

NONCOMMUTATIVE FACTORIZATIONS OF HIGHER SINE FUNCTIONS IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper we describe new noncommutative factorizations of functions related to d -th tensor powers of Carlitz's $\mathbb{F}_q[\theta]$ -module for $d \geq 1$, called higher sine functions, related to previous works of the second named author. In [34] factorizations of this type have been constructed for operators which are combinations of powers of a Frobenius endomorphism with coefficients “in $\text{End}(\text{End}(\mathbb{G}_a^d))$ ”. In the present paper we succeed in determining factorizations with coefficients “in $\text{End}(\mathbb{G}_a^d)$ ” which are not easily deducible from [34]. One key ingredient in obtaining this is an application of a “motivic pairing” that the first named author introduced in [18]. Another key ingredient is the notion of “ Δ -matrix” which comes into play in the analysis of the coefficients of the factorizations. Our results can be applied to explicitly describe analogues of shuffle q^n -powers for multiple polylogarithms at one, and to multiple zeta values of Thakur. All the identities we prove occur at the finite level.

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1. INTRODUCTION

The present paper deals with factorizations of certain “higher sine functions” (defined by (1.15) below). These functions are associated to tensor powers of Carlitz’s module (see Anderson and Thakur’s [3]), counterparts of Tate twists $\mathbb{Q}(d)$ in positive characteristic function field arithmetic. The choice of the terminology “sine” indicates that these are normalizations of exponentials of these modules. The factorizations we are interested in hold in certain non-commutative algebras of formal series in powers of the Frobenius endomorphism, and are called ‘non-commutative’ because the factors do not commute each other. In order to introduce the reader to these themes and to our results, we start with a reminder of classical investigations by Euler, adopting for this and other aspects, as a reference, the book [8] of Frésan and Burgos Gil.

Euler’s setting. Euler’s factorization of the sine function

$$\begin{aligned} \frac{\sin(\pi z)}{\pi z} &= \prod_{k \geq 1} \left(1 - \frac{z^2}{k^2}\right) \\ &= \sum_{n \geq 0} (-1)^n \frac{\pi^{2n}}{(2n+1)!} z^{2n} \end{aligned}$$

allows, by comparison of the coefficients of z^2 , to deduce Euler’s formula

$$\zeta(2) = \frac{\pi^2}{6},$$

with ζ Riemann’s zeta function. For the coefficients of higher powers of z , this formula also gives explicit identities for Euler-Zagier multiple zeta values of ‘parallel weight’

$$(1.1) \quad \zeta(\underbrace{2, \dots, 2}_{n \text{ times}}) = \frac{\pi^{2n}}{(2n+1)!}.$$

To get explicit formulas for special values of ζ at even positive integers one uses the cotangent. We have:

$$\begin{aligned} \cot(\pi z) &= \pi^{-1} \frac{d}{dz} \log(\sin(\pi z)) \\ &= \frac{1}{\pi z} \left(1 - 2 \sum_{k \geq 1} \zeta(2k) z^{2k}\right) \in \frac{1}{\pi z} - (\pi z) \mathbb{Q}[[\pi z^2]]. \end{aligned}$$

From this one deduces Euler's 1735 formula involving Bernoulli's numbers B_n :

$$(1.2) \quad \zeta(2k) = \frac{(-1)^{k-1} B_{2k}}{2(2k)!} (2\pi)^{2k}, \quad k \in \mathbb{N}^*, \quad \frac{x}{e^x - 1} =: \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

(\mathbb{N}^* denotes the set of positive integers). In the above discussion, the key point is that $\sin(z) \in \mathbb{Q}[[z]]$ and that the factors of $\sin(\pi z)/(\pi z)$ are in $\mathbb{Q}[z]$.

Carlitz's setting. Two centuries after Euler's investigations, Carlitz [11] explored parallel structures in the framework of global function fields of positive characteristic. Let \mathbb{F}_q be the finite field with q elements and characteristic $p > 0$. Carlitz discovered formulas analogous of (1.2) and from his work it is easy to deduce analogues of (1.1). Carlitz was guided by analogies that can be summarized in the following table:

\mathbb{Z}	$A := \mathbb{F}_q[\theta]$
\mathbb{Q}	$K := \text{Frac}(A)$
\mathbb{R}	$K_\infty := \widehat{K}_{ \cdot } = \mathbb{F}_q((\theta^{-1}))$
\mathbb{C}	$\mathbb{C}_\infty := \widehat{K}_\infty^{\text{sep}}$ (completion of a separable closure)
\mathbb{N}^*	$A^+ := \{a \in A : a \text{ monic}\}$

In this table K denotes the fraction field of A , the \mathbb{F}_q -algebra of polynomials in an indeterminate θ with coefficients in the finite field \mathbb{F}_q , K_∞ denotes the local field which is the completion $\widehat{K}_{|\cdot|}$ of K at the infinity place, to which we associated a norm $|\cdot|$ (with the property that $|\theta| > 1$ and uniquely determined by this value in θ), that has $\frac{1}{\theta}$ as a uniformizer. Also, \mathbb{C}_∞ denotes the completion of a separable closure of K_∞ that we fix once and for all; it plays the role of the field of complex numbers in the paper; it is complete and algebraically closed though not locally compact, unlike \mathbb{C} . Finally, \mathbb{N}^* denotes the set of positive integers, and the choice of the set of monic polynomials as an analogue of it means that we made the choice of a sign function $K_\infty^\times \rightarrow \mathbb{F}_q$. Further analogies can be noticed observing that both \mathbb{Z} and A are euclidean, and \mathbb{Z} is discrete and co-compact in \mathbb{R} , while A is discrete and co-compact in K_∞ (the infinity place is the only one carrying this property).

Let us write, with $z \in \mathbb{C}_\infty$,

$$\sin_A(z) := z \prod_{a \in A^+} \left(1 - \frac{z^{q-1}}{a^{q-1}}\right) = z \prod_{b \in A \setminus \{0\}} \left(1 - \frac{z}{b}\right).$$

The product converges to an entire \mathbb{F}_q -linear function $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$, hence surjective. We call this function *Carlitz's sine function*. It is the analogue of the factor \sin_z in

$$\sin(\pi z) := \pi z \underbrace{\prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right)}_{\sin_z(x)}.$$

The crucial tools that Carlitz introduced are the *Carlitz module* C (analogue of \mathbb{G}_m), and its *exponential function* $\exp_C(z)$, which belongs to $K[[z]]$. Using them we see that the

factorization of \sin_A produces identities related to *Thakur's multiple zeta values*, introduced in [38], the simplest of which is an analogue of the formula $\zeta(2) = \frac{\pi^2}{6}$, and the use of an analogue \cot_A of the classical cotangent allowed Carlitz to discover identities for so-called *Carlitz's zeta values*, analogous to Euler's formulas for values of ζ at positive even integers. We quickly review these basic facts.

Carlitz's module. The Carlitz module can be viewed as an analogue in function field arithmetic of the functor \mathbb{G}_m from commutative rings to abelian groups, see [16] and [32]. Let B be an A -algebra. Carlitz's module $C(B)$ over B is the \mathbb{F}_q -vector space B with the A -module structure determined by the multiplication by θ given by:

$$C_\theta(b) = \theta b + \tau(b) = \theta b + b^q, \quad b \in B,$$

where τ is the \mathbb{F}_q -linear endomorphism of $\mathbb{G}_a(B)$ given by $c \mapsto c^q$. Carlitz's module C over \mathbb{C}_∞ is *uniformizable*. There exists an entire \mathbb{F}_q -linear endomorphism $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ such that $\frac{d}{dz}(\exp_C(z)) = 1$ and, for all $z \in \mathbb{C}_\infty$:

$$C_\theta(\exp_C(z)) = \exp_C(\theta z).$$

This is easy to prove because we can actually identify \exp_C with an explicit formal series

$$\exp_C = \sum_{k \geq 0} \frac{1}{D_k} \tau^k \in K[[\tau]],$$

with $D_0 = 1$ and

$$(1.3) \quad D_k = (\theta^{q^k} - \theta) D_{k-1}^q, \quad k \geq 1.$$

Here and in all the following, given B an \mathbb{F}_q -algebra, we denote by $B[\tau]$ (resp. $B[[\tau]]$) the skew ring of finite sums (resp. formal series) $\sum_{i \geq 0} b_i \tau^i$ with coefficients $b_i \in B$, with the unique ring structure arising from the commutation rule $\tau b = \tau(b)\tau = b^q \tau$. Note the fundamental property that

$$(1.4) \quad \frac{1}{D_k} \in K, \quad k \geq 0,$$

which is obvious from the construction. Analogously, we have

$$\exp(z) = e^z = \sum_{k \geq 0} \frac{z^k}{k!} \in \mathbb{Q}[[z]],$$

so that $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \in \mathbb{Q}[[z]]$. A simple computation of Newton polygons (at the infinite place of K) shows that there exists $\tilde{\pi} \in \mathbb{C}_\infty$ (in fact $\tilde{\pi} \in \mathbb{C}_\infty \setminus K_\infty$ if $q \neq 2$) such that $\text{Ker}(\exp_C) = \tilde{\pi}A$. We deduce right away that, viewing \sin_A no longer as an \mathbb{F}_q -linear, entire function, but as a formal series in powers of τ ,

$$(1.5) \quad \sin_A = \tilde{\pi}^{-1} \exp_C \tilde{\pi} = \sum_{i \geq 0} \frac{\tilde{\pi}^{q^i - 1}}{D_i} \tau^i \in K_\infty[[\tau]].$$

Note that $\tilde{\pi}$ is defined up to a factor in \mathbb{F}_q^\times , but \sin_A does not depend on this choice. We can compute $\tilde{\pi}$ by means of the following formula:

$$(1.6) \quad \tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i \geq 1} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1} \in K_\infty[(-\theta)^{\frac{1}{q-1}}]^\times,$$

which can be applied to prove its transcendence: $\tilde{\pi} \in \mathbb{C}_\infty \setminus K^{\text{ac}}$ (K^{ac} denotes the algebraic closure of K in \mathbb{C}_∞). From now on, $(-\theta)^{\frac{1}{q-1}}$ designates a chosen $(q-1)$ -th root of $-\theta$ in \mathbb{C}_∞ ; this fixes a choice of $\tilde{\pi}$ in (1.6). Thanks to (1.4) one recovers Carlitz's original result (see [11]), where

$$\zeta_A(n) := \sum_{a \in A^+} \frac{1}{a^n} \in K_\infty, \quad n \geq 1.$$

Theorem (Carlitz). *For all $k \geq 1$ we have*

$$(1.7) \quad \zeta_A(k(q-1)) \in K^\times \tilde{\pi}^{k(q-1)}.$$

We mention, for completeness, that in his work, Carlitz discussed certain analogues of Bernoulli numbers in K called *Bernoulli-Carlitz elements*. The theory can be further developed along the lines of Kummer's "ideal numbers" which can be here encoded in Taelman's class modules and class number formulas that have connections with Bernoulli-Carlitz elements. See for example [16, 4, 36, 37] (non-exhaustive suggestions).

Non-commutative factorization of \sin_A . In this paper we are interested in another type of factorization of \sin_A . The *noncommutative factorization* of \sin_A no longer sees \sin_A as an entire function but as an element of the noncommutative algebra $K_\infty[[\tau]]$, and can be proved essentially applying Carlitz's theory:

$$(1.8) \quad \sin_A = \prod_{i \geq 0}^{\leftarrow} (1 - \mathcal{L}_i \tau),$$

where $\mathcal{L}_i := l_i^{1-q}$ and the sequence $(l_i)_{i \geq 0}$ is given by the recursion

$$(1.9) \quad l_0 = 1, \quad l_i = (\theta - \theta^{q^i})l_{i-1}, \quad i > 0$$

and the arrow on top of the product sign means that the factors are nested from right to left, so that to compute the product, one starts with the first factor $(1 - \tau)$, and then multiplies it on the left by the other factors: $\cdots (1 - \mathcal{L}_1 \tau)(1 - \tau)$. See [34, Proposition 4.4.9]. There is a direct connection to Thakur's multiple zeta values which is described in [34, Remark 4.4.11], and this paper contains generalizations of these observations. Incidentally, the sequence (1.9) characterizes the *Carlitz's logarithm*

$$(1.10) \quad \log_C := \sum_{j \geq 0} \frac{\tau^j}{l_j} \in K[[\tau]],$$

which is the formal inverse $\log_C = \exp_C^{-1}$ of \exp_C in $K[[\tau]]$. Another way to rewrite (1.8) is by using function composition. On every non-empty bounded subset B of \mathbb{C}_∞ (e. g. a disk containing 0) we have that, uniformly,

$$(1.11) \quad \sin_A(z) = \lim_{k \rightarrow \infty} \left((1 - \mathcal{L}_{k-1}\tau) \circ \cdots \circ (1 - \mathcal{L}_1\tau) \circ (1 - \tau)(z) \right), \quad z \in B,$$

where τ acts as the q -power Frobenius. Similar noncommutative factorizations, in the framework of rank-one Drinfeld A -modules associated to a ring A of regular functions over a smooth projective, geometrically irreducible curve, away from a fixed but arbitrarily chosen closed point, have been obtained in [12] (with ‘‘power sums’’ in the place of the coefficients \mathcal{L}_i).

Tensor powers of Carlitz’s module. In order to present the results of our paper, we now introduce the main ingredients. Let B be an A -algebra and $d \geq 1$. We define an A -module structure $C^{\otimes d}(B)$ over $\mathbb{G}_a^d(B)$ identified with $B^{d \times 1}$ by setting, for $b \in B^{d \times 1}$ (column matrix with d entries in B), the multiplication by θ to be the image of b by Carlitz’s module C_θ if $d = 1$, or by

$$(1.12) \quad C_\theta^{\otimes d} := \begin{pmatrix} \theta & 1 & 0 & \cdots & 0 & 0 \\ 0 & \theta & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & \theta & 1 \\ \tau & 0 & 0 & \cdots & 0 & \theta \end{pmatrix} = \theta + N + e_{d,1}\tau \in \text{End}_{\mathbb{F}_q}(\mathbb{G}_a^d(K)),$$

where $e_{i,j}$ denotes the elementary $d \times d$ matrix having its only non-zero coefficient being equal to one, in the line i and column j , and where $N = e_{1,2} + e_{2,3} + \cdots + e_{n-1,n}$, that is:

$$(1.13) \quad N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This is the d -th tensor power of Carlitz’s module $C^{\otimes d}$, and is an *Anderson A -module* (or *Anderson t -module*) of rank one and dimension d [16, Chapter 5] and [32, §3.4]. It was introduced by Anderson and Thakur in [3], see also Brownawell and Papanikolas’ [7]. It provides an algebro-geometric structure supporting a function field analogue of Tate’s twist $\mathbb{Z}(d)$, see [3, §1.11]. In particular there are isomorphisms of Galois modules $\text{Ker}(C_a^{\otimes d}) \cong \text{Ker}(C_a)^{\otimes d}$ for all $d \geq 1$ and $a \in A$. Moreover $C^{\otimes d}(\mathbb{C}_\infty)$ is *uniformizable*. This surprising property ⁽¹⁾ was proved by Anderson and Thakur [3, §2.2]; it means that there exists an exponential function $\exp_{C^{\otimes d}}$ fitting into the exact sequence (2.1), see section §2.2.

¹There are no signs that similar uniformizations behind $\mathbb{Z}(d)$ exist.

Higher sine functions. Based on the one dimensional theory we can construct higher dimensional analogues of Carlitz's sine function \sin_A . We choose a generator $\Pi \in \mathbb{C}_\infty^{d \times 1}$ of the free rank one A -module whose elements are the zeroes of the entire function $\exp_{C^{\otimes d}}$. We construct, starting from Π , a matrix $\widehat{\Pi}$ in the following way:

$$(1.14) \quad \Pi =: \begin{pmatrix} \widetilde{\pi}_{d-1} \\ \widetilde{\pi}_{d-2} \\ \vdots \\ \widetilde{\pi}_0 \end{pmatrix}, \quad \widehat{\Pi} := \begin{pmatrix} \widetilde{\pi}_0 & \widetilde{\pi}_1 & \cdots & \widetilde{\pi}_{d-1} \\ 0 & \widetilde{\pi}_0 & \cdots & \widetilde{\pi}_{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{\pi}_0 \end{pmatrix} \in \mathrm{GL}_d(\mathbb{C}_\infty).$$

Finally we define the *sine function of order d* to be

$$(1.15) \quad \sin_A^{\otimes d} := \widehat{\Pi}^{-1} \exp_{C^{\otimes d}} \widehat{\Pi} \in \mathrm{End}_{K_\infty}(\mathbb{G}_a^n(K_\infty))[[\tau]].$$

This formal series, again independent of the choice of Π , represents a surjective vector-valued \mathbb{F}_q -linear entire function of d variables. The kernel of $\sin_A^{\otimes d}$ is:

$$\mathrm{Ker}(\sin_A^{\otimes d}) = \left\{ a(\theta + N) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} : a \in A \right\},$$

it inherits a structure of free rank one A -module from $\mathrm{Lie}(C^{\otimes d})$ (see Lemma 3.23). If $d = 1$, we see that $\sin_A^{\otimes d} = \sin_A$, Carlitz's sine function.

A noncommutative factorization of $\sin_A^{\otimes n}$. The first main result that we present is a noncommutative factorization of $\sin_A^{\otimes d}$. To state it we need to introduce a class of partial differential operators that we use in the paper (the basic properties are collected in §4). For a field extension F/\mathbb{F}_q , and x an indeterminate over F we denote by $(\mathcal{D}_{x,j})_{j \geq 0}$ the unique family of F -linear higher divided derivatives over $F(x)$ such that for all $i, j \in \mathbb{Z}$, $0 \leq i \leq j$,

$$\mathcal{D}_{x,j}(x^i) = \binom{i}{j} x^{i-j}.$$

Given now x, z two independent indeterminates over F and $f \in F(x, z)$, we set:

$$\Delta_{x,z}(f) := \begin{pmatrix} \mathcal{D}_{x,d-1}(f) & \mathcal{D}_{x,d-1}(\mathcal{D}_{z,1}(f)) & \cdots & \mathcal{D}_{x,d-1}(\mathcal{D}_{z,d-1}(f)) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{x,1}(f) & \mathcal{D}_{x,1}(\mathcal{D}_{z,1}(f)) & \cdots & \mathcal{D}_{x,1}(\mathcal{D}_{z,d-1}(f)) \\ f & \mathcal{D}_{z,1}(f) & \cdots & \mathcal{D}_{z,d-1}(f) \end{pmatrix} \in F(x, z)^{d \times d}.$$

Consider three independent indeterminates x, y, z . For $d \geq 1$ we set ⁽²⁾

$$(1.16) \quad f_d(x, y, z) := \frac{(z-y)^d - (x-y)^d}{z-x} = \sum_{j=0}^{d-1} (z-y)^j (x-y)^{d-1-j} \in \mathbb{F}_p[x, y, z].$$

²Note that this is a polynomial of $\mathbb{Z}[x, y, z]$ reduced modulo p to give $f_d \in \mathbb{F}_p[x, y, z]$.

We also set $f_0 = 0$. This sequence was used crucially in [35], it is denoted by S_d in that paper. In [35, Theorem E] the second author observed a connection between specializations of these polynomials, the *Bernoulli-Carlitz elements* $\text{BC}_{n(q-1)}$ and the *Anderson-Thakur polynomials* $H_{(q-1)n-1}$. In the notation of *ibid.* we have the identity in $K[Y^q]$:

$$(\theta - \theta^q) \text{BC}_{n(q-1)} f_n(\theta, Y^q, \theta^q) = H_{(q-1)n-1}(Y).$$

This formula can also be recovered from our results.

