sums and L-series. *Invent. Math.* 200 (2015), no. 2, 653–669.

[5] D. Brownawell and M. Papanikolas. A rapid introduction to Drinfeld modules, *t*-modules and *t*-motives. In G. Böckle, D. Goss, U. Hartl, and M. Papanikolas, editors, *t-motives: Hodge structures, transcendence and other motivic aspects"*, *EMS Series of Congress Reports*, pages 3–30. European Mathematical Society, 2020.

[6] L. Carlitz. On certain functions connected with polynomials in Galois field. *Duke Math. J.*, 1(2):137–168, 1935.

[7] C.-Y. Chang, N. Green, and Y. Mishiba. Taylor coefficients of *t*-motivic multiple zeta values and explicit formulae. Math. Annalen, (2020), 379(2):1-50.

[8] C.-Y. Chang and Y. Mishiba. On a conjecture of Furusho over function fields. *Inventiones mathematicae.* 223, 49-102 (2021)

[9] A. El-Guindy and M. Papanikolas. Explicit formulas for Drinfeld modules and their periods. *J. Number Theory* 133 (2013), no. 6, 1864–1886.

[10] A. El-Guindy and M. Papanikolas. Identities for Anderson generating functions for Drinfeld modules. *Monatshefte für Mathematik.* 173 (2014), no. 4, 471-493.

[11] H. Furusho, Y. Komori, K. Matsumoto, H. Tsumura. Desingularization of complex multiple zeta-functions. *American Journal of Mathematics*, Volume 139, Number 1, February 2017, pp. 147-173

[12] D. Goss. *Basic Structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1996.

[13] N. Green, Tensor powers of rank 1 Drinfeld modules and periods, *Journal of Number Theory*, Journal of Number Theory Volume 232, (2022), Pages 204-241.

[14] N. Green. Special zeta values using tensor powers of Drinfeld modules. *Math. Res. Letters*, vol. 26, (2019), no. 6, 1629 – 1676.

[15] N. Green and T. Ngo Dac. On log-algebraic identities for Anderson *t*-modules and characteristic *p* multiple zeta values. *available at https://arxiv.org/abs/2007.11060, (41 pages)*, 2020.

[16] O. Gezmis, F. Pellarin, *Trivial multiple zeta values in Tate algebras*, https://arxiv.org/abs/2008.07144, 2020.

[17] U. Hartl and A. K. Juschka. Pink's theory of Hodge structures and the Hodge conjectures over function fields. In G. Böckle, D. Goss, U. Hartl, and M. Papanikolas, editors, *t-motives: Hodge structures, transcendence and other motivic aspects"*, *EMS Series of Congress Reports*, pages 31–182. European Mathematical Society, 2020.

[18] A. Maurischat. Abelian equals A-finite for Anderson A-modules. *available at https://arxiv.org/abs/2110.11114*, (33 pages).

[19] C. Namoijam and M. Papanikolas. Hyperderivatives of periods and quasi-periods for Anderson t-modules. *Memoirs of the American Mathematical Society.* Accepted for publication. *available at https://arxiv.org/abs/2103.05836* (116 pages).

[20] M. Papanikolas. Log-algebraicity on tensor powers of the Carlitz module and special values of Goss *L*-functions. *work in progress, 167 pages (last version: 28 April 2015).*

[21] F. Pellarin. Values of certain *L*-series in positive characteristic. *Ann. of Math.*, 176(3):2055–2093, 2012.

[22] S. K. Sinha, *Periods of t-motives and transcendence*, Duke Math. J. **88** (1997), no. 3, 465–535.

[23] D. Thakur, Zeta measure associated to $F_q[T]$. *J. Number Theory* 35 (1990), no. 1, 1–17.

[24] D. Zagier. Appendix. The Mellin Transform and Related Analytic Techniques. Available at: https://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf.

Email address: ngreen@latech.edu