Define, for $i \geq 0$,

$$(1.17) \quad \mathcal{L}_i := \Delta_{x,z} \left(f_d(x, y, z) \left(\frac{(x - \theta^q) \cdots (x - \theta^{q^i})}{(z - \theta^{q^2}) \cdots (z - \theta^{q^{i+1}})} \right)^d \right) \Big|_{\substack{x=\theta \\ y=\theta^{q^{i+1}} \\ z=\theta^q}} \in \text{GL}_d(K).$$

We show:

Theorem A. *The product below converges and the following identity holds:*

$$\sin_A^{\otimes d} = \prod_{i \geq 0}^{\leftarrow} (1 - \mathcal{L}_i \tau) \in \text{End}_{K_\infty} (\mathbb{G}_a^d(K_\infty))[[\tau]].$$

Note that if $d > 0$, \mathcal{L}_0 is not the identity matrix. In the case $d = 1$ we find (1.8). This result (see Theorem 4.10 in the paper) is a consequence of an identity given for finite products of the above infinite product, which we state below as Theorem B, which is proved by using the theory of *motivic pairings* introduced by the first author in [18], and requires a few additional tools to be stated. In [31], Papanikolas introduces certain higher versions of *Carlitz's polynomials* that we denote by E_k (in the case $d = 1$ those are the polynomials $D_n^{-1} f_{<n}(x)$ in [32, Lemma 5.4.2], the polynomials $D_d^{-1} e_d(x)$ in [16, Theorem 3.1.5], and the polynomials Ψ_d in [3, (3.4.1)]). They can be identified with elements of the algebra $\text{End}_K(\text{End}_K(\mathbb{G}_a^d(K)))[\tau]$ (see also [35]; the definition is recalled in our §3.1). We further introduce, in §3.4, a certain normalization

$$\mathbf{E}_k \in \text{End}_K (\mathbb{G}_a^d(K))[\tau], \quad k \geq 0$$

of *Carlitz's operator* E_k restricted over a certain subspace of $\text{Lie}(C^{\otimes d})(\mathbb{C}_\infty)$ (see (3.14)). The relationship with the normalizations introduced in [35] will be discussed in Remark 3.21. The following theorem is restated as Theorem 4.9 in the text.

Theorem B. *The following factorization holds:*

$$\mathbf{E}_k = (1 - \mathcal{L}_{k-1} \tau)(1 - \mathcal{L}_{k-2} \tau) \cdots (1 - \mathcal{L}_0 \tau).$$

Since $\lim_{k \rightarrow \infty} \mathbf{E}_k = \sin_A^{\otimes d}$ (see Proposition 3.20), Theorem B implies Theorem A. From both Theorems A and B, choosing $d = 1$, we recover corresponding results of [34], see [35, Proposition 4.4.9] and its proof.

The reader may be interested in comparing with the results of [35]. By [35, Theorem 5.2 and the exact sequence (5.3)], just like in the case $d = 1$, the function $\sin_A^{\otimes n}$ is the exponential function of the Anderson module of rank one and dimension d denoted by

$\tilde{\phi}$ in *ibid.*, defined by $\tilde{\phi}_\theta = \widehat{\Pi}^{-1}C_\theta^{\otimes d}\widehat{\Pi}$. A comparison between the factorization of [35] and that of this paper is made in Remark 3.22. It seems difficult to go directly from the factorizations of [35] to the main formulas of the present paper, and the techniques we use here are substantially different than those of [35].

In particular, Theorem B says that

$$\mathbf{E}_k = (1 - \mathcal{L}_{k-1}\tau)\mathbf{E}_{k-1}, \quad k \geq 1.$$

By using the formula (3.14) and comparing the coefficients of the different powers of τ we deduce recursive formulas for the coefficients Q_i of the exponential $\exp_{C^{\otimes d}} = \sum_{i \geq 0} Q_i \tau^i$ (reviewed in §2.2). These are:

$$\begin{aligned} Q_j \left(\mathbf{d}_t((t - \theta^{q^{k-j}})^d)_{t=\theta} \right)^{(j)} - \mathbf{d}_t((t - \theta^{q^k})^d)_{t=\theta} Q_j = \\ = -\Delta_{x,z}(f_d(x, y, z)) \Big|_{\substack{x=\theta \\ y=\theta^{q^k} \\ z=\theta^q}} Q_{j-1}^{(1)}, \quad 1 \leq j \leq k \end{aligned}$$

(\mathbf{d}_t is defined in §2.3). Compare with the formula [3, (2.2.2)] by Anderson and Thakur:

$$Q_i \mathbf{d}_t(t)_{t=\theta^{q^i}} - \mathbf{d}_t(t)_{t=\theta} Q_i = -\Delta_{x,z}(1) Q_{i-1}^{(1)}, \quad i \geq 1.$$

If $j = k$ one term of our formula vanishes and we get the formula

$$\mathbf{d}_t((t - \theta^{q^k})^d)_{t=\theta} Q_j = -\Delta_{x,z}(f_d(x, y, z)) \Big|_{\substack{x=\theta \\ y=\theta^{q^k} \\ z=\theta^q}} Q_{k-1}^{(1)}, \quad 0 \leq j \leq k$$

from which it is easy to deduce that Q_i determines an automorphism for all i . This property is not easy to deduce directly from Anderson and Thakur formula and our formula can be compared with a formula by Papanikolas, reproduced in our Proposition 2.1.

Application to Carlitz multiple polylogarithms. As previously noticed, the factorization (1.8) (this is the case $d = 1$) and its ‘finite variants’ imply the existence of non-trivial K -linear dependence relations among multiple zeta values of Thakur, see [34, 14]. At once (1.8) implies that non-trivial linear dependence relations hold among values of Anderson-Thakur polylogarithms. In this paper we describe a family of linear dependence relations of *Carlitz multiple polylogarithms* ⁽³⁾, see Theorem C below. While comparing them with the existing literature, we will also recall the above mentioned results.

Consider, for $r > 0$, positive integers $n_1, \dots, n_r \in \mathbb{N}^*$. Consider also integers m_1, \dots, m_r such that $m_i < n_i \frac{q}{q-1}$ for all i . It is easy to see that the series

$$(1.18) \quad \mathcal{L} \begin{pmatrix} m_1 & \dots & m_r \\ n_1 & \dots & n_r \end{pmatrix} := \sum_{i_1 > \dots > i_r \geq 0} \frac{\theta^{m_1 q^{i_1} + m_2 q^{i_2} + \dots + m_r q^{i_r}}}{l_{i_1}^{n_1} l_{i_2}^{n_2} \dots l_{i_r}^{n_r}}$$

³The extensive use of which has been inaugurated by Chang in [9] with the purpose of proving an analogue of Goncharov’s conjecture for Thakur’s multiple zeta values, rooted on the idea of polylogarithm as the reader can find in [3, §3], see also [16], and ultimately, in the work of Carlitz.

converge to non-zero elements of K_∞ . These are special values of *Carlitz multiple polylogarithms*, studied in wider generality in Chang's [9] (the author uses these series to show that the K -algebra of Thakur's multiple zeta values is graded by weights). In Theorem C below, we describe certain explicit families of non-trivial K -linear dependence relations among these values. To describe the coefficients of these linear relations expand, for indeterminates y_1, \dots, y_r , the polynomials

$$f_d(\theta, y_1, \theta^{q^r}) f_d(\theta^q, y_2, \theta^{q^r}) \cdots f_d(\theta^{q^{r-1}}, y_r, \theta^{q^r}) = \sum_{\underline{m}} c_{\underline{m}} y_1^{m_1} y_2^{m_2} \cdots y_r^{m_r} \in A[y_1, y_2, \dots, y_r],$$

where the finite sum runs over r -tuples $\underline{m} = (m_1, m_2, \dots, m_r) \in \mathbb{N}^r$ and the coefficients $c_{\underline{m}}$, uniquely defined, are in A . From the definition (1.16) it is visible that if \underline{m} is such that there exists j with $n_j \geq d$, then $c_{\underline{m}} = 0$. So the non-zero coefficients are in natural correspondence with the points of a hypercube and there are at most d^r non-zero such coefficients.

Our next result describes certain linear dependence relations with coefficients in A (see Corollary 5.4):

Theorem C. *For any choice of $d, r \geq 1$ the following formula holds:*

$$\mathcal{L} \begin{pmatrix} 0 \\ dq^r \end{pmatrix} = (-1)^r [r]^{1-d} D_r^d \sum_{\underline{m}} c_{\underline{m}} \mathcal{L} \begin{pmatrix} 0 & m_1 q & \cdots & m_r q^r \\ d & d(q-1) & \cdots & d(q-1)q^{r-1} \end{pmatrix}.$$

To prove this result we proceed by induction on $r \geq 0$ with a study of the projection of the matrix identities of Theorem B on the coefficients situated on the d -th line first column of the matrices representing the endomorphisms in the canonical basis. The main result is Theorem 5.1, of which Corollary 5.4 is a consequence. Projecting the initial matrix identities of Theorem B to scalar identities of multiple polylogarithms is quite a delicate recursive procedure that is described in §5. In the case $d = 1$ it is clear that $f_d = 1$. Then the sum in the right-hand side of the formula of Theorem C reduces to the unique contribution of $\underline{m} = (0, \dots, 0)$ independently on r . We deduce the formula

$$\mathcal{L} \begin{pmatrix} 0 \\ q^r \end{pmatrix} = (-1)^r D_r^{-1} \mathcal{L} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & q-1 & \cdots & (q-1)q^{r-1} \end{pmatrix}$$

which agrees, mutatis mutandis, with [14, Theorem 7.2] and a formula in [24]. The reason is that when $m := \max\{m_1, \dots, m_r\} = 0$, any series as in (1.18) is a K -linear combination of multiple zeta values of Thakur. In fact the unital K -algebras of multiple zeta values and of multiple Carlitz polylogarithms 'at one' agree as a consequence of Ngo Dac's fundamental result [28, Theorem A] (analogue of Brown's theorem in [5]).

In the case $r = 1$ and arbitrary d it is not difficult to compute the coefficients $c_{\underline{n}}$. Indeed by (7.3), The formula of Theorem C becomes, in this case:

$$\mathcal{L} \begin{pmatrix} 0 \\ dq \end{pmatrix} = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} (\theta^{d-i} - \theta^{q(d-i)}) \mathcal{L} \begin{pmatrix} 0 & iq \\ d & d(q-1) \end{pmatrix}.$$

Linear relations between multiple polylogarithms at one. To conclude the introduction of our main results, we review our results in §7, where the reader can find complete definitions, basic notations and tools. The main result of this part is Theorem D below (see the more precise Theorem 7.6). In this result we show that the right-hand side of the identity of Theorem C is a K -linear combination of multiple polylogarithms at one, that is, series as in (1.18) with $m_1 = \cdots = m_r = 0$ (with r which can vary). It is possible to write this expansion in a completely explicit way. To simplify notations, write

$$\mathcal{L} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} = \mathcal{L}(n_1, n_2, \dots, n_r) = \mathcal{L}(\mathbf{n}),$$

so that we can stress the dependence on the *array* $\mathbf{n} = (n_1, \dots, n_r)$ (that is multi-index with positive integers as entries).

Theorem D. *For any choice of $d, r \geq 1$ Theorem C selects a non trivial expansion of $\mathcal{L}(dq^r) = \mathcal{L}(d)^{q^r}$ as a linear combination of multiple polylogarithms at one of higher depths: it exhibits arrays $\mathfrak{k}_{\underline{h}}$ of weight dq^r and elements $\kappa_{\underline{h}} \in K$, with $\underline{h} \in \{0, \dots, d-1\}^r$, such that*

$$\mathcal{L}(dq^r) = \sum_{\underline{h}} \kappa_{\underline{h}} \mathcal{L}(\mathfrak{k}_{\underline{h}}).$$

Note that there are several independent linear relations as in the above statement. The main point is that our non-commutative factorization selects one of them. The statement will be clarified in the paper. Theorem D is a consequence of Theorem 7.6, which describes the arrays more precisely using the following ideas. Multiple polylogarithms at one are series involving multiple sums as in §7. Analogously, multiple zeta values of Thakur are series involving multiple power sums defined in §7.3. Universal identities of such finite sums imply identities for multiple polylogarithms at one, or multiple zeta values of Thakur. When this occurs, we say that the identities occur at the level of finite sums. For instance, the results in [28] all occur at the level of finite sums, and Carlitz's identities (1.7) do not occur at the level of finite sums. In particular, to get Theorem D it suffices to take a limit in our identities of finite sums. The construction of the arrays $\mathfrak{k}_{\underline{h}}$ is completely explicit but the complete description is postponed to §7, where the tools that we sketch now will be described. It involves the *triangle product* (see [22, 30]) of arrays which are *concatenations* of powers of the array $(q-1)$ of depth one for the *shuffle product*, intertwined with other arrays of depth one. These identities can be also transposed into identities for multiple power sums and Thakur's multiple zeta values thanks to the work of Ngo Dac [28], but they are better suited for the formalism of multiple polylogarithms at one.

1.1. Motivic viewpoint. We describe a motivic interpretation of our formulas. In particular, we place them in the context of Drinfeld module “cycle integration” developed by Gekeler, Deligne, Anderson and others (see [2], [13] and [17]). Framed in this context, we interpret our formulas as providing an analogue of the integral representation for multiple polylogarithms and the resulting integral shuffle relations, restricted to q^n -th powers.

We first briefly explain the classical theory. Cycle integration was originally conceived as a way of comparing singular cohomology (also called Betti cohomology) with de Rham

cohomology, and as a means of producing periods from these two cohomology theories. The simplest example of this is the cohomology of the punctured complex plane. The first singular homology group is equal to the fundamental group of the punctured plane, which is generated by a single clockwise loop about the origin. The first de Rham cohomology group is generated by the differential form $\frac{dt}{t}$. Then the pairing between the two cohomology theories, which recovers the periods of the punctured plane, is given by cycle integration. For example, if we let γ denote the clock-wise, unit-circular path about the origin, then

$$\int_{\gamma} \frac{dt}{t} = 2\pi i.$$

Similarly, other meaningful arithmetic values may be identified as periods arising from cycle integration (see for example [23]). The periods important to this work are values of multiple polylogarithms. For our purposes here, we simplify to the iterated integral representation of the single polylogarithm (we refer the reader to [8] for a full account on multiple polylogarithms). For $|t| < 1$ we have

$$\sum_{n_1 \geq 1} \frac{t^{n_1}}{n_1^{s_1}} = \int_{\Delta(t)} \frac{dt_1}{1-t_1} \cdots \frac{dt_{s-1}}{1-t_{s-1}} \frac{dt_s}{t_s},$$

where $\Delta(t)$ is the simplex $\{(t_1, \dots, t_s) \in \mathbb{R}^s \mid t \geq t_1 \geq t_2 \geq \dots \geq t_s \geq 0\}$. This formula has a motivic interpretation by viewing the differentials $\frac{dt_i}{1-t_i}$ and $\frac{dt_j}{t_j}$ as being elements of the de Rham cohomology of the twice punctured plane $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and viewing the simplex $\Delta(t)$ as an element of the *relative* homology group of the twice punctured plane (mod a specified subgroup). See [8, Examples 3.346 and 3.348] for a more thorough analysis.

One key property that is derived from these iterated integral expressions is the integral shuffle relations for multiple polylogarithms. Namely, if we multiply two such iterated integrals together, one can “shuffle”, or rearrange, the simplices that define the area of integration to obtain depth-preserving \mathbb{Z} -linear relations between multiple polylogarithms (see for example [8, Example 1.118] and the ensuing discussion). We propose that the formulas of this paper provide a characteristic p analogue of such relations. Of course, other structures emerge beyond integration - which does not seem to be appropriate in this characteristic $p > 0$ setting - and our formulas are not proved using integration, rather they arise as a pairing between cohomology theories which mimics cycle integration.

We now wish to make a comparison between the characteristic 0 theory described above and the new positive characteristic function field formulas contained in our paper. There is a heavy amount of theory in the area of cohomology, periods and cycle integration of Drinfeld modules, so in our limited exposition here we will give a light overview following the account of Hartl and Juschka [19]. The first definitions of de Rham and Betti cohomology and the period pairing were given by Anderson [1] and Gekeler [13]. Given a Drinfeld module ϕ with period lattice Λ , define the Betti homology of ϕ to be

$$H_{B,1} = \Lambda.$$

The definition of the de Rham cohomology of ϕ is slightly more involved. First we define the space of biderivations of ϕ , denoted $D(\phi)$, which are certain families in $\mathbb{C}_{\infty}[\tau]\tau$ which

satisfy a natural difference equation related to ϕ (see [19, 5.43] for a succinct summary). Next, we identify a subspace of “strictly inner” biderivations, denoted $D_{\text{si}}(\phi)$. The de Rham cohomology of ϕ is then defined to be

$$H_{\text{dR}}^1(\phi) = D(\phi)/D_{\text{si}}(\phi),$$

which should remind the reader of the classical definition of de Rham cohomology given by smooth differential forms quotiented by exact forms.

The pairing between Betti homology and de Rham cohomology provides an analogue of cycle integration for Drinfeld modules, and recovers the periods and quasi-periods of a Drinfeld module. Given $\eta \in H_{\text{dR}}^1(\phi)$ and $\gamma \in H_{\text{B},1}$, we define a pairing

$$\int_{\gamma} \eta = - \sum_{n=0}^{\infty} \eta \left(\exp_{\phi} \left(\frac{\gamma}{\theta^{n+1}} \right) \right) \theta^n,$$

where \exp_{ϕ} is the exponential function associated to ϕ (see §2.2). Readers familiar with the theory of Anderson’s generating functions will recognize that this pairing consists of a difference operator applied to an Anderson generating function, which is then evaluated at θ (see [27] and [6]).

In order to apply this theory to the formulas in our paper, we must transport the above constructions to the setting of Anderson’s t -motives. We comment that Brownawell, Chang, Papanikolas and Wei study a similar theory extensively in the case of dual t -motives in the recent [6]; the counterpart of this theory for t -motives is not as well studied. For our treatment we follow the definitions and theory of [19, Sec. 2.3.5]. We define the Tate algebra to be

$$(1.19) \quad \mathbb{T} = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_{\infty}[[t]] : |b_i| \rightarrow 0 \right\},$$

where we recall that $|\cdot|$ is the norm of \mathbb{C}_{∞} . The Tate algebra admits an action of the Frobenius, which we denote by τ , that acts $\mathbb{F}_q[t]$ -linearly. Let M be the t -motive associated to an abelian t -module ϕ , which is isomorphic to $\mathbb{C}_{\infty}[t]^r$, for some $r \in \mathbb{Z}_+$, as a $\mathbb{C}_{\infty}[t]$ -module, but which also carries a (non-standard) action of the Frobenius, which we denote by τ_M (see Definition 3.1). Thus, $(M \otimes_{\mathbb{C}_{\infty}[t]} \mathbb{T})$ admits a diagonal action of the Frobenius, which we also denote by τ_M . Define the Betti cohomology of M with coefficients in \mathbb{C}_{∞} as the set of τ_M -invariant elements,

$$H_B^1(M) = (M \otimes_{\mathbb{C}_{\infty}[t]} \mathbb{T})^{\tau_M} = \{m \in M \otimes_{\mathbb{C}_{\infty}[t]} \mathbb{T} \mid \tau_M m = m\}.$$

We then define the de Rham cohomology with coefficients in \mathbb{C}_{∞} as

$$H_{\text{dR}}^1(M) = M/(t - \theta)M.$$

We define the comparison isomorphism from Betti to de Rham cohomology by first defining the natural multiplication map

$$\alpha : (M \otimes_{\mathbb{C}_{\infty}[t]} \mathbb{T})^{\tau_M} \otimes_{\mathbb{F}_q[t]} \mathbb{T} \rightarrow M \otimes_{\mathbb{C}_{\infty}[t]} \mathbb{T},$$

defined on simple tensors for $\lambda \in (M \otimes_{\mathbb{C}_\infty[t]} \mathbb{T})^{\tau M}$ and $z \in \mathbb{T}$ by

$$\lambda \otimes z \mapsto z \cdot \lambda.$$

Finally, we define the comparison map

$$h_{\text{B,dR}} : H_B^1(M) \rightarrow H_{\text{dR}}^1(M)$$

to be the map

$$h_{\text{B,dR}}(\lambda) = \alpha(\lambda)^{(1)} \pmod{(t - \theta)},$$

where if $f = \sum_i f_i t^i \in \mathbb{T}$, $f^{(1)} = \tau(f) = \sum_i f_i^q t^i \in \mathbb{T}$ (see §2.1). In particular, when M is a rank 1 t -motive (which is the case we consider in this paper), then we have $M \cong \mathbb{C}_\infty[t]$ as $\mathbb{C}_\infty[t]$ -modules, and so $M \otimes_{\mathbb{C}_\infty[t]} \mathbb{T} \cong \mathbb{T}$. In this case $M/(t - \theta)M \cong \mathbb{C}_\infty$ and the map $h_{\text{B,dR}}$ is given simply by

$$h_{\text{B,dR}}(\lambda) = \lambda^{(1)}(\theta).$$

We now restrict our attention to the case where M is the t -motive associated with the d -th tensor power of the Carlitz module, which is the setting of our main theorems. We give an alternate expression for the comparison isomorphism which connects with the formulas of our paper.

Let $\Omega = 1/\omega^{(1)} \in \mathbb{T}$, defined in (2.4), which satisfies

$$(1.20) \quad (t - \theta)^d (\Omega)^d = (\Omega^{(-1)})^d$$

(see [33, §3.3.5]), and so $\tau_M((\Omega^{(-1)})^d) = (\Omega^{(-1)})^d$, which implies $(\Omega^{(-1)})^d \in H_B^1(M)$. In fact, a quick calculation shows that that it is a generator, $H_B^1(M) = (\Omega^{(-1)})^d \mathbb{F}_q[t]$. In [18, Definition 2.14] the first author defined certain motivic maps δ_0^M and $\delta_{1,z}^M$, which we describe in (3.2) and (3.4). We now give an equivalent definition of the comparison isomorphism, which is proven in Lemma 3.3, namely

$$h_{\text{B,dR}}(\lambda) = \left(\delta_0^M \left(\frac{1}{t - \theta} \lambda \right) \right)_d,$$

where $(\cdot)_d$ denotes projection onto the d -th coordinate. Since we have a canonical isomorphism of $\mathbb{C}_\infty[t]$ -modules

$$H_{\text{dR}}^1(M) = M/(t - \theta)M \cong \frac{1}{t - \theta} M/M,$$

we view the above expression as a pairing between the element $\frac{1}{t - \theta} \in H_{\text{dR}}^1(M)$ and $\lambda \in H_B^1(M)$.

For example, for fixed d , we have

$$h_{\text{B,dR}}((\Omega^{(-1)})^d) = \left(\delta_0^M \left(\frac{1}{t - \theta} (\Omega^{(-1)})^d \right) \right)_d = -\frac{1}{\pi^d}.$$

which agrees with [19, Example 2.3.39] for $d = 1$ and with [6, Proposition 5.2.3] for the case of dual t -motives.

In the next result we propose an extension of the above formula which places our formulas from this paper in the context of the cycle integration described above (using some settings and notations as explained in §6; see Theorem 6.7).

Theorem E *We have the following formulas for $k \geq 0$:*

$$\delta_{1,\mathbf{z}}^M \left(\tau_M^k \frac{1}{t-\theta} (\tilde{\pi}^d \Omega^{(-1)})^d \right) = \mathcal{L} \left(\begin{array}{c} 0 \\ dq^k \end{array} \right),$$

where τ_M acts on $h \in \mathbb{T}$ as $\tau_M h = (t-\theta)^d h^{(1)}$.

We view Theorem E as expressing the polylogarithm $\mathcal{L} \left(\begin{array}{c} 0 \\ dq^k \end{array} \right)$ as a period, equal to the pairing between $\frac{1}{t-\theta}$, viewed as an element of the De Rham cohomology, and $(\Omega^{(-1)})^d$, an element of the Betti cohomology, modified with the extra operator τ_M^k , which acts naturally on both cohomologies, and with $\tilde{\pi}^d$ as a normalizing factor. We are also able to express the noncommutative factorization described in Theorem D in terms of elements from the t -motive evaluated under the $\delta_{1,\mathbf{z}}^M$ map, expressed in Proposition 6.8. Finally, we give an equality of these expressions inside the t -motive, which descends to the \mathbb{C}_∞ -valued identities described in Theorem C after evaluation under the map $\delta_{1,\mathbf{z}}^M$. This is described in the diagram found in Remark 6.11.

Thus, our new theorems can be viewed as an extension of cycle integration, where we transition from evaluating under δ_0^M to $\delta_{1,\mathbf{z}}^M$. Whether these formulas can be described using cycles from a relative homology class as in the characteristic 0 case, and how the map $\delta_{1,\mathbf{z}}^M$ fits into the broader context of cycle integration are natural questions. In [18] and [15] the first author uses the map $\delta_{1,\mathbf{z}}^M$ as an analogue of integration to construct an algebraic version of the Mellin transform related to zeta and L -functions. Thus it seems natural this map would appear in our analogue of cycle integration. These questions are a topic for future work.

The fact that it is this t -motive structure which leads to the new linear relations between Carlitz multiple polylogarithms, and the fact that the linear relations we produce in Theorem C preserve the depth of the Carlitz multiple polylogarithm, lead us to view our new formulas as a characteristic p analogue of the integral shuffle relations. We caution the reader that our identities do not describe a shuffle product structure - but rather a structure derived from the Frobenius operator. In this paper, we only observe the Frobenius structure acting on single polylogarithms (as in Theorem E), but ongoing investigations of the authors seem to indicate that this structure exists on the full set of multiple polylogarithms. This is a topic for future study. However, we also remind the reader that we prove our formulas at the level of finite sums, so they are actually stronger than the classical integral shuffle relations.

2. SOME BACKGROUND

We begin by presenting the necessary tools to work with the tensor powers $C^{\otimes d}$ of Carlitz's module C , especially its exponential and logarithm functions $\exp_{C^{\otimes d}}, \log_{C^{\otimes d}}$. Then we present some unpublished formulas due to Papanikolas allowing to compute its matrix

coefficients. We will then have at our disposal a toolbox of matrices and relations that will allow us to state and prove our main results.

2.1. Main notations and conventions. In all the following:

- \mathbb{N} denotes the set of natural integers and \mathbb{N}^* denotes the subset of positive integers.
- There is a common notation for all the multiplicative neutral elements of all rings (commutative and non-commutative) considered in this paper. In particular we write 1 for all identity matrices.
- \mathbb{F}_q denotes a finite field with q elements and characteristic p .
- $A = \mathbb{F}_q[\theta]$ denotes the \mathbb{F}_q -algebra of polynomials in the indeterminate θ with coefficients in \mathbb{F}_q .
- $K = \mathbb{F}_q(\theta)$ denotes the fraction field of A .
- K_∞ denotes the completion of K at the infinity place.
- \mathbb{C}_∞ denotes the completion of a separable closure of K_∞ .
- $|\cdot|$ is a fixed norm over \mathbb{C}_∞ such that $|\theta| > 1$.
- Given an \mathbb{F}_q -algebra B , we write $\tau(b) = b^q$ for $b \in B$. At the same time we denote by τ the \mathbb{F}_q -linear endomorphism that this map represents. We often write $f^{(1)}$ instead of $\tau(f)$. More generally, we write $f^{(k)}$ for $\tau^k(f)$.
- $B[\tau]$ (resp. $B[[\tau]]$) denotes the skew ring of finite linear combinations with coefficients in B of powers of τ (resp. the skew ring of formal series in powers of τ with coefficients in B).
- If B is any ring, $B^{r \times s}$ denotes the set of matrices with r lines and s columns, with its rs entries in B .
- In this paper $x, x', y, z, t \dots$ denote independent indeterminates or variables. In general t is reserved for a central variable, that is such that $\tau t = t\tau$. x, y, z, \dots are not central, unless otherwise specified.
- $(\cdot)^\top$ denotes the matrix transposition and $(\cdot)^\perp$ the antitransposition (see §2.3).
- If $M \in B^{r \times s}$, $M^{(k)}$ denotes the matrix obtained applying τ^k on all its entries.
- $[m] = \theta^{q^m} - \theta \in A$ if $m > 0$. If $m = 0$ we set $[0] = 1$.
- Empty products are by convention set to one and empty sums are set to zero.
- $\binom{i}{j}$ the reduction modulo p of the binomial coefficient when it is well defined.
- \mathbb{T} is the Tate algebra completion of $\mathbb{C}_\infty[t]$ for the Gauss norm.

2.2. The tensor powers of Carlitz's module. We recall the definition of $C^{\otimes d}$ from (1.12). There exists a unique exact sequence of A -modules

$$(2.1) \quad 0 \rightarrow \text{Ker}(\exp_{C^{\otimes d}}) \hookrightarrow \text{Lie}(C^{\otimes d})(\mathbb{C}_\infty) \xrightarrow{\exp_{C^{\otimes d}}} C^{\otimes d}(\mathbb{C}_\infty) \rightarrow 0$$

that we explain now. (1) $\text{Lie}(C^{\otimes d})$ is the Lie functor associated to $C^{\otimes d}$, uniquely defined, for B an A -algebra, by the multiplication by θ which is the left multiplication by the endomorphism

$$(2.2) \quad \mathbf{d}_\theta(\theta) := \theta \cdot 1 + N = \theta + N \in \text{End}_K(\mathbb{G}_a^d(K)),$$

where 1 is the identity matrix of $K^{d \times d}$ and N is defined in (1.13). Then, the multiplication by a general element $a \in A$ for the module structure of $\text{Lie}(C^{\otimes d})$ is given by the evaluation endomorphism

$$\mathbf{d}_\theta(a) = a(\theta + N).$$

In this paper we often have families of objects $(\text{Obj}^{(d)})_d$ depending on $d \in \mathbb{N}$, but we mainly study them for fixed choices of d . If we want to stress the dependence on d we use the notation $\text{Obj}^{(d)}$. If not, we simply write Obj . For example, we may write $N^{(d)}$ instead of N with the purpose of giving importance to the dependence in $d \in \mathbb{N}^*$.

(2) The map $\exp_{C^{\otimes d}}$ is entire and surjective, and is represented by a formal series ⁽⁴⁾

$$\exp_{C^{\otimes d}} = \sum_{i \geq 0} Q_i \tau^i \in \text{End}_K(\mathbb{G}_a^d(K))[[\tau]],$$

for a family of automorphisms $(Q_i)_{i \geq 0} \in \text{GL}_d(K)$ with $Q_0 = 1$ (this normalization makes $\exp_{C^{\otimes d}}$ to be the uniquely determined by (2.1)). The reason for which these are automorphisms is explained in §2.3.1.

(3) The kernel of $\exp_{C^{\otimes d}}$ can be computed explicitly. There exists

$$\Pi \in K_\infty[(-\theta)^{\frac{1}{q-1}}]^{d \times 1} \setminus \{0\}$$

such that $\text{Ker}(\exp_{C^{\otimes d}})$ is the free rank-one submodule of $\text{Lie}(C^{\otimes d})(\mathbb{C}_\infty)$ generated by Π (see [3, §2.5]). The last entry of Π belongs to $\mathbb{F}_q^\times \tilde{\pi}^d$. If d is prime with the characteristic p of \mathbb{F}_q , the d entries of Π are known to be algebraically independent over K (see Maurischat's [25, Theorem 8.1]; Corollary 8.5 *ibid.* allows to compute the transcendence degree in the general case) ⁽⁵⁾. We already fixed a fundamental period $\tilde{\pi}$ of Carlitz's module, in (1.6). From now on, we choose Π to be a generator of $\text{Ker}(\exp_{C^{\otimes d}})$ such that

$$\Pi = \begin{pmatrix} \tilde{\pi}_{d-1} \\ \vdots \\ \tilde{\pi}_1 \\ \tilde{\pi}_0 \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ * \\ \tilde{\pi}^d \end{pmatrix} \in K_\infty[(-\theta)^{\frac{1}{q-1}}]^{d \times 1}.$$

The matrix $\widehat{\Pi}$ introduced in (1.14) can also be computed easily by using [26, (3)]:

$$(2.3) \quad \mathbf{d}_x(\tau(\omega)(x))_{x=\theta}^d = \widehat{\Pi},$$

where ω is *Anderson-Thakur function*

$$(2.4) \quad \omega(x) := (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{x}{\theta^{q^i}}\right)^{-1},$$

⁴The formal exponential in the terminology of [3]; in our paper, we often identify exponential functions etc. with the corresponding formal counterparts.

⁵We are thankful to Andreas Maurischat for bringing his work to our attention.

with the element $(-\theta)^{\frac{1}{q-1}}$ chosen so that the residue of ω at $t = \theta$ equals $-\tilde{\pi}$, which defines a rigid analytic function $D_{\mathbb{C}_\infty}^\circ(0, |\theta|) \rightarrow \mathbb{C}_\infty$, where

$$D_{\mathbb{C}_\infty}^\circ(0, |\theta|) = \{z \in \mathbb{C}_\infty : |z| < |\theta|\}$$

and τ is the $\mathbb{F}_q(x)$ -linear extension of $c \mapsto c^q$. An alternative proof of (2.3) is proposed in [35, Proof of Lemma 4.6].

2.3. Papanikolas' d -matrices, ∂ -matrices, and anti-transposition. For an \mathbb{F}_q -algebra B , the centralizer $\mathcal{Z}(B)$ of N in $\text{End}(\mathbb{G}_a^d(B))$ can be identified with $\mathbb{G}_a^d(B)$ via the projection $[\cdot]_d$ on the last column in the canonical basis, determining a structure of A -algebra over $\mathbb{G}_a^d(B)$. We denote by h the inverse of the isomorphism $\mathcal{Z} \xrightarrow{[\cdot]_d} \mathbb{G}_a^d$. Alternatively, if $Z \in \mathbb{G}_a^d(B)$, we write \widehat{Z} instead of $h(Z)$. Explicitly,

$$Z =: \begin{pmatrix} z_{d-1} \\ \vdots \\ z_1 \\ z_0 \end{pmatrix}, \quad \widehat{Z} := \begin{pmatrix} z_0 & z_1 & \cdots & z_{d-1} \\ 0 & z_0 & \cdots & z_{d-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z_0 \end{pmatrix}.$$

We also note that the elementary commutation rule holds

$$(2.5) \quad \widehat{Z}U = \widehat{U}Z, \quad U, Z \in \mathbb{G}_a^d.$$

We review from Papanikolas' [31] other tools: the notions of d -matrix and ∂ -matrix, and the operation of anti-transposition, as well as certain formulas for the coefficients of the inverse series

$$\log_{C^{\otimes d}} := \exp_{C^{\otimes d}}^{-1}.$$

This reference being in this moment an unpublished monograph, we recall all we need here, with slightly modified notations that are suitable for our purposes. With F any field containing \mathbb{F}_q and t an indeterminate over F , we have the injective ring homomorphism

$$F(t) \xrightarrow{\mathbf{d}_t} \mathcal{Z}(F(t))$$

defined by the truncation at the order d of Taylor series expansions in the variable t in the following way:

$$f \mapsto \sum_{i \geq 0} \mathcal{D}_{t,i}(f) N^i \in \mathcal{Z}(F(t)),$$

where we set $N^0 = 1$ and we recall that $\mathcal{D}_{t,i}(f)$ denotes the i -th higher divided derivatives of f in the variable t . A \mathbf{d}_t -matrix (or, more simply, a d -matrix) is any matrix contained in the image of this homomorphism. Note that this definition agrees with (2.2) for $f \in \mathbb{F}_q[t]$.

Papanikolas also defines ∂_t -matrices, or more simply, ∂ -matrices. These are variants of Wronski matrices. Given an s -tuple $(f_1, \dots, f_s) \in F(t)^{1 \times s}$, we set

$$\partial_t(f_1, \dots, f_s) := \begin{pmatrix} \mathcal{D}_{t,d-1}(f_1) & \mathcal{D}_{t,d-1}(f_2) & \cdots & \mathcal{D}_{t,d-1}(f_s) \\ \mathcal{D}_{t,d-2}(f_1) & \mathcal{D}_{t,d-2}(f_2) & \cdots & \mathcal{D}_{t,d-2}(f_s) \\ \vdots & \vdots & & \vdots \\ \mathcal{D}_{t,1}(f_1) & \mathcal{D}_{t,1}(f_2) & \cdots & \mathcal{D}_{t,1}(f_s) \\ f_1 & f_2 & \cdots & f_s \end{pmatrix} \in F(t)^{d \times s}.$$

Papanikolas noticed that for $g \in F(t)$,

$$(2.6) \quad \partial_t(gf_1, \dots, gf_s) = \mathbf{d}_t(g) \partial_t(f_1, \dots, f_s).$$

In other words, the map

$$F(t) \xrightarrow{\partial_t} F(t)^{d \times 1}$$

which sends $f \in F(t)$ to the column matrix $\partial_t(f)$ (one column) is left $F(t)$ -linear via \mathbf{d}_t . This follows directly from (2.5) and Leibnitz's rule.

Here and in all the following, $(\cdot)^\top$ denotes the matrix transposition. *Anti-transposition* is another fundamental tool introduced in Papanikolas' work [31]. to define it, it is practical to first introduce the *reverse* $(f_1, \dots, f_s)^R$ of a row sequence (f_1, \dots, f_s) of objects. This is just the row sequence (f_s, \dots, f_1) . Now, given a matrix with entries in a ring R ,

$$M = (M^1, \dots, M^d) \in R^{d \times s}$$

so that M^j is the j -th column of M , the anti-transpose M^\perp is defined by

$$M^\perp = \left(\left((M^1, \dots, M^d)^R \right)^\top \right)^R \in R^{s \times d}.$$

We get the same construction if we apply instead an analogous reverse operation on column sequences, an this on the rows as anti-transposition and transposition commute. We also have $(M^\perp)^\perp = M$ ⁽⁶⁾. Given matrices M_1, M_2 such that $M_1 M_2$ is well defined,

$$(M_1 M_2)^\perp = M_2^\perp M_1^\perp.$$

2.3.1. Some formulas by Papanikolas. We describe Papanikolas explicit formulas allowing to compute explicitly the coefficients $Q_i \in \text{End}_K(\mathbb{G}_a^d)$ of $\exp_{C^{\otimes d}} = \sum_i Q_i \tau^i$ as well as the coefficients $P_j \in \text{End}_K(\mathbb{G}_a^d)$ of its inverse formal series

$$\log_{C^{\otimes n}} = \exp_{C^{\otimes d}}^{-1} = \sum_j P_j \tau^j \in \text{End}_K(\mathbb{G}_a^d(K))[[\tau]],$$

the associated *logarithm* (uniquely defined by $\exp_{C^{\otimes d}} \log_{C^{\otimes n}} = \log_{C^{\otimes n}} \exp_{C^{\otimes d}} = 1$). To describe these formulas we introduce a few more notations. Consider indeterminates x, y, t .

We set

$$H_{x,y} := \partial_x((x-y)^{d-1}, \dots, x-y, 1) \in \text{GL}_d(F(x,y)).$$

⁶In the case $s = d$ the anti-transposition is the adjunction operator for the symmetric bilinear form which associates to a couple (f, g) of elements of $R^{\oplus d}$ the standard scalar product of f and the reverse of g .

For elements $a, b \in F$, we also set

$$H_{a,b} := \left(H_{x,y} \right)_{\substack{x=a \\ y=b}} \in \mathrm{GL}_d(F).$$

Since $H_{x,y}$ is the idempotent matrix representing the identity endomorphism over the vector space $F(x,y)^{d \times 1}$ with respect to the bases $(\partial_x(x^i))_{0 \leq i \leq d-1}$ (target) and $(\partial_y(y^i))_{0 \leq i \leq d-1}$ (source), one sees easily that, given any $a, b, c \in F$,

$$(2.7) \quad (H_{a,b})^{-1} = H_{b,a}$$

and

$$(2.8) \quad H_{a,b} H_{b,c} = H_{a,c}.$$

These properties have been observed in [35] ⁽⁷⁾.

We also define, with x, t independent indeterminates, the matrices

$$\mathcal{T}_k(x, t) := \mathbf{d}_x \left((x-t) \cdots (x-t^{q^{k-1}}) \right)^{-d} \in \mathcal{Z}(F(x, t)),$$

and when it is well defined, for $a, b \in F$, we set

$$\mathcal{T}_k(a, b) := \mathcal{T}_k(x, t)_{\substack{x=a \\ t=b}}.$$

Then we have:

Proposition 2.1 (Papanikolas).

$$Q_i = \left((H_{x,y})^\perp \mathcal{T}_i(x, t) \right)_{\substack{x=\theta^{q^i} \\ z=\theta \\ t=\theta}} = (H_{\theta^{q^i}, \theta})^\perp \mathcal{T}_i(\theta^{q^i}, \theta),$$

$$P_i = (\mathcal{T}_i(x, t) H_{x,z})_{\substack{x=\theta \\ z=\theta^{q^i} \\ t=\theta^q}} = \mathcal{T}_i(\theta, \theta^q) H_{\theta, \theta^{q^i}}.$$

See [31, Proposition 4.3.6]. Papanikolas' proof of the second formula is also reproduced in [35, §7.2]. We are going to use the matrices $\mathcal{T}_i(\theta, \theta^q)$ often, so that we introduce an ad hoc notation for them:

$$\Gamma_i := \mathcal{T}_i(\theta, \theta^q) = \mathbf{d}_x \left((x - \theta^q) \cdots (x - \theta^{q^i}) \right)_{x=\theta}^{-d} \in \mathrm{GL}_d(K).$$

In particular,

$$(2.9) \quad P_i = \Gamma_i H_{\theta, \theta^{q^i}} \in \mathrm{GL}_d(K), \quad i \geq 0.$$

⁷Another way to check these identities is to first proving them in the case $d = 2$, and noticing that for general d (here we use the notation $(\cdot)^{(d)}$ because we consider several values for d), $H_{x,y}^{(d)}$ is a symmetric power:

$$H_{x,y}^{(d)} = \mathrm{Sym}^{(d-1)}(H_{x,y}^{(2)}).$$

3. CARLITZ OPERATORS, MOTIVIC PAIRINGS, APPLICATIONS

The main result in this section establishes the noncommutative factorization of the higher dimensional Carlitz operators described in the introduction, culminating in Proposition 3.19. This factorization is the main new innovation of this paper and leads directly to the applications on relations between polylogarithms discussed in the introduction. In order to develop the factorization, we study the Carlitz operators (mentioned just after the statement of Theorem A, see §3.1 below) from the point of view of t -motives. In Section 3.1 we define and study the basic properties of these operators. Then in Section 3.2 we review useful properties of t - and dual t -motives and the theory developed by the first author in [18]. Then in Section 3.3 we use this motivic theory to prove the noncommutative factorization of these higher Carlitz operators. Finally, in Section 3.4 we modify these operators and their factorization such that they give formulas in $\text{End}_K(\mathbb{G}_a^d(K))[\tau]$ rather than operators in $\text{End}_K(\text{End}_K(\mathbb{G}_a^d(K)))[\tau]$ as in [34].

3.1. Carlitz’s operators. As already mentioned in the introduction Papanikolas [31] introduces a natural generalization of Carlitz’s polynomials from the theory of Carlitz’s module [16, Chapter 3], see also [34, §4.4.2]. The construction mimics in higher dimensions Carlitz’s construction [16, Theorem 4.1.5]. For $W \in \text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d(\mathbb{C}_\infty))$ we expand:

$$\exp_{C^{\otimes d}} W \log_{C^{\otimes d}} = \sum_{k \geq 0} E_k(W) \tau^k \in \text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d(\mathbb{C}_\infty))[[\tau]].$$

The coefficients of this expansion can be explicitly described:

$$E_k(W) = \sum_{i=0}^k Q_i W^{(i)} P_{k-i}^{(i)}.$$

Note that for $d > 1$, E_k , \mathbb{F}_q -linear, does not represent an endomorphism of \mathbb{G}_a^d but rather an element of $\text{End}_{\mathbb{F}_q}(\text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d(\mathbb{C}_\infty)))$. In [35, §2] we write, alternatively

$$E_k = \sum_{i=0}^k (Q_i \otimes P_{k-i}^{(i)}) \tau^i \in \text{End}_{\mathbb{F}_q}(\text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d(\mathbb{C}_\infty))),$$

where τ is such that $\tau \cdot (U \otimes V) = (U^{(1)} \otimes V^{(1)}) \tau$. If $d = 1$ we recover the classical definition of Carlitz’s polynomials. If z varies in \mathbb{C}_∞ , $E_k(z)$ behaves as the evaluation of an \mathbb{F}_q -linear polynomial of degree q^k with kernel $A(< k)$, the \mathbb{F}_q -vector space of the elements of A that have degree in θ which is $< k$. Recall that Carlitz (see [34, Proposition 4.4.8 (3)] and [16, §3.2]) proved that, in the case $d = 1$ and using our notations,

$$\lim_{k \rightarrow \infty} P_k^{-1} E_k = \text{sin}_A.$$

In this simpler situation, $P_k^{-1} = l_k \in A$, where the sequence l_k has been introduced in (1.9). This led the second author to study in [35] the normalization

$$(3.1) \quad \mathcal{E}_k(W) := P_k^{-1} E_k(W)$$

in the general case $d \geq 1$ (by Proposition 2.1, P_k is invertible for all k). He proved (see [35, Theorem A]) that the sequence of functions $(\mathcal{E}_k)_{k \geq 0}$, as $k \rightarrow \infty$, tends to the exponential function $\exp_{\tilde{\phi}}$ of an Anderson A -module $\tilde{\phi}$ of dimension d^2 and rank d which is isomorphic to $(C^{\otimes d})^{\oplus d}$ (direct sum of d copies of $C^{\otimes d}$). The structure of this A -module is described in [35, Theorem 5.2]. In [35, Theorem B] it is proved that the restriction $\exp_{\tilde{\phi}}|_{\mathcal{Z}(\mathbb{C}_\infty)}$ of $\exp_{\tilde{\phi}}$ to $\mathcal{Z}(\mathbb{C}_\infty)$ has a noncommutative factorization extending (1.8) to the case $d \geq 1$. The factors in the factorization are of the form

$$\tilde{\gamma}_k + \tilde{\delta}_k \tau$$

with $\tilde{\gamma}_k, \tilde{\delta}_k$ explicit sequences of elements of $\text{End}_K(\text{End}_K(\mathbb{G}_a^n))$.

We now present another type of normalization of the operators E_k that give different properties, and radically simpler factorizations, with factors of the type

$$1 - \mathcal{L}_k \tau,$$

as in the introduction. They will give us our Theorems A and B.

3.2. Motivic pairings. We briefly recall the definition of the t -motive and dual t -motive attached to the d -th tensor power $C^{\otimes d}$ of the Carlitz module C . These objects have similarities so we give the definitions simultaneously. We will not use t -motives extensively, so we do not give a full account of their theory here (refer to [7] for a full account), but we do use these specific t -motive and dual t -motive in our proof of the main theorems in this section.

We denote by $\mathbb{C}_\infty[t, \tau_M]$ (denoted τ_M to differentiate it from the Frobenius τ acting on \mathbb{C}_∞) the noncommutative $\mathbb{C}_\infty[t]$ -algebra generated by formal finite $\mathbb{C}_\infty[t]$ -linear combinations $\sum_i c_i \tau_M^i$ with the obvious sum and the unique product defined by $\tau_M c = c^{(1)} \tau_M$ for $c \in \mathbb{C}_\infty[t]$ and $\tau_M^i \tau_M^j = \tau_M^{i+j}$ for all $i, j \geq 0$. Note that $\mathbb{C}_\infty[t, \tau_M]$ contains the \mathbb{C}_∞ -algebra $\mathbb{C}_\infty[\tau_M]$. Similarly, we define $\mathbb{C}_\infty[t, \sigma_N]$ by setting $\sigma_N c = c^{(-1)} \sigma_N$, for $c \in \mathbb{C}_\infty[t]$ (it contains $\mathbb{C}_\infty[\sigma_N]$).

We choose an integer $d \geq 1$.

Definition 3.1. The t -motive M (resp. the dual t -motive N) associated to $C^{\otimes d}$ is the left $\mathbb{C}_\infty[t, \tau_M]$ -module (respectively, the left $\mathbb{C}_\infty[t, \sigma_N]$ -module) determined by the free rank one $\mathbb{C}_\infty[t]$ -module $M = N = \mathbb{C}_\infty[t]$ with the left multiplication by τ_M (resp. by σ_N) given, for $m \in M$ and $n \in N$, by

$$\tau_M m = (t - \theta)^d m^{(1)}, \quad \sigma_N n = (t - \theta)^d n^{(-1)}.$$

We comment that - with suitable restrictions - the category of dual t -motives (t -motives) is equivalent (antiequivalent) to the category of t -modules (see [7]).

Both M and N are free $\mathbb{C}_\infty[t]$ -modules of rank 1 (by definition) and M is a free $\mathbb{C}_\infty[\tau_M]$ -module of rank d , while N is a free $\mathbb{C}_\infty[\sigma_N]$ -module of rank d . Indeed both M and N have the basis

$$(1, (t - \theta), \dots, (t - \theta)^{d-1})$$

(for the $\mathbb{C}_\infty[\tau_M]$ -module structure of M and for the $\mathbb{C}_\infty[\sigma_N]$ -module structure of N). In order to maintain consistency with the notation in [18] we set

$$g_k = (t - \theta)^{k-1}, \quad h_k = (t - \theta)^{n-k}$$

for $1 \leq k \leq d$ and consider

$$\mathbf{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix}$$

as a $\mathbb{C}_\infty[\tau_M]$ -basis for M and

$$\mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}$$

as a $\mathbb{C}_\infty[\sigma_N]$ -basis for N (later we will reuse this notation to define certain column matrices with entries in $A[t]$).

The maps δ_0^M, δ_0^N and $\delta_{1,\mathbf{z}}^M$. Following [18, §2], define (left) \mathbb{C}_∞ -linear maps

$$\delta_0^M : M \rightarrow \mathbb{C}_\infty^{d \times 1}, \quad \delta_{1,\mathbf{z}}^M : M \rightarrow \mathbb{C}_\infty, \quad \text{and} \quad \delta_0^N : N \rightarrow \mathbb{C}_\infty^{d \times 1}$$

in the following way. First, to define δ_0^M , recall that $M = \mathbb{C}_\infty[t]$ as a left $\mathbb{C}_\infty[t]$ -module. Every element m of M has a Taylor expansion at θ :

$$m = \sum_{i \geq 0} (\mathcal{D}_{t,i}(m))(\theta)(t - \theta)^i$$

(the sum is finite). We set:

$$(3.2) \quad \delta_0^M(m) = \begin{pmatrix} m(\theta) \\ (\mathcal{D}_{t,1}(m))(\theta) \\ \vdots \\ (\mathcal{D}_{t,d-1}(m))(\theta) \end{pmatrix} = \left(\left((\boldsymbol{\partial}_t^{(d-1)}(m))^\top \right)^\mathbb{R} \right)^\top \in \mathbb{C}_\infty^{d \times 1}.$$

In other words, δ_0^M is the column vector determined by the truncation to the order d of the Taylor series of m in θ , or equivalently, the reverse of the column matrix $\boldsymbol{\partial}_t^{(d-1)}(m)_{t=\theta}$. Note that this map is a particular case of the motivic map described in [18, Definitions 2.13 and 2.14], the reader can see that it is related to the residue map [3, (2.5.6)]. It can be alternatively computed by expanding m in the $\mathbb{C}_\infty[\tau_M]$ -basis \mathbf{g} of M : expand

$$m = \sum_{j=0}^k \sum_{i=0}^{d-1} \alpha_i^j \tau_M^j (t - \theta)^i,$$

with $a_i^j \in \mathbb{C}_\infty$ for some $k \geq 0$. It is easy to check that

$$(3.3) \quad \delta_0^M(m) = \begin{pmatrix} a_0^0 \\ a_1^0 \\ \vdots \\ a_{d-1}^0 \end{pmatrix},$$

which agrees with the reverse of the column matrix $\boldsymbol{\theta}_t^{(d-1)}(m)_{t=\theta}$. For fixed $\mathbf{z} \in \mathbb{C}_\infty^d$, we also define a related map $\delta_{1,\mathbf{z}}^M : M \rightarrow \mathbb{C}_\infty$ (which is not used until §6) by setting

$$(3.4) \quad \delta_{1,\mathbf{z}}^M(m) = \begin{pmatrix} a_0^0 \\ a_1^0 \\ \vdots \\ a_{d-1}^0 \end{pmatrix}^\top \mathbf{z} + \begin{pmatrix} a_0^1 \\ a_1^1 \\ \vdots \\ a_{d-1}^1 \end{pmatrix}^\top \mathbf{z}^{(1)} + \cdots + \begin{pmatrix} a_0^k \\ a_1^k \\ \vdots \\ a_{d-1}^k \end{pmatrix}^\top \mathbf{z}^{(k)}.$$

We extend $\delta_{1,\mathbf{z}}^M$ to certain elements of $M \otimes_{\mathbb{C}_\infty[[t]]} \mathbb{T} \cong \mathbb{T}$ in the following way. First, note that we have an injection $M \otimes_{\mathbb{C}_\infty[[\tau_M]]} \mathbb{C}_\infty[[\tau_M]] \hookrightarrow \mathbb{C}_\infty[[t]]$ given by the isomorphism $M \otimes_{\mathbb{C}_\infty[[\tau_M]]} \mathbb{C}_\infty[[\tau_M]] = \mathbb{C}_\infty[[\tau_M]]^d$ using the basis \mathbf{g} , then writing $m \in M \otimes_{\mathbb{C}_\infty[[\tau_M]]} \mathbb{C}_\infty[[\tau_M]]$ as

$$m = \sum_{j=0}^{\infty} \sum_{i=0}^{d-1} a_i^j \tau_M^j (t - \theta)^i \in \mathbb{C}_\infty[[t]].$$

We then let $M_{\mathbf{z}}\langle \tau_M \rangle$ be the subspace of $M \otimes_{\mathbb{C}_\infty[[\tau_M]]} \mathbb{C}_\infty[[\tau_M]]$ for which

$$\begin{pmatrix} a_0^k \\ a_1^k \\ \vdots \\ a_{d-1}^k \end{pmatrix}^\top \mathbf{z}^{(k)} \rightarrow 0$$

as $k \rightarrow \infty$. We define an extension of $\delta_{1,\mathbf{z}}^M$ to $M_{\mathbf{z}}\langle \tau_M \rangle$ by setting

$$(3.5) \quad \delta_{1,\mathbf{z}}^M(m) = \begin{pmatrix} a_0^0 \\ a_1^0 \\ \vdots \\ a_{d-1}^0 \end{pmatrix}^\top \mathbf{z} + \begin{pmatrix} a_0^1 \\ a_1^1 \\ \vdots \\ a_{d-1}^1 \end{pmatrix}^\top \mathbf{z}^{(1)} + \cdots.$$

By abuse of notation, we will sometimes consider $\delta_{1,\mathbf{z}}^M$ to have domain inside \mathbb{T} using the natural isomorphism described above when the image lands inside $\mathbb{T} \subset \mathbb{C}_\infty[[t]]$.

The construction is analogous to define δ_0^N , (compare again with [18, Definitions 2.13 and 2.14]), but we must take into account the $\mathbb{C}_\infty[[\sigma_N]]$ -basis \mathbf{h} . Therefore, expanding an element $n \in N$ in Taylor series at θ or in the basis \mathbf{h} writing $n = \sum_{i=0}^{d-1} b_i (t - \theta)^i$ we write, with obvious notations taking into consideration the left ideal $\mathbb{C}_\infty[[\sigma_N]]\sigma_N$ of $\mathbb{C}_\infty[[\sigma_N]]$ (the

higher derivatives now occur in reverse order):

$$(3.6) \quad \delta_0^N(n) := \begin{pmatrix} (\mathcal{D}_{t,d-1}(n))(\theta) \\ (\mathcal{D}_{t,d-2}(n))(\theta) \\ \vdots \\ n(\theta) \end{pmatrix} = \begin{pmatrix} b_{d-1}^0 \\ b_{d-2}^0 \\ \vdots \\ b_0^0 \end{pmatrix} \in \mathbb{C}_\infty^{d \times 1}.$$

This agrees with $\partial_t^{(d-1)}(n)_{t=\theta}$.

We can see M and N as subsets of $\mathbb{C}_\infty[t, \tau_M, \tau_M^{-1}]$ - and $\mathbb{C}_\infty[t, \sigma_N, \sigma_N^{-1}]$ -modules by setting $M_K = N_K = \mathbb{C}_\infty(t)$ and

$$\tau_M^{-1}m = \frac{1}{(t - \theta^{1/q})^d} m^{(-1)} \in \mathbb{C}_\infty(t), \quad m \in \mathbb{C}_\infty(t)$$

$$\sigma_N^{-1}n = \frac{1}{(t - \theta^q)^d} n^{(1)} \in \mathbb{C}_\infty(t), \quad n \in \mathbb{C}_\infty(t).$$

The maps δ_0^M, δ_0^N extend in unique way to the subspace of elements in $\mathbb{C}_\infty(t)$ which are regular at θ , so in particular we may evaluate them at the elements $\tau_M^{-k}m$ and $\sigma_N^{-j}n$ for all $j, k \geq 0$ and all $m \in M$ and $n \in N$.

Lemma 3.2. *The maps δ_0^M and δ_0^N have the following properties:*

- (1) Both δ_0^M and δ_0^N are \mathbb{C}_∞ -linear
- (2) $\delta_0^M(tm) = (\theta + N)^\top \delta_0^M(m)$.
- (3) $\delta_0^N(tn) = (\theta + N) \delta_0^N(n)$
- (4) For all $m \in \tau_M M$ and $n \in \sigma_N N$, we have $\delta_0^M(m) = 0$ and $\delta_0^N(n) = 0$.

Proof. These properties follow from the more general [18, Proposition 2.15]. These statements all follow from [19, Propositions 2.4.3 and 2.5.8]. \square

Lemma 3.3. *For $\lambda \in (\Omega^{(-1)})^d \mathbb{F}_q[t]$ we have*

$$\left(\delta_0^M \left(\frac{1}{t - \theta} \lambda \right) \right)_d = \lambda^{(1)}(\theta).$$

Proof. First recall (1.20). Thus

$$\frac{1}{t - \theta} (\Omega^{(-1)})^d = (t - \theta)^{d-1} (\Omega)^d,$$

Then, since $\Omega(\theta) = 1/\tilde{\pi}$, we apply the alternate expression for δ_0^M from (3.3) and find that the bottom coordinate of $\delta_0^M((t - \theta)^{d-1} (\Omega)^d)$ is $(1/\tilde{\pi})^d$. This shows that the theorem holds for $\lambda = (\Omega^{(-1)})^d$. We conclude that it holds for any $\lambda \in (\Omega^{(-1)})^d \mathbb{F}_q[t]$ by observing that

$$\left(\delta_0^M \left(a(t) (\Omega^{(-1)})^d \right) \right)_d = \left(a(\theta + N) \cdot \delta_0^M \left((\Omega^{(-1)})^d \right) \right)_d = a(\theta) \left(\delta_0^M \left((\Omega^{(-1)})^d \right) \right)_d,$$

for any $a(t) \in \mathbb{F}_q[t]$. Finally, $\mathbb{F}_q[t]$ is fixed under Frobenius twisting, so $a(\theta) = a^{(1)}(\theta)$, finishing the proof. \square

3.2.1. *Pairings.* Recall the commutator $\mathcal{Z}(\mathbb{C}_\infty) \subset \mathbb{C}_\infty^{d \times d}$ of N introduced at the beginning of §2.3.

Definition 3.4. Fix $\ell \geq 0$ and $W \in \mathcal{Z}(\mathbb{C}_\infty)$. For $x \in \mathbb{C}_\infty[t, \tau_M]$ and $y \in \mathbb{C}_\infty[t, \sigma_N]$, we define the pairing

$$E_\ell : \mathbb{C}_\infty[t, \tau_M] \times \mathbb{C}_\infty[t, \sigma_N] \rightarrow \mathbb{C}_\infty^{d \times d},$$

$$E_\ell(x, y; W) = \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} (xg_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (yh_m))^{(j)} \right)_{k,m=1}^d \in \mathbb{C}_\infty^{d \times d}.$$

Proposition 3.5. For all $\ell \geq 1$ the pairing E_ℓ satisfies the following properties:

- (1) $E_\ell(tx, y; W) = E_\ell(x, ty; W)$
- (2) $E_\ell(\theta^{q^\ell} x, y; W) = E_\ell(x, \theta y; W)$
- (3) $E_\ell(x, \sigma_N y; W) = E_{\ell-1}(x, y; W)$
- (4) $E_\ell(\tau_M x, y; W) = E_{\ell-1}(x, y; W)^{(1)}$
- (5) $E_\ell(1, 1; W) = \sum_{j=0}^{\ell} Q_j W^{(j)} P_{\ell-j}^{(j)}$.

Proof. Part (1) follows from Lemma 3.2 Parts (2) and (3), from the fact that t commutes with τ_M and σ_N and the fact that W is in $\mathcal{Z}(\mathbb{C}_\infty)$. The proof of Part (2) is a simple computation using the fact that $\tau_M z = z^q \tau_M$ and $\sigma_N z = z^{1/q} \sigma_N$ for all $z \in \mathbb{C}_\infty$. Parts (3) and (4) are a direct computation using Lemma 3.2 Parts (1) and (4). Part (5) follows from [18, Corollaries 3.8 and 4.5]. \square

Definition 3.6. We also introduce a modified version of E_ℓ which satisfies simpler properties, by setting

$$E'_\ell(x, y; W) = \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} (xg_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (yh_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d.$$

Proposition 3.7. E'_ℓ satisfies the following properties, for $W \in \mathcal{Z}(\mathbb{C}_\infty)$.

- (1) $E'_\ell(tx, y; W) = E'_\ell(x, ty; W)$
- (2) $E'_\ell(\theta x, y; W) = E'_\ell(x, \theta y; W) = \theta E'_\ell(x, y; W)$
- (3) $E'_\ell(1, 1, W) = E_\ell(1, 1, W) H_{\theta^{q^\ell}, \theta}$

Proof. Parts (1) and (2) follow similarly as in the proof of Proposition 3.5. For part (3), we recall from Proposition 3.5 that $E_\ell(1, 1; W) = \sum_{j=0}^{\ell} Q_j W^{(j)} P_{\ell-j}^{(j)}$. Additionally, from Proposition 2.1 we have $P_\ell = \Gamma_\ell H_{\theta, \theta^{q^\ell}}$. Then, from (2.7) and (2.8) we see that

$$\begin{aligned} P_{\ell-j}^{(j)} H_{\theta^{q^\ell}, \theta} &= (\Gamma_{\ell-j})^{(j)} (H_{\theta, \theta^{q^{\ell-j}}})^{(j)} H_{\theta^{q^\ell}, \theta} \\ &= (\Gamma_{\ell-j})^{(j)} (H_{\theta^{q^j}, \theta^{q^\ell}}) H_{\theta^{q^\ell}, \theta} \\ &= (\Gamma_{\ell-j})^{(j)} H_{\theta^{q^j}, \theta} \\ &= (\Gamma_{\ell-j})^{(j)} (H_{\theta, \theta^{q^{\ell-j}}})^{(j)}. \end{aligned}$$

Finally, a careful comparison with definitions shows that this is the same as

$$\left(\delta_0^N (\sigma_N^{j-\ell} (h_1)^{(-\ell)})^{(j)}, \dots, \delta_0^N (\sigma_N^{j-\ell} (h_d)^{(-\ell)})^{(j)} \right).$$

□

Proposition 3.8. *For all $W \in \mathcal{Z}(\mathbb{C}_\infty)$ we have that $E'_\ell(W)^\perp = E'_\ell(W)$.*

Proof. By Proposition 3.7 we see that $E'_\ell((t - \theta), 1; W) = E'_\ell(1, (t - \theta); W)$. Recalling the definitions of g_k and h_k , we see that $(t - \theta)g_k = g_{k+1}$ for $1 \leq k \leq d - 1$ and $(t - \theta)h_k = h_{k-1}$ for $2 \leq k \leq d$. The proposition then follows from the definition of E'_ℓ . □

Definition 3.9. Set, for $W \in \mathcal{Z}(\mathbb{C}_\infty)$,

$$F_\ell(x, y; W) = \Gamma_\ell^{-1} E'_\ell(x, y; W).$$

Going forward we will often set $x = y = 1$ in the above pairings. We will thus suppress that notation, and simply write $E_\ell(W) := E_\ell(1, 1; W)$ or simply E_ℓ (if the value of W is understood from the context) and similarly for E'_ℓ and F_ℓ . We record here that

$$(3.7) \quad F_\ell(W) = \Gamma_\ell^{-1} E_\ell(W) H_{\theta^{q^\ell}, \theta}.$$

Remark 3.10. We make a brief comment about the utility of the three pairings E_ℓ , E'_ℓ and F_ℓ . Each is natural from a certain viewpoint, and thus it is worthwhile to study each of them in turn. First, E_ℓ is the most natural generalization of Carlitz's polynomials, in view of 3.5 (5). On the other hand, E'_ℓ provides the normalization allowing to obtain the non-commutative factorization of Theorem 3.13. Finally, F_ℓ is a final normalization such that the limit $\ell \rightarrow \infty$ exists.

3.3. A recursive formula for E'_ℓ . In this section we derive a recursive formula (a certain non-commutative factorization) for E'_ℓ which results in our Theorem B (see Theorem 4.9) giving new families of linear relations between function field multiple polylogarithms (see §5 for details on multiple polylogarithms, the main result of this subsection is Theorem 3.13). Such recursive formulas are not unique in general; the second author considers an infinite family of similar recursive formulas in [35]. The relationship between these different noncommutative factorizations will be discussed in Remark 3.22.

Definition 3.11. For $1 \leq \ell \leq d$ we define matrices $M_\ell \in \text{GL}_d(K)$ as follows. First, we expand in unique way for $1 \leq \ell \leq d$

$$(t - \theta^{q^\ell})^d = (\theta - \theta^{q^\ell})^d + a_1(t - \theta) + a_2(t - \theta)^2 + \dots + a_{d-1}(t - \theta)^{d-1} + (t - \theta)^d,$$

for coefficients $a_i \in K$ depending on ℓ (note that the coefficients a_i can be written as evaluations of hyperderivatives). We then set

$$M_\ell = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{d-1} & 1 & 0 & \dots & 0 \\ a_{d-2} & a_{d-1} & 1 & \dots & 0 \\ a_{d-3} & a_{d-2} & a_{d-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & 1 \end{pmatrix}.$$

Remark 3.12. The matrices M_ℓ can be equivalently defined as follows. Write

$$\mathbf{g} = (g_1, \dots, g_d)^\top = \begin{pmatrix} 1 \\ t - \theta \\ \vdots \\ (t - \theta)^{d-1} \end{pmatrix} \in \mathbb{C}_\infty[t]^{d \times 1}.$$

Then

$$(3.8) \quad (t - \theta^{q^\ell})^d \cdot \mathbf{g} = \left(\mathbf{d}_t((t - \theta^{q^\ell})^d) \right)_{t=\theta} \mathbf{g} + (t - \theta)^d \cdot M_\ell \cdot \mathbf{g}.$$

In all the following we also write

$$(3.9) \quad H := H_{\theta^q, \theta},$$

to simplify our formulas.

Theorem 3.13. *We have the following recursive formula for $E'_\ell(W)$, with $W \in \mathcal{Z}(\mathbb{C}_\infty)$:*

$$E'_\ell(W) = -\mathbf{d}_t \left((t - \theta^{q^\ell})^{-d} \right)_{t=\theta} \left(M_\ell \cdot H^\perp (E'_{\ell-1}(W))^{(1)} H - E'_{\ell-1}(W) \right).$$

Proof. Our starting point is the identity

$$(3.10) \quad E'_\ell((t - \theta^{q^\ell})^d, 1; W) = E'_\ell(1, (t - \theta^{q^\ell})^d; W),$$

which follows from Proposition 3.7 (where it is crucial that $W \in \mathcal{Z}(\mathbb{C}_\infty)$). The right-hand side of (3.10) gives

$$\begin{aligned} & \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(g_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} ((t - \theta^{q^\ell})^d h_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(g_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (t - \theta)^d (h_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(g_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} \circ \sigma_N (h_m)^{(1-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(g_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell+1} (h_m)^{(1-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= E'_{\ell-1}(1, 1; W). \end{aligned}$$

We recall that $\sigma_N z = (t - \theta)z^{(-1)}$ for $z \in \mathbb{C}_\infty$ and that $\delta_0^N(\sigma_N(h_i)) = 0$, we have used these facts in the last two lines.

On the other hand, the left-hand side of (3.10) gives (\mathbf{e}_k is the k -th standard basis vector of $\mathbb{C}_\infty^{d \times 1}$)

$$\begin{aligned} & \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} ((t - \theta^{q^\ell})^d \mathbf{g}_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (h_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} (\mathbf{e}_k^\top (t - \theta^{q^\ell})^d \mathbf{g}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (h_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d. \end{aligned}$$

We then use the expansion for $(t - \theta^{q^\ell})$ from (3.8) and Proposition 3.7 Part (2) to write this as

$$\begin{aligned} &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} (\mathbf{e}_k \mathbf{d}_t ((t - \theta^{q^\ell})^{-d})_{t=\theta} \mathbf{g} + (t - \theta)^d \cdot M_\ell \cdot \mathbf{g}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (h_m)^{(-\ell)})^{(j)} \right)_{k,m=1}^d \\ &= \mathbf{d}_t \left((t - \theta^{q^\ell})^{-d} \right)_{t=\theta} E'_\ell(1, 1; W) + M_\ell E'_\ell((t - \theta)^d, 1; W). \end{aligned}$$

The first term of the above equality will become the second term in the right-hand side of the statement of the proposition. We must deal with the second term. We first observe that for $k \geq 0$

$$(3.11) \quad (t - \theta^{(-1)})^k = (\theta - \theta^{(-1)})^k + \mathcal{D}_{t,1}(t - \theta^{(-1)})_{t=\theta}^k (t - \theta) + \cdots + \mathcal{D}_{t,k-1}(t - \theta^{(-1)})_{t=\theta}^k (t - \theta)^{k-1} + (t - \theta)^k.$$

We observe that the coefficients of the powers of $(t - \theta)$ in (3.11) twisted give the columns of H . Then using the fact that $g_k = (t - \theta)^{k-1}$, that $h_m = (t - \theta)^{d-m}$ and (3.11) we write, with

$$\mathbf{h} = (h_1, \dots, h_d)^\top = \begin{pmatrix} (t - \theta)^{d-1} \\ (t - \theta)^{d-2} \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}_\infty[t]^{d \times 1}$$

and recalled that H is defined in (3.9),

$$\begin{aligned}
E_\ell((t - \theta)^d, 1; W) &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j} \circ \tau_M (g_k^{(-1)}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (h_m^{(-\ell)})^{(j)}) \right)_{k,m=1}^d \\
&= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{1-j} (g_k^{(-1)}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (h_m^{(-1)})^{(1-\ell)})^{(j)} \right)_{k,m=1}^d \\
&= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{1-j} (\mathbf{e}_k^\top (H^{(-1)})^\perp \mathbf{g}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell} (\mathbf{h}^\top H^{(-1)} \mathbf{e}_m)^{(1-\ell)})^{(j)} \right)_{k,m=1}^d \\
&= H^\perp \cdot E'_{\ell-1}(1, 1; W)^{(1)} \cdot H
\end{aligned}$$

Putting all these calculations together yields the theorem. \square

Corollary 3.14. *We have the following recursive formula for $F_\ell(W)$, for all $W \in \mathcal{Z}(\mathbb{C}_\infty)$*

$$F_\ell(W) = -\Gamma_{\ell-1}^{-1} \cdot M_\ell \cdot H^\perp \cdot (\Gamma_{\ell-1} F_{\ell-1}(W))^{(1)} \cdot H + F_{\ell-1}(W).$$

Proof. This follows from Theorem 3.13 and the Definition 3.9 of F_ℓ after noticing that $\mathbf{d}_t \left((t - \theta^{q^\ell})^{-d} \right)_{t=\theta} \cdot \Gamma_\ell^{-1} = \Gamma_{\ell-1}^{-1}$. \square

Definition 3.15. In order to simplify notation, we define a map for $k \geq 0$

$$L_k : \text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d) \rightarrow \text{End}_{\mathbb{C}_\infty}(\mathbb{G}_a^d),$$

$$L_k(W) = \Gamma_k^{-1} \cdot M_{k+1} \cdot H^\perp \cdot \Gamma_k^{(1)} \cdot W^{(1)} \cdot H.$$

In this notation we have $F_\ell(W) = (1 - L_{\ell-1})(F_{\ell-1}(W))$.

Using the notation above, we have the following noncommutative factorization of the operator F_ℓ .

Corollary 3.16. *For all $W \in \mathcal{Z}(\mathbb{C}_\infty)$ we have:*

$$F_\ell(W) = \left((1 - L_{\ell-1}) \circ (1 - L_{\ell-2}) \circ \cdots \circ (1 - L_1) \circ (1 - L_0) \right)(W).$$

Remark 3.17. Corollary 3.16 is very similar to the noncommutative factorization given for the Carlitz sine function, as described in (1.11) (when one lets the parameter tend to infinity). We remark that one key difference is that our L_0 is not equal to the identity if $d > 1$.

We deduce the following recursive formula for Papanikolas' generalization of Carlitz polynomials, restricted to $\mathcal{Z}(\mathbb{C}_\infty)$.

Corollary 3.18. *Consider $W \in \mathcal{Z}(\mathbb{C}_\infty)$. Then*

$$(3.12) \quad E_\ell(W) = \sum_{j=0}^{\ell} Q_j W^{(j)} P_{\ell-j}^{(j)} = \Gamma_k \left((1 - L_{\ell-1}) \cdots (1 - L_0)(W) \right) H_{\theta, \theta^{q^\ell}}.$$

3.4. One sided noncommutative factorizations. In the recursive formulas we have obtained so far (Theorem 3.13, Corollaries 3.14, 3.16, 3.18), there is always multiplication on the left and right by nonzero matrices. We can say that the recursive formulas take place in $\text{End}_{\mathbb{F}_q}(\text{End}_K(\mathbb{G}_a^d(K)))$.

An important observation we make in this subsection is that, with an appropriate normalization, it is possible to neutralize the factors which multiply on the right so that we can obtain recursive formulas, and therefore noncommutative factorizations, in $\text{End}_K(\mathbb{G}_a^d(K))[\tau]$. This is made possible by the formulas (2.8).

Define

$$(3.13) \quad \tilde{\mathcal{L}}_k := \Gamma_k^{-1} M_{k+1} H^\perp (\Gamma_k)^{(1)} \in \text{GL}_d(K).$$

Write:

$$(3.14) \quad \mathbf{E}_k := \Gamma_k^{-1} \sum_{j=0}^k Q_j (\Gamma_{k-j})^{(j)} \tau^j \in \text{End}_K(\mathbb{G}_a^d(K))[\tau].$$

We have:

Proposition 3.19. *The following identity holds in $\text{End}_K(\mathbb{G}_a^d(K))[\tau]$:*

$$\mathbf{E}_k = (1 - \tilde{\mathcal{L}}_{k-1} \tau) \cdots (1 - \tilde{\mathcal{L}}_1 \tau) (1 - \tilde{\mathcal{L}}_0 \tau).$$

Proof. Write $W = \widehat{Z} \in \mathcal{Z}(\mathbb{C}_\infty)$ with $Z \in \mathbb{G}_a^d(\mathbb{C}_\infty)$. Recall that $L_k(W) = \mathcal{L}_k W^{(1)} H_{\theta^q, \theta}$. We note that

$$(3.15) \quad F_k(W) = \Gamma_k^{-1} E_k(W) H_{\theta^{q^k}, \theta} = \Gamma_k^{-1} \sum_{j=0}^k Q_j (\Gamma_{k-j})^{(j)} W^{(j)} H_{\theta^{q^j}, \theta}.$$

The first identity is clear by (3.7). The second identity follows easily combining (2.8) and (2.7), as well as Proposition 2.1, which implies

$$P_{k-j}^{(j)} = (\Gamma_{k-j})^{(j)} H_{\theta^{q^j}, \theta^{q^k}}$$

(note that W and $(\Gamma_k)^{(j)}$ commute). From Corollary 3.16 we get that

$$(3.16) \quad F_\ell(W) = (1 - L_{\ell-1}) \circ (1 - L_{\ell-2}) \circ \cdots \circ (1 - L_1) \circ (1 - L_0)(W) \\ = \sum_{j=0}^{k-1} (-1)^j \sum_{0 \leq i_1 < \cdots < i_j \leq k-1} (L_{i_j} \circ \cdots \circ L_{i_1})(W).$$

We observe that by Definition 3.15 the term of the above sum involving $W^{(j)}$ is of the form

$$M W^{(j)} H^{(j-1)} \cdots H^{(1)} H = M W^{(j)} H_{\theta^{q^j}, \theta}$$

by iterating (2.8), where $M \in K^{d \times d}$ is some explicitly computable matrix. Finally, observe that

$$(3.17) \quad W^{(j)} H_{\theta^{q^j}, \theta} = (W H_{\theta, 0})^{(j)} H_{0, \theta}.$$

We compute the projection on the last column of $F_k(W)$, which we denote $[F_k(W)]_d$. We use the identities (3.15) and (3.17). With e_d the d -th component of the canonical basis of \mathbb{G}_a^d , we observe that if $M = (*, \dots, *, M_d)$ and $H = (*, \dots, *, e_d)$ are two matrices of $R^{d \times d}$ having as last columns M_d and e_d , then $MH = (*, \dots, *, M_d)$; its last column is M_d . The projection $[\cdot]_d$ on the last column $R^{d \times d} \rightarrow R^{d \times 1}$ (R any ring) induces an isomorphism from $\mathcal{Z}(\mathbb{C}_\infty)$ to $\mathbb{C}_\infty^{d \times 1}$. We have $[W]_d = Z$. Composing the inverse of this isomorphism with the right multiplication by $H_{\theta, 0}$ and then projecting again on the last column yields

$$[E'_k(W)]_d = \Gamma_k^{-1} \sum_{j=0}^k Q_j (\Gamma_{k-j})^{(j)} \tau^j(Z)$$

because $H_{a,b}$ is lower triangular with the last column equal to e_d . Now recall that if M and $W = (*, \dots, *, W_d)$ are two matrices of $R^{d \times d}$, $[MW]_d = MW_d$. Hence from (3.16) we deduce, projecting on the last column, the requested identity of endomorphisms in $\text{End}_K(\mathbb{G}_a^d(K))[\tau]$. \square

The sequence of operators $(\mathbf{E}_k)_k$ has a limit, represented by an \mathbb{F}_q -linear, entire, surjective function. This is described in the next result.

Proposition 3.20. *We have:*

$$\lim_{k \rightarrow \infty} \mathbf{E}_k = \widehat{\Pi}^{-1} \exp_{C^{\otimes d}} \widehat{\Pi} = \sin_A^{\otimes d}.$$

Proof. Select an integer $j \geq 0$. We aim at computing the limit, for $k \rightarrow \infty$, of the sequence

$$(\Gamma_k^{-1} Q_j (\Gamma_{k-j})^{(j)})_k.$$

Going back to the definition of Γ_k , this is equal to

$$\mathbf{d}_x \left((x - \theta^q) \cdots (x - \theta^{q^k}) \right)_{x=\theta}^d Q_j \left(\mathbf{d}_z \left((z - \theta^q) \cdots (z - \theta^{q^{k-j}}) \right)_{z=\theta}^{-d} \right)^{(j)}.$$

We now transform this expression so that we let copies of Anderson-Thakur function ω (2.4) appear, which will allow us to conclude. The previous expression is obviously identical to

$$\begin{aligned} \mathbf{d}_x \left((-\theta)^{q+\dots+q^k} \left(1 - \frac{x}{\theta^q} \right) \cdots \left(1 - \frac{x}{\theta^{q^k}} \right) \right)_{x=\theta}^d Q_j \times \\ \times \left(\mathbf{d}_z \left((-\theta)^{q+\dots+q^{k-j}} \left(1 - \frac{z}{\theta^q} \right) \cdots \left(1 - \frac{z}{\theta^{q^{k-j}}} \right) \right)_{z=\theta}^{-d} \right)^{(j)}. \end{aligned}$$

There are scalar coefficients that behave like constants with respect to the differential operators involved. Note that

$$(-\theta)^{q+\dots+q^k} \left((-\theta)^{q+\dots+q^{k-j}} \right)^{-q^j} = (-\theta)^{-\frac{q}{q-1}} (-\theta)^{\frac{q^{j+1}}{q-1}}.$$

This allows to reformulate the last expression as:

$$\begin{aligned} \mathbf{d}_x \left((-\theta)^{-\frac{q}{q-1}} \left(1 - \frac{x}{\theta^q} \right) \cdots \left(1 - \frac{x}{\theta^{q^k}} \right) \right)_{x=\theta}^d Q_j \times \\ \times \left(\mathbf{d}_z \left((-\theta)^{-\frac{q}{q-1}} \left(1 - \frac{z}{\theta^q} \right) \cdots \left(1 - \frac{z}{\theta^{q^{k-j}}} \right) \right)_{z=\theta}^{-d} \right)^{(j)}. \end{aligned}$$

It is now easy to recognize the convergence of all the sequences of matrices taken individually. Recall the factorization of Anderson-Thakur's omega function (2.4). For $k \rightarrow \infty$ we have

$$\lim_{k \rightarrow \infty} \mathbf{d}_x \left((-\theta)^{-\frac{q}{q-1}} \left(1 - \frac{x}{\theta^q} \right) \cdots \left(1 - \frac{x}{\theta^{q^k}} \right) \right)_{x=\theta}^d = \mathbf{d}_x (\tau(\omega)(x))_{x=\theta}^{-d},$$

where $\tau x = x\tau$ so that here, x is central (the operator \mathbf{d}_x extends in unique way to formal series in ascending powers of x , and similarly for \mathbf{d}_z). By (2.3) the limit is therefore equal to $\widehat{\Pi}^{-1}$. In a similar way we can compute the second limit, corresponding to the product in the right. We therefore obtain

$$\lim_{k \rightarrow \infty} \Gamma_k^{-1} Q_j (\Gamma_{k-j})^{(j)} = \widehat{\Pi}^{-1} Q_j \widehat{\Pi}^{(j)}.$$

□

Remark 3.21. Similarly, we can prove:

$$\lim_{k \rightarrow \infty} F_k(Z) = \widehat{\Pi}^{-1} \exp_{C^{\otimes d}} (Z \widehat{\Pi} H_{\theta,0}) H_{0,\theta}.$$

Remark 3.22. We discuss the relation between our factorization of E_ℓ to that of $E_{\phi,k}$ of [35]. Observe first that, with the notations of *ibid.*, $E_k = E_{\phi,k}$, but we choose different normalizations of these operators, as is seen by comparing (3.1) with (3.7). Besides that, the factorizations are obtained using fundamentally different methods. Namely, the factorization of $E_{\phi,k}$ is obtained by the second author by noting that for $W \in \mathcal{Z}(\mathbb{C}_\infty)$ we have

$$C_\theta^{\otimes d} \exp_{C^{\otimes d}} (W \log_{C^{\otimes d}}) = \exp_{C^{\otimes d}} (W \log_{C^{\otimes d}}) C_\theta^{\otimes d}.$$

One then collects like powers of Frobenius and gets identities between $E_{\phi,k}$ and $E_{\phi,k-1}$. In the language of our motivic constructions, the above identity is equivalent to observing that

$$E_k(t, 1; W) = E_k(1, t; W).$$

Indeed, it is shown in [18, Lemma 2.10(2) and Example 3.13] that $t\mathbf{g} = C_\theta^{\otimes d}\mathbf{g}$. Thus

$$\begin{aligned}
E_k(t, 1; W) &= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(tg_k))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell}(h_m))^{(j)} \right)_{k,m=1}^d \\
&= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(\mathbf{e}_k^\top C_t^{\otimes d} \mathbf{g}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell}(h_m))^{(j)} \right)_{k,m=1}^d \\
&= \left(\sum_{j=0}^{\ell} (\delta_0^M (\tau_M^{-j}(\mathbf{e}_k^\top (\theta I_d + N + e_{d,1}^{(d)} \tau) \mathbf{g}))^{(j)})^\top W^{(j)} \delta_0^N (\sigma_N^{j-\ell}(h_m))^{(j)} \right)_{k,m=1}^d \\
&= (\theta + N)E_k(1, 1; W) + e_{d,1}E_k(\tau_M, 1; W) \\
&= (\theta + N)E_k(1, 1; W) + e_{d,1}E_{k-1}(1, 1; W)^{(1)},
\end{aligned}$$

where we remind the reader of the definitions of the matrices N and $e_{d,1}$ in (1.13) and just after. Similarly,

$$E_k(1, t; W) = E_k(1, 1; W)(\theta^{q^k} + N) + E_{k-1}(1, 1; W)E.$$

Putting these two identities together gives [35, (6.2)].

The second author also describes alternate factorizations of $E_{\phi,k}$, parametrized by $a \in A$. These identities are similarly obtained from the equality

$$E_k(a, 1; W) = E_k(1, a; W).$$

The final connection then comes from noticing that the starting point for our factorization in Theorem 3.13 is the identity (which fundamentally uses the different normalization)

$$E'_k((t - \theta^{q^\ell})^d, 1; W) = E'_k(1, (t - \theta^{q^\ell})^d; W),$$

which by the \mathbb{C}_∞ -bilinearity of E'_k (see Prop. 3.7 (2)) may be written as

$$\sum_{m=0}^d (-1)^m \binom{d}{m} \theta^{q^\ell} E'_k((t^{d-m}, 1; W).$$

Thus, our factorization can be viewed as an A -linear combination of a different normalization of the second author's factorizations from [35]. However, it is not obvious that such linear combinations of factorizations of $E_{\phi,k}$ is again a factorization of $E_{\phi,k}$. The methods we use here show that this is true.

We also add, for completeness, as it was claimed in the introduction:

Lemma 3.23. *We have the following identity of A -modules free of rank one induced by $\text{Lie}(C^{\otimes d})$:*

$$\text{Ker}(\text{sin}_A^{\otimes d}) = \left\{ \mathbf{d}_\theta(a) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} : a \in A \right\}.$$

Proof. Recall that $\text{sin}_A^{\otimes d}(Z) = \widehat{\Pi}^{-1} \exp_{C^{\otimes d}} \widehat{\Pi} Z$. By [3, Corollary 2.5.9] $\exp_{C^{\otimes d}}(\widehat{\Pi} Z) = 0$ if and only if $\widehat{\Pi} Z \in \{\mathbf{d}_\theta(a) : a \in A\} \Pi$. By (2.5) we have $\widehat{\Pi} Z = \widehat{Z} \Pi$ so that $Z \in \text{Ker}(\text{sin}_A^{\otimes d})$ if and only if there exists $a \in A$ such that $\widehat{Z} = \mathbf{d}_\theta(a)$. Now, recalling that $[\cdot]_d$ denotes the projection on the last column, $Z = [\widehat{Z}]_d = \mathbf{d}_\theta(a) = \mathbf{d}_\theta(a) \mathbf{d}_\theta(1)$. \square

4. Δ -MATRICES

In order to complete the proof of Theorem B in the introduction we use Proposition 3.19, but we still need to determine alternative expressions for the coefficients $\widetilde{\mathcal{L}}_k$. In this section we discuss the main properties of the matrix valued differential operators introduced in the introduction, that provide the appropriate tools in view of our results. The formalism that we introduce needs at least three independent indeterminates. This has been already partly employed in [35], but the use of Δ -matrices is new.

Let F be a field. Most of the results we are going to state and prove in this section are identities of matrices in $\text{GL}_d(\mathbb{Z}[x, y, z, \dots])$ that reduce modulo p , the characteristic of \mathbb{F}_q , and we can set $F = \mathbb{F}_q$ or $F = K$. We recall the Δ -operator of the introduction.

Definition 4.1. For $f \in F(x, z)$ we set

$$\begin{aligned} \Delta_{x,z}^{(d)}(f) &:= \begin{pmatrix} \mathcal{D}_{x,d-1}(f) & \mathcal{D}_{x,d-1}(\mathcal{D}_{z,1}(f)) & \cdots & \mathcal{D}_{x,d-1}(\mathcal{D}_{z,d-1}(f)) \\ \mathcal{D}_{x,d-2}(f) & \mathcal{D}_{x,d-2}(\mathcal{D}_{z,d-1}(f)) & \cdots & \mathcal{D}_{x,d-2}(\mathcal{D}_{z,1}(f)) \\ \vdots & \vdots & & \vdots \\ f & \mathcal{D}_{z,1}(f) & \cdots & \mathcal{D}_{z,d-1}(f) \end{pmatrix} = \\ &= \left(\mathcal{D}_{x,d-i}(\mathcal{D}_{z,j-1}(f)) \right)_{1 \leq i, j \leq d} \in F(x, z)^{d \times d}. \end{aligned}$$

We call it the $\Delta_{x,z}$ -matrix associated to f . Loosely, we may speak about Δ -matrices. A Δ -matrix $\Delta_{x,z}(f)$ is uniquely determined by its root f , which is the coefficient in the d -th row and 1st column.

Observe that, by the fact that the divided higher derivatives in x and z commute,

$$\Delta_{x,z}(f)^\perp = \Delta_{z,x}(f).$$

The following is the basic elementary criterion to recognize Δ -matrices.

Lemma 4.2. *A matrix $M \in F(x, z)^{d \times d}$ is a $\Delta_{x,z}$ -matrix if and only if M is a ∂_x -matrix and its antitranspose M^\perp is a ∂_z -matrix.*

Proof. One implication is obvious. Suppose that M is a ∂_x -matrix and at the same time M^\perp is a ∂_z -matrix. We can write (a) $M = \partial_x(m_0, \dots, m_{d-1})$ and (b) $M^\perp = \partial_z(m'_0, \dots, m'_{d-1})$ for elements $m_0, \dots, m_{d-1}, m'_0, \dots, m'_{d-1} \in F(x, z)$. We deduce that $m_0 = m'_0$. We are going to show that $f := m_0 = m'_0$ is the root of M as a Δ -matrix. From (a) we see that $m'_i = \mathcal{D}_{x,i}(f)$ and from (b) that $m_j = \mathcal{D}_{z,j}(f)$, and this for all $0 \leq i, j \leq d-1$. Now given two elements $i, j \in \{1, \dots, d\}$, the (i, j) -entry $m_{i,j}$ of M is, by (a),

$$m_{i,j} = \mathcal{D}_{x,d-i}(m_{j-1}) = \mathcal{D}_{x,d-i}(\mathcal{D}_{z,j-1}(f)).$$

From (b) we also see that

$$m_{i,j} = \mathcal{D}_{z,j-1}(m'_{d-i}) = \mathcal{D}_{z,j-1}(\mathcal{D}_{x,d-i}(f)).$$

The above expressions agree thanks to the fact that for all i, j , the operators $\mathcal{D}_{x,i}$ and $\mathcal{D}_{z,j}$ commute. \square

Our next task is to describe basic compatibility properties among Δ -matrices.

Lemma 4.3. *Let us consider two indeterminates x, z independent over F . Suppose that $f \in F(x)$, $g \in F(x, z)$ and $h \in F(z)$. Then*

$$\mathbf{d}_x(f)\Delta_{x,z}(g)\mathbf{d}_z(h) = \Delta_{x,z}(fgh).$$

Proof. By Leibnitz's formula, $\mathbf{d}_x(f)\Delta_{x,z}(g)\mathbf{d}_z(h) = \Delta_{x,z}(fg)\mathbf{d}_z(h)$. Taking the anti-transpose of $\Delta_{x,z}(fg)\mathbf{d}_z(h)$ we get

$$(\Delta_{x,z}(fg)\mathbf{d}_z(h))^\perp = \mathbf{d}_z(h)^\perp \Delta_{x,z}(fg)^\perp = \mathbf{d}_z(h)\Delta_{x,z}(fg)^\perp = \Delta_{x,z}(fgh)^\perp$$

because $\Delta_{x,z}(fg)^\perp$ is a ∂_z -matrix. \square

Hence the map

$$F(x, z) \xrightarrow{\Delta_{x,z}} F(x, z)^{d \times d}$$

is left $F(x)$ -linear and right $F(z)$ -linear via \mathbf{d}_x and \mathbf{d}_z . It defines an F -linear map

$$F(x, z) \rightarrow \text{Bil}^+ \left(F(x) \times F(z) \rightarrow F(x, z)^{d \times d} \right),$$

into bilinear maps (for \mathbf{d}_x and \mathbf{d}_z) which are symmetric in the sense that if $\iota_{x,z}$ is the involution of $F(x, z)$ that exchanges the indeterminates x, z , B is in the target space if and only if $\iota_{x,z}(B^\perp) = B$ (this is condensed in the notation Bil^+ explaining the $+$ sign).

There is an analogue of (2.8) for the operators Δ .

Lemma 4.4. *Let us consider three independent indeterminates x, y, z over F . We consider $f, g \in F(x, y, z)$. Suppose moreover that $f \in F(x, y)$ and $g \in F(y, z)$. Then*

$$\Delta_{x,y}(f)\Delta_{y,z}(g) = \Delta_{x,z}(\mathcal{D}_{y,d-1}(fg)).$$

Proof. If M is a ∂_x -matrix and M' is a matrix whose entries do not depend on the variable x , then MM' , if well defined, is a ∂_x -matrix. Hence both left- and right-hand sides of the identity are ∂_x -matrices. Both the matrices $(\Delta_{x,y}(f)\Delta_{y,z}(g))^\perp = \Delta_{z,y}(g)\Delta_{y,x}(f)$ and $\Delta_{u,w}(\mathcal{D}_{v,d-1}(fg))^\perp = \Delta_{w,u}(\mathcal{D}_{v,d-1}(fg))$ are ∂_z -matrices. But by Leibnitz's rule, the

bottom left coefficients of the identity of the lemma agree, and therefore the identity is verified. \square

4.1. How to construct Δ -matrices. Suppose that, over a field F , we have two independent indeterminates x, z and, for any d , two d -tuples of elements $a_1, \dots, a_d \in F(x)$ and $b_1, \dots, b_d \in F(z)$. We set

$$\varphi := \sum_{i=1}^d a_i b_{d-i+1} \in F(x, z).$$

The first way to construct Δ -matrices is given by the next lemma.

Lemma 4.5. *The following formula holds:*

$$\partial_x(a_1, \dots, a_d)(\partial_z(b_1, \dots, b_d))^\perp = \Delta_{x,z}(\varphi).$$

Proof. Set $U := \partial_x(a_1, \dots, a_d)$ and $V = \partial_z(b_1, \dots, b_d)$. Then UV^\perp is a ∂_x -matrix and $(UV^\perp)^\perp = VU^\perp$ is a ∂_z -matrix so that UV^\perp is a $\Delta_{x,z}$ -matrix and, in the two matrices UV^\perp and $\Delta_{x,z}(\varphi)$, the entries in the d -th row and first column. \square

In the opposite direction, a $\Delta_{x,z}$ -matrix needs not be the left product of the anti-transpose of a ∂_z -matrix by a ∂_x -matrix. Already if $d = 1$, there are examples of elements of $F(x, z)$ which are not products of elements in $F(x)$ and elements in $F(z)$. By the fact that $F[x, z] = F[x] \otimes_F F[z]$, every $\Delta_{x,z}$ -matrix with polynomial root is a finite linear combination of such matrices.

From Lemma 4.5 we deduce the next formula, where x, y, z, t are independent variables over a field F :

$$(4.1) \quad H_{x,y}(H_{z,t})^\perp = \Delta_{x,z} \left(\sum_{i=1}^d (x-y)^{d-i} (z-t)^{i-1} \right).$$

Recall the sequence of polynomials $(f_d)_d$ in (1.16) and set

$$f'_d := \sum_{j=1}^d \binom{d}{j} (x-y)^{d-j} (z-x)^{j-1} \in \mathbb{F}_q[x, y, z].$$

We also set $f'_0 = 0$. We have:

Lemma 4.6. *For all $d \geq 0$, $f'_d(z, y, z) = f_d(x, y, z) = \sum_{i=1}^d (x-y)^{d-i} (z-y)^{i-1}$.*

Proof. It suffices to show the recursive formula $f'_d = (x - y)f'_{d-1} + (z - y)^{d-1}$ for $d \geq 1$ because the lemma follows easily from it by induction. But

$$\begin{aligned}
f'_d - (x - y)f'_{d-1} &= \\
&= \sum_{j=1}^d \left(\binom{d}{j} - \binom{d-1}{j} \right) (x - y)^{d-j} (z - x)^{j-1} \\
&= \sum_{j=1}^d \binom{d-1}{j-1} (x - y)^{d-j} (z - x)^{j-1} \\
&= \sum_{i=0}^{d-1} \binom{d-1}{i} (x - y)^{d-1-i} (z - x)^i \\
&= (x - y + z - x)^{d-1} = (z - y)^{d-1}.
\end{aligned}$$

□

We define

$$M_{x,y} := (1 + (x - y)N^\top)^d \in \mathrm{GL}_d(\mathbb{F}_p[x, y]).$$

For elements $a, b \in K$, we set $M_{a,b} := (M_{x,y})_{\substack{x=a \\ y=b}} \in \mathrm{GL}_d(K)$. Note that

$$(4.2) \quad M_{\theta, \theta^{q^\ell}} = M_\ell,$$

with the family of matrices $(M_\ell)_\ell$ defined in (3.11).

Lemma 4.7. $M_{x,y}H_{z,x}^\perp = \Delta_{x,z}(f_d)$.

Proof. By construction, the lower left coefficient of $M_{x,y}(H_{z,x})^\perp$ is f'_d that we know being equal to f_d thanks to Lemma 4.6. Since $M_{x,y} \in \mathbb{F}_p[x, y]^{d \times d}$ does not depend on z , by the fact that $H_{z,x}$ is a ∂_z -matrix it is easy to see that $(M_{x,y}(H_{z,x})^\perp)^\perp = H_{z,x}(M_{x,y})^\perp = H_{z,x}M_{x,y}$ is a ∂_z -matrix. Now, to show that $M_{x,y}(H_{z,x})^\perp$ is a ∂_x -matrix, it suffices to show that its first column C is, because the operators $\mathcal{D}_{x,i}$ and $\mathcal{D}_{z,j}$ commute for all i, j . This follows from the following identities which can be proved by induction on d for all i ,

$$\mathcal{D}_{x,d-i}(f_d) = \sum_{j=1}^d \binom{d}{d-i+j} (x - y)^{i-j} (z - x)^{j-1},$$

noticing that the right-hand side coincides with the i -th coefficient of C . □

By a computation of degrees in the variables involved in f_d both $M_{x,y}(H_{z,x})^\perp$ and $\Delta_{x,z}(f_d)$ are lower triangular with ones over the diagonal. We easily deduce from Lemmas 4.3, 4.7 and the identity (4.2):

Corollary 4.8. For all $k \geq 0$ the matrix $\tilde{\mathcal{L}}_k = \Gamma_k^{-1} M_{k+1} H^\perp \Gamma_k^{(1)}$ is the evaluation at $x = \theta, y = \theta^{k+1}, z = \theta^q, t = \theta$ of the Δ -matrix $\Delta_{x,z}(g_{d,k})$ with

$$g_{d,k} = \frac{c_{d,k}(x, t)}{c_{d,k}(z, t^q)} f_d(x, y, z) \in \mathbb{F}_p(x, y, z, t),$$

where

$$c_{d,k}(x, t) := \prod_{i=1}^k (x - t^{q^i})^d.$$

We can finally state and prove:

Theorem 4.9. The following identity holds in $\text{End}_K(\mathbb{G}_a^d(K))[\tau]$:

$$\mathbf{E}_k = (1 - \mathcal{L}_{k-1}\tau) \cdots (1 - \mathcal{L}_1\tau)(1 - \mathcal{L}_0\tau),$$

where

$$(4.3) \quad \mathcal{L}_i = \Delta_{x,z} \left(f_d(x, y, z) \left(\frac{(x - \theta^q) \cdots (x - \theta^{q^i})}{(z - \theta^{q^2}) \cdots (z - \theta^{q^{i+1}})} \right)^d \right) \begin{matrix} x=\theta \\ y=\theta^{q^{i+1}} \\ z=\theta^q \end{matrix}.$$

This is Theorem B in the introduction.

Proof. This follows directly from a combination of Proposition 3.19 and Corollary 4.8. \square

Finally, we reach a proof of Theorem A of the introduction.

Theorem 4.10. We have the following factorization:

$$\sin_A^{\otimes d} = \prod_{i \geq 0}^{\leftarrow} (1 - \mathcal{L}_i \tau) \in \text{End}_{K_\infty}(\mathbb{G}_a^d(K_\infty))[[\tau]],$$

where \mathcal{L}_i is given in (4.3).

Proof. By Proposition 3.20 we get that

$$\sin_A^{\otimes d} = \prod_{k \geq 0}^{\leftarrow} (1 - \mathcal{L}_k \tau) \in \text{End}_{K_\infty}(\mathbb{G}_a^d(K_\infty))[[\tau]].$$

Coefficientwise, this product is convergent. We can conclude with Theorem 4.9. \square

5. APPLICATION TO NEW IDENTITIES OF MULTIPLE POLYLOGARITHMS

In this section we show our Theorem C as a consequence of Theorem 5.1 (the main result of this section), through Corollary 5.4. Recall that we wrote:

$$(5.1) \quad \mathcal{L}_k = \Gamma_k^{-1} \Delta_{x,z}(f_d(x, y, z)) \begin{matrix} x=\theta \\ y=\theta^{q^{k+1}} \\ z=\theta^q \end{matrix} \Gamma_k^{(1)}, \quad k \geq 0.$$

Matrix multiple sums. We define, for $k \geq m \geq 1$,

$$\mathcal{L}_{<k}(m) := \sum_{k > i_1 > \dots > i_m \geq 0} \mathcal{L}_{i_1} \mathcal{L}_{i_2}^{(1)} \dots \mathcal{L}_{i_m}^{(m-1)} \in K^{d \times d}.$$

Note that for all $k \geq 0$,

$$(5.2) \quad \mathcal{L}_{<k}(m) = \mathcal{L}_{k-1} \mathcal{L}_{<k-1}(m-1)^{(1)} + \mathcal{L}_{<k-1}(m), \quad m > 1$$

$$(5.3) \quad \mathcal{L}_{<m}(m) = \mathcal{L}_{m-1} \mathcal{L}_{m-2}^{(1)} \dots \mathcal{L}_0^{(m-1)}, \quad m > 1.$$

Recall the main identity of Theorem 4.9. We deduce the following identity.

$$(5.4) \quad \mathcal{L}_{<k}(m) = (-1)^m \Gamma_k^{-1} Q_m \Gamma_{k-m}^{(m)}, \quad k \geq m \geq 1.$$

We recall that

$$l_n = (\theta - \theta^q)(\theta - \theta^{q^2}) \dots (\theta - \theta^{q^n}), \quad n \geq 0.$$

For all $M \in K^{d \times e}$ we denote by $[M]_1$ the first column of M and by $[M]_{d,1}$ the last coefficient of $[M]_1$. The identity (5.4) implies

$$[\mathcal{L}_{<k}(m)]_1 = (-1)^m \Gamma_k^{-1} [Q_m]_1 l_{k-m}^{-dq^m}.$$

Recall from [31, Corollary 4.2.4]:

$$(5.5) \quad [Q_m]_1 = D_m^{-d} \begin{pmatrix} 1 \\ [m] \\ \vdots \\ [m]^{d-1} \end{pmatrix}$$

where $[m] = \theta^{q^m} - \theta$ for $m \geq 1$ and

$$D_m = [m][m-1]^q \dots [1]^{q^{m-1}}, \quad m \geq 0.$$

This also follows directly from Proposition 2.1. We also set $[0] = 1$. The column vector $[\mathcal{L}_{<k}(m)]_1$ is determined by the coefficient $[\mathcal{L}_{<k}(m)]_{d,1}$ in the following sense.

Define, for $m \geq 1$ and $k \geq 0$:

$$U_{k,m} := [m]^{1-d} l_k^{-d} \Gamma_k^{-1} \begin{pmatrix} 1 \\ [m] \\ \vdots \\ [m]^{d-1} \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ * \\ 1 \end{pmatrix} \in K^{d \times 1}.$$

Then

$$\Gamma_k^{-1} [Q_m]_1 = l_k^d [m]^{d-1} D_m^{-d} U_{k,m}$$

and we deduce the following two identities, corresponding to (5.4):

$$(5.6) \quad [\mathcal{L}_{<k}(m)]_1 = [\mathcal{L}_{<k}(m)]_{d,1} U_{k,m}$$

$$(5.7) \quad = [m]^{d-1} D_m^{-d} l_k^d l_{k-m}^{-dq^m} U_{k,m}.$$

Scalar multiple sums. Now set

$$(5.8) \quad \lambda_{i,j} := [j-1]^{(1-d)q} l_i^{d(1-q)} f_d(\theta, \theta^{q^{i+1}}, \theta^{q^j}) \in K, \quad i \geq 0, \quad j \geq 1,$$

and the multiple sum

$$(5.9) \quad \lambda_{<k}(m) := \sum_{k > i_1 > \dots > i_m \geq 0} \lambda_{i_1, m} \lambda_{i_2, m-1}^q \cdots \lambda_{i_m, 1}^{q^{m-1}}, \quad k \geq m \geq 1.$$

Note the identities

$$(5.10) \quad \lambda_{<k}(m) = \lambda_{k-1, m} \left(\lambda_{<k-1}(m-1) \right)^{(1)} + \lambda_{<k-1}(m), \quad m > 1$$

$$(5.11) \quad \begin{aligned} \lambda_{<m}(m) &= \lambda_{m-1, m} \left(\lambda_{<m-1}(m-1) \right)^{(1)} \\ &= \lambda_{m-1, m} \lambda_{m-2, m-1}^{(1)} \cdots \lambda_{0, 1}^{(m-1)}, \quad m > 1. \end{aligned}$$

Theorem 5.1. *For $k \geq m \geq 1$ we have*

$$(5.12) \quad [\mathcal{L}_{<k}(m)]_{d,1} = \lambda_{<k}(m),$$

$$(5.13) \quad = (-1)^m l_k^d [m]^{d-1} D_m^{-d} l_{k-m}^{-dq^m}.$$

The equality (5.13) is a direct consequence of (5.4). We focus on (5.12). Let us first study it in the case $m = 1$. Note that in this case we have

$$\lambda_{i,1} = l_i^{d(1-q)} f_d(\theta, \theta^{q^{i+1}}, \theta^q).$$

Now

$$(5.14) \quad \begin{aligned} [\mathcal{L}_{<k}(m)]_{d,1} &= \left[\sum_{k > j} \mathcal{L}_j \right]_{d,1} \\ &= \sum_{k > j} \left[(\Gamma_j^{(d)})^{-1} \Delta_{x,z}(f_d)_{x=\theta, y=\theta^{q^{j+1}}, z=\theta^q} (\Gamma_j^{(d)})^{(1)} \right]_{d,1} \\ &= \sum_{k > j} \lambda_{j,1}, \quad \forall k \geq 1. \end{aligned}$$

To prove the identity (5.12) in the case $m > 1$ we will use induction, but we first need two preliminary lemmas. The next one deals in fact with an identity in $\mathbb{Z}[x, y, z, z']$ with independent indeterminates x, y, z, z' that reduces modulo p . Here we need induction over $d \geq 1$. In conformity with our convention, we indicate the dependence in the parameter d by writing $\Delta_{x,z}^{(d)}$ instead of $\Delta_{x,z}$, $\partial_x^{(d)}$ instead of ∂_x etc. (it is the only place in the paper where we do this).

Lemma 5.2. *We have, for variables x, y, z, z' ,*

$$\Delta_{x,z}^{(d)}(f_d(x, y, z)) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix} = \partial_x^{(d)}(f_d(x, y, z')).$$

Proof. We proceed by induction on $d \geq 1$. The case $d = 1$ is trivial as $f_d = 1$. Now, recall from the proof of Lemma 4.6 that

$$f_d(x, y, z) = (x - y)f_{d-1}(x, y, z) + (z - y)^{d-1}, \quad d > 1.$$

Applying the operator $\Delta_{x,z}^{\langle d \rangle}$ and using its compatibility formulas with d_x - and d_z -matrices Lemma 4.3 we obtain:

$$\Delta_{x,z}^{\langle d \rangle}(f_d(x, y, z)) = \mathbf{d}_x^{\langle d \rangle}(x - y)\Delta_{x,z}^{\langle d \rangle}(f_{d-1}(x, y, z)) + e_{d,1}^{\langle d \rangle}\mathbf{d}_z^{\langle d \rangle}((z - y)^{d-1}).$$

Note indeed that $e_{d,1}^{\langle d \rangle}$, the elementary matrix of $K^{d \times d}$ having all the coefficients equal to zero except the coefficient on the d -th row and first column, is a Δ -matrix as it equals $\Delta_{x,z}^{\langle d \rangle}(1)$. Now we multiply the above identity on the right by the column vector $(1, z' - z, \dots, (z' - z)^{d-1})^\top$. First observe that

$$\begin{aligned} \Delta_{x,z}^{\langle d \rangle}(f_{d-1}(x, y, z)) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix} &= \\ &= \begin{pmatrix} 0 & \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-2} \end{pmatrix} \\ \Delta_{x,z}^{\langle d-1 \rangle}(f_{d-1}(x, y, z)) \end{pmatrix} = \boldsymbol{\partial}_x^{\langle d \rangle}(f_{d-1}(x, y, z')) \end{aligned}$$

by the fact that $\Delta_{x,z}^{\langle d \rangle}(f_{d-1}(x, y, z))$ has a block decomposition with the first row and the last column zero and the remaining coefficients corresponding to $\Delta_{x,z}^{\langle d-1 \rangle}(f_{d-1}(x, y, z))$ as a submatrix, and the induction hypothesis. Secondly, observe that the top coefficient of

$$\mathbf{d}_z^{\langle d \rangle}((z - y)^{d-1}) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix}$$

equals $(y - z')^{d-1}$ by an obvious consequence of Taylor's series expansions. Hence

$$e_{n,1}^{\langle d \rangle}\mathbf{d}_z^{\langle d \rangle}((z - y)^{d-1}) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (y - z')^{d-1} \end{pmatrix} = \boldsymbol{\partial}_x^{\langle d \rangle}((y - z')^{d-1})$$

and the lemma follows because we get

$$\begin{aligned} \Delta_{x,z}^{(d)}(f_d(x, y, z)) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix} &= \\ &= \partial_x(f_{d-1}(x, y, z')) + \partial_x^{(d)}((y - z')^{d-1}) = \partial_x^{(d)}((y - z')^{d-1})(f_d(x, y, z')). \end{aligned}$$

□

From the above lemma we deduce the next statement. We can again omit the symbols $(\cdot)^{(d)}$, taking into account Lemma 5.2, we can now fix the value of d again.

Lemma 5.3. *We have that*

$$\mathcal{L}_i U_{i,j}^{(1)} = [j]^{(1-d)q} l_i^{-dq} \Gamma_i^{-1} \partial_x(f_d(x, y, z')) \Big|_{\substack{x=\theta \\ y=\theta^{q^{i+1}} \\ z'=\theta^{q^{j+1}}}},$$

so that

$$[\mathcal{L}_i U_{i,j}^{(1)}]_{d,1} = [j]^{(1-d)q} l_i^{d(1-q)} f_d(\theta, \theta^{q^{i+1}}, \theta^{q^{j+1}}) = \lambda_{i,j+1}.$$

Proof. We have

$$\begin{aligned} \mathcal{L}_i U_{i,j}^{(1)} &= \Gamma_i^{-1} \Delta_{x,z}(f_d) \Big|_{\substack{x=\theta \\ y=\theta^{q^{i+1}} \\ z'=\theta^q}} \Gamma_i^{(1)} [j]^{(1-d)q} l_i^{-dq} (\Gamma_i^{(1)})^{-1} \begin{pmatrix} 1 \\ [j]^q \\ \vdots \\ [j]^{(d-1)q} \end{pmatrix} \\ &= \Gamma_i^{-1} \left(\Delta_{x,z}(f_d(x, y, z)) \begin{pmatrix} 1 \\ z' - z \\ \vdots \\ (z' - z)^{d-1} \end{pmatrix} \right) \Big|_{\substack{x=\theta \\ y=\theta^{q^{i+1}} \\ z=\theta^q \\ z'=\theta^{q^{j+1}}}} [j]^{(1-d)q} l_i^{-dq} \\ &= \Gamma_i^{-1} \partial_x(f_d(x, y, z')) \Big|_{\substack{x=\theta \\ y=\theta^{q^{i+1}} \\ z'=\theta^{q^{j+1}}}} [j]^{(1-d)q} l_i^{-dq} \end{aligned}$$

by Lemma 5.2. □

Proof of Theorem 5.1. Without loss of generality, we suppose $m > 1$. As a first step we begin by showing the formula (5.12) in the case $k = m$ for all $m \geq 1$. To do this we use induction on $m \geq 1$, knowing that the formula is true in the case $k = m = 1$ by (5.14). Let us suppose that the formulas are true for the integer $m - 1$, namely:

$$[\mathcal{L}_{< m-1}(m-1)]_{d,1} = \lambda_{< m-1}(m-1) = (-1)^{m-1} [m-1]^{d-1} l_{m-1}^d D_{m-1}^{-d}.$$

It is easy to show that, with the assumption of the second identity,

$$(5.15) \quad \lambda_{<m}(m) = (-1)^m [m]^{d-1} l_m^d D_m^{-d}.$$

Indeed, by (5.11),

$$\begin{aligned} \lambda_{<m}(m) &= \lambda_{m-1,m} \lambda_{<m-1}(m-1) \\ &= (-1)^{m-1} [m-1]^{q(1-d)} l_{m-1}^{d(1-q)} f_d(\theta, \theta^{q^m}, \theta^{q^m}) [m-1]^{d-1} l_{m-1}^d D_{m-1}^{-d} \\ &= [m-1]^{q(1-d)} l_{m-1}^{d(1-q)} (\theta - \theta^{q^m})^{d-1} [m-1]^{(d-1)q} l_{m-1}^{dq} D_{m-1}^{-dq} \\ &= l_{m-1}^d (\theta - \theta^{q^m})^{d-1} D_{m-1}^{-dq}, \end{aligned}$$

where the third equality holds because $f_d(x, z, z) = (x - z)^{d-1}$ implying $f_d(\theta, \theta^{q^m}, \theta^{q^m}) = (\theta - \theta^{q^m})^{d-1}$. This implies (5.15). We have, by (5.3), (5.4), (5.7) and (5.15),

$$\begin{aligned} [\mathcal{L}_{<m}(m)]_{d,1} &= (-1)^m [\Gamma_m^{-1} Q_m]_{d,1} \\ &= (-1)^m [m]^{d-1} l_m^d D_m^{-d} \\ &= \lambda_{<m}(m). \end{aligned}$$

This proves the identity

$$(5.16) \quad [\mathcal{L}_{<m}(m)]_{d,1} = \lambda_{<m}(m), \quad m \geq 1$$

and proves the theorem in the case $k = m \geq 1$. It remains to show the general formula (5.12):

$$[\mathcal{L}_{<k}(m)]_{d,1} = \lambda_{<k}(m), \quad k \geq m \geq 1.$$

Again, we can suppose that $m > 1$ and proceed this time by double induction, for $m \geq 1$ as a primary parameter, and then, for each m given, for $k \geq m$ as a secondary parameter. Hence we can suppose that the formula holds for all couples of integers (k, m) with $k \geq m - 1 \geq 1$. We have seen in (5.16) that the formula also holds in the case $(k, m) = (m, m)$. By the induction hypothesis on the secondary parameter $k \geq m$ we can suppose that $[\mathcal{L}_{<k-1}(m)]_{d,1} = \lambda_{<k-1}(m)$. Apply (5.2) and then compute $[\cdot]_{d,1}$. We get:

$$\begin{aligned} [\mathcal{L}_{<k}(m)]_{d,1} &= [\mathcal{L}_{k-1} U_{k-1, k-1}^{(1)}]_{d,1} \lambda_{<k-1}(m-1)^{(1)} + [\mathcal{L}_{<k-1}(m)]_{d,1} \\ &= [m-1]^{(1-d)q} l_{k-1}^{d(1-q)} f_d(\theta, \theta^{q^k}, \theta^{q^m}) \lambda_{<k-1}(m-1)^{(1)} + \lambda_{<k-1}(m) \\ &= \lambda_{k-1,m} \lambda_{<k-1}(m-1)^{(1)} + \lambda_{<k-1}(m) \\ &= \lambda_{<k}(m), \end{aligned}$$

where the second equality is a consequence of Lemma 5.3 and the last equality follows from (5.10). This completes the proof of our theorem. \square

Consequences for multiple polylogarithms. Write, for all $k \geq 0$,

$$\mathcal{L}_{<k} \begin{pmatrix} m_1 & \cdots & m_r \\ n_1 & \cdots & n_r \end{pmatrix} := \sum_{k > i_1 > \cdots > i_r \geq 0} \frac{\theta^{m_1 q^{i_1} + m_2 q^{i_2} + \cdots + m_r q^{i_r}}}{l_{i_1}^{n_1} l_{i_2}^{n_2} \cdots l_{i_r}^{n_r}}$$

We deduce, from Theorem 5.1, with the notations of Theorem C of the introduction:

Corollary 5.4. *For all $r \geq 1$ and for all $k \geq r$,*

$$\mathcal{L}_{<k-r} \begin{pmatrix} 0 \\ dq^r \end{pmatrix} = (-1)^r [r]^{1-d} D_r^d \sum_m c_m \mathcal{L}_{<k} \begin{pmatrix} 0 & m_1 q & \cdots & m_r q^r \\ d & d(q-1) & \cdots & d(q-1)q^{r-1} \end{pmatrix}.$$

Proof. Recall the definitions (5.8) and (5.9). By Theorem 5.1 we have, for all $i \geq r$,

$$l_{i-r}^{-dq^i} = (-1)^r [r]^{1-d} D_r^d l_i^{-d} \lambda_{<i}(r), \quad i \geq r.$$

It is plain that

$$\lambda_{<i}(r) = \sum_m c_m \mathcal{L}_{<i} \begin{pmatrix} m_1 q & \cdots & m_r q^r \\ d(q-1) & \cdots & d(q-1)q^{r-1} \end{pmatrix}.$$

The formula of our corollary follows immediately. \square

Taking the limit for $k \rightarrow \infty$ allows us to deduce Theorem C.

6. MOTIVIC INTERPRETATION OF IDENTITIES AND INTEGRAL SHUFFLE RELATION

The identities we prove in the previous three sections, culminating in Corollary 5.4, are all proved at the level of finite sums, meaning we get linear relations amongst multiple sums which result in linear relations amongst multiple polylogarithms when we take the limit. In this section, we explain how these identities can be seen as naturally coming from identities happening amongst elements of a t -motive, from which we recover the scalar identities using the $\delta_{1,z}^M$ “cycle integration” map mentioned in the introduction. Thus, we view these identities as being a characteristic p version of q^s -power aspects of integral shuffle relations: our identities do not provide a general shuffle product structure for polylogarithms, only a structure resulting from the Frobenius endomorphism.

Definition 6.1. Let $b_0 = 1$ and set $b_k = (t - \theta) \dots (t - \theta^{q^{k-1}})$ and define

$$\mathbf{b}_k(t) = \begin{pmatrix} b_k(t)^d \\ b_k(t)^{d-1} b_{k+1}(t) \\ b_k(t)^{d-2} b_{k+1}(t)^2 \\ \vdots \\ b_k(t) b_{k+1}(t)^{d-1} \end{pmatrix}.$$

Also define

$$I_i(W) = \sum_{k=0}^i E_k(W) \mathbf{b}_k(t) \in \mathbb{C}_\infty[[t]]^d,$$

for $W \in \mathcal{Z}(\mathbb{C}_\infty)$ as in Section 3.2.

Definition 6.2. Recall the definition of the t -motive M associated with the d th tensor power of the Carlitz module from Definition 3.1, which is a free $\mathbb{C}_\infty[t]$ -module of rank 1 and a free $\mathbb{C}_\infty[\tau]$ -module of rank d . We define $\widehat{M} := M \otimes_{\mathbb{C}_\infty[t]} \mathbb{T} \cong \mathbb{T}$ with diagonal τ -action.

Lemma 6.3. *If $h \in \widehat{M}$, then $\tau_M h \in \widehat{M}$.*

Proof. Recall that $\tau_M h = (t - \theta)^d h^{(1)}$. Thus, the lemma follow from the fact that the Frobenius acts as a contraction on the unit ball in \mathbb{C}_∞ . \square

In order to explain the origin of our motivic identities, we write $I_i(W)$ in two different ways, each time, collecting common Frobenius twists of W . First, writing in the formula for E_k from Proposition 3.5(5) gives

$$\begin{aligned} I_i(W) &= \sum_{k=0}^i \sum_{j=0}^k Q_j W^{(j)} P_{k-j}^{(j)} \mathbf{b}_k \\ &= \sum_{k=0}^i \sum_{j=0}^k Q_j W^{(j)} b_j (P_{k-j} \mathbf{b}_{k-j})^{(j)} \\ &= \sum_{k=0}^i \sum_{j=0}^k Q_j W^{(j)} \tau_M^j (P_{k-j} \mathbf{b}_{k-j}), \end{aligned}$$

where in the last two lines we used the definition of τ_M^j acting on M and the fact that $b_k = b_j \cdot b_{k-j}^{(j)}$ for all $0 \leq j \leq k$. We switch the order of summation and collect common powers of τ_M to get

$$\begin{aligned} I_i(W) &= \sum_{m=0}^i \sum_{n=m}^i Q_m W^{(m)} \tau_M^m (P_{n-m} \mathbf{b}_{n-m}) \\ (6.1) \quad &= \sum_{m=0}^i Q_m W^{(m)} \tau_M^m \left(\sum_{n=0}^{i-m} P_n \mathbf{b}_n \right). \end{aligned}$$

Now, compare this with the normalization for E_k from Definition (3.7) to get

$$\begin{aligned} I_i(W) &= \sum_{k=0}^i E_k(W) \mathbf{b}_k \\ &= \sum_{k=0}^i \Gamma_k \cdot F_k(W) \cdot H_{\theta, \theta^k} \cdot \mathbf{b}_k. \end{aligned}$$

We then substitute the noncommutative factorization for F_k from Corollary 3.16 to get

$$\begin{aligned}
 &= \sum_{k=0}^i \Gamma_k \cdot [(1 - L_{k-1}) \circ (1 - L_{k-2}) \circ \cdots \circ (1 - L_1) \circ (1 - L_0)(W)] \cdot H_{\theta, \theta q^k} \cdot \mathbf{b}_k \\
 &= \sum_{k=0}^i \Gamma_k \sum_{j=0}^{k-1} (-1)^j \sum_{0 \leq i_1 < \cdots < i_j < k} (L_{i_j} \circ \cdots \circ L_{i_1})(W) \cdot H_{\theta, \theta q^k} \cdot \mathbf{b}_k \\
 (6.2) \quad &= \sum_{m=0}^i (-1)^m \sum_{n=m}^{i-1} \Gamma_n \sum_{0 \leq i_1 < \cdots < i_m < n} (L_{i_m} \circ \cdots \circ L_{i_1})(W) \cdot H_{\theta, \theta q^n} \cdot \mathbf{b}_n.
 \end{aligned}$$

Notice that, for fixed m , the last line of the above equation includes an m -fold Frobenius twist of the variable W coming from Definition 3.15. So, the motivic identities discussed in this section come from equating the bottom coordinates of (6.1) and (6.2), comparing common powers of the Frobenius. To recover the scalar identities of the previous sections, we evaluate both sides under the $\delta_{1, \mathbf{z}}^M$ map. We begin by studying the identities arising from (6.1). Write:

$$(6.3) \quad W_0 = we_{1,d} \in \mathcal{Z}(\mathbb{C}_\infty),$$

for $w \in \mathbb{C}_\infty$.

Proposition 6.4. *Let W_0 be as above. Then we have*

$$\left(\lim_{i \rightarrow \infty} I_i(W_0) \right)_d = \sum_{m=0}^{\infty} \frac{1}{D_m^d} [m]^{d-1} w^{(m)} \tau_M^m \left(\frac{\tilde{\pi}^d}{(t-\theta)} \left(\Omega^{(-1)} \right)^d \right) \in \widehat{M}.$$

Proof. We start with the identity (6.1). Using the formula for P_k from Proposition 2.1, we see that the bottom coordinate of $P_n \mathbf{b}_n$ is

$$b_n^d \frac{1}{\ell_n \ell_{n-1}^{d-1}} + b_n^{d-1} b_{n+1} \frac{1}{\ell_n^2 \ell_{n-1}^{d-2}} + \cdots + b_n b_{n+1}^{d-1} \frac{1}{\ell_n^d}.$$

Finally, taking the limit as $i \rightarrow \infty$ and comparing with [18, (5.3)] gives

$$\lim_{i \rightarrow \infty} \sum_{n=0}^i b_n^d \frac{1}{\ell_n \ell_{n-1}^{d-1}} + b_n^{d-1} b_{n+1} \frac{1}{\ell_n^2 \ell_{n-1}^{d-2}} + \cdots + b_n b_{n+1}^{d-1} \frac{1}{\ell_n^d} = \frac{\tilde{\pi}^n}{(t-\theta) \omega_C^n},$$

where we interpret any term with a negative index as being 0.

We then compute that $Q_m W_0^{(m)}$ is a matrix with all zeros, except the right-most column, that by (5.5) equals

$$w^{(m)} \begin{pmatrix} \frac{1}{D_m^d} \\ [m] \\ D_m^d \\ \vdots \\ \frac{[m]^{d-1}}{D_m^d} \end{pmatrix}.$$

Putting all these formulas together, multiplying out these matrices and taking the limit as $i \rightarrow \infty$ then gives the bottom coordinate of $\lim_{i \rightarrow \infty} I_i(W_0)$ as the formal sum

$$\sum_{m=0}^{\infty} \frac{1}{D_m^d} [m]^{d-1} w^{(m)} \tau_M^m \left(\frac{\tilde{\pi}^d}{(t-\theta)} \left(\Omega^{(-1)} \right)^d \right),$$

as claimed. Finally, we compute $1/(t-\theta)(\Omega^{(-1)})^d = (t-\theta)^{d-1} \Omega^d$ by (1.20), and it is proved in several sources that $\Omega \in \mathbb{T}$ (see [33, §3.3.4]). \square

Lemma 6.5. *Recall the definition of $\delta_{1,\mathbf{z}}^M$ from (3.5). For $h \in \widehat{M}$, if $\delta_{1,\mathbf{z}}^M(h) = c \in \mathbb{C}_\infty$, then we have*

$$\delta_{1,\mathbf{z}}^M(\tau_M^k(h)) = c^{q^k}.$$

Proof. Then if $\mathbf{z} = (z_1, \dots, z_d)^\top$, we have that $\delta_{1,\mathbf{z}}^M(a\tau_M^l(t-\theta)^m) = az_{m+1}^{q^l}$ for $a \in \mathbb{C}_\infty$, for $l \geq 0$ and for $0 \leq m \leq d-1$. So we have that

$$\tau_M^k(a\tau_M^l(t-\theta)^m) = a^{q^k} \tau_M^{k+m}(t-\theta)^m = a^{q^k} (z_{m+1}^{q^m})^{q^k}.$$

The result then follows by extending using \mathbb{C}_∞ -linearity. \square

Lemma 6.6. *Set $\mathbf{z} = (0, \dots, 0, 1)^\top$. Then for $0 \leq j \leq d-2$ we have*

$$\delta_{1,\mathbf{z}}^M(b_k(t)^{d-j} b_{k+1}(t)^j) = 0$$

and for $j = d-1$ it equals 1.

Proof. Recall the definition of the map $\delta_{1,\mathbf{z}}^M$ from (3.4) and of the $\mathbb{C}_\infty[\tau_M]$ -basis elements g_k from Definition 3.1. Then, observe that

$$b_k(t)^{d-j} b_{k+1}(t)^j = \tau_M^k(g_{j+1}),$$

for $k \geq 0$ and $0 \leq j \leq d-1$. Thus we apply Lemma 6.5 to conclude

$$\delta_{1,\mathbf{z}}^M(b_k(t)^{d-j} b_{k+1}(t)^j) = \delta_{1,\mathbf{z}}^M(\tau_M^k(g_{j+1})) = \delta_{1,\mathbf{z}}^M(g_{j+1})^{q^k},$$

and the lemma follows from (3.4), since $\mathbf{z} = (0, \dots, 0, 1)^\top$. \square

Theorem 6.7. *Let $\mathbf{z} = (0, \dots, 0, 1)^\top$. We have*

$$\delta_{1,\mathbf{z}}^M \left(\tau_M^k \left(\frac{\tilde{\pi}^d}{(t-\theta)} \left(\Omega^{(-1)} \right)^d \right) \right) = \mathcal{L} \begin{pmatrix} 0 \\ dq^k \end{pmatrix}.$$

Proof. Recall the definition of $M_{\mathbf{z}}\langle \tau_M \rangle$ from (3.5). It follows from [18, (5.4)] that

$$\tau_M^k \left(\frac{\tilde{\pi}^d}{(t-\theta)} \left(\Omega^{(-1)} \right)^d \right) \in M_{\mathbf{z}}\langle \tau_M \rangle.$$

Then, combining [18, Theorem 5.10] and Lemma 6.5 we get

$$\delta_{1,\mathbf{z}}^M \left(\tau_M^k \left(\frac{\tilde{\pi}^d}{(t-\theta)} \left(\Omega^{(-1)} \right)^d \right) \right) = (\text{Log}_{C^{\otimes n}}(\mathbf{z}))_d^{q^k}$$

where we use that $\delta_{1,\mathbf{z}}^M$ is \mathbb{C}_∞ -linear, and that $\Omega^{(-1)} = 1/\omega_C$. The theorem follows using the fact that the d th coordinate of the logarithm of the d th tensor power of Carlitz evaluated

at \mathbf{z} is the polylogarithm $\mathcal{L} \binom{0}{d}$ (this can be seen from [3, Remark, p. 172], for example) and Lemma 6.5. \square

Now we study the identities arising from (6.2). Extend the definition of $\delta_{1,\mathbf{z}}^M$ to act on \widehat{M}^d by acting coordinatewise.

Proposition 6.8. *Let $\mathbf{z} = (0, \dots, 0, 1)^\top$. We have*

$$\begin{aligned} \delta_{1,\mathbf{z}}^M \left(\lim_{i \rightarrow \infty} (-1)^k \sum_{n=k}^{i-1} \Gamma_n \sum_{0 \leq i_1 < \dots < i_k < n} (L_{i_k} \circ \dots \circ L_{i_1})(W_0) \cdot H_{\theta, \theta^{q^n}} \cdot \mathbf{b}_n \right)_d \\ = w^{q^k} (-1)^k \sum_{\underline{m}} c_{\underline{m}} \mathcal{L} \binom{0 \quad m_1 q \quad \dots \quad m_k q^k}{d \quad d(q-1) \quad \dots \quad d(q-1)q^{k-1}}. \end{aligned}$$

We prove this proposition with a series of lemmas.

Lemma 6.9. *We have*

$$(L_{i_k} \circ \dots \circ L_{i_1})(W) \cdot H_{\theta, \theta^{q^n}} = \mathcal{L}_{i_k} \cdot \mathcal{L}_{i_{k-1}}^{(1)} \dots \mathcal{L}_{i_1}^{(k-1)} W^{(k)} H_{\theta^{q^k}, \theta^{q^n}}.$$

Proof. This follows quickly from Definition 3.15, Definition 5.1 and (2.8). \square

Lemma 6.10. *Let \mathbf{b}_n be as in Definition (6.1) and $\mathbf{z} = (0, \dots, 0, 1)^\top$. Then for any $a, b \in \mathbb{Z}$ and any $n \geq 0$ we have*

$$\delta_{1,\mathbf{z}}^M (H_{\theta^a, \theta^b} \cdot \mathbf{b}_n) = (0, \dots, 0, 1)^\top.$$

Proof. This follows because H_{θ^a, θ^b} is a lower triangular matrix with 1's along the diagonal and from Lemma 6.6. \square

Proof of Proposition 6.8. Using Lemmas 6.9 and 6.10 and the \mathbb{C}_∞ -linearity of $\delta_{1,\mathbf{z}}^M$ we get

$$\begin{aligned} \delta_{1,\mathbf{z}}^M \left(\lim_{i \rightarrow \infty} (-1)^k \sum_{n=k}^{i-1} \Gamma_n \sum_{0 \leq i_1 < \dots < i_k < n} (L_{i_k} \circ \dots \circ L_{i_1})(W_0) \cdot H_{\theta, \theta^{q^n}} \cdot \mathbf{b}_n \right)_d \\ = \delta_{1,\mathbf{z}}^M \left(\lim_{i \rightarrow \infty} (-1)^k \sum_{n=k}^{i-1} \Gamma_n \sum_{0 \leq i_1 < \dots < i_k < n} \mathcal{L}_{i_k} \cdot \mathcal{L}_{i_{k-1}}^{(1)} \dots \mathcal{L}_{i_1}^{(k-1)} W_0^{(k)} H_{\theta^{q^k}, \theta^{q^n}} \cdot \mathbf{b}_n \right)_d \\ = \left(\lim_{i \rightarrow \infty} (-1)^k \sum_{n=k}^{i-1} \Gamma_n \sum_{0 \leq i_1 < \dots < i_k < n} \mathcal{L}_{i_k} \cdot \mathcal{L}_{i_{k-1}}^{(1)} \dots \mathcal{L}_{i_1}^{(k-1)} W_0^{(k)} \delta_{1,\mathbf{z}}^M \left(H_{\theta^{q^k}, \theta^{q^n}} \cdot \mathbf{b}_n \right) \right)_d \\ = \left(\lim_{i \rightarrow \infty} (-1)^k \sum_{n=k}^{i-1} \Gamma_n \mathcal{L}_{<n}(k) \cdot W_0^{(k)} \cdot (0, \dots, 0, 1)^\top \right)_d. \end{aligned}$$

The Proposition then follows by observing that the bottom right coordinate of Γ_n equals l_n^{-d} and from Theorem 5.1. \square

Remark 6.11. We summarize the results of this section with a diagram illustrating how our identities occur in \widehat{M} , then are mapped down to identities in \mathbb{C}_∞ using the cycle integration map $\delta_{1,\mathbf{z}}^M$:

$$\begin{array}{ccc}
\frac{1}{D_m^d} [m]^{d-1} \tau_M^m \left(\frac{\tilde{\pi}^d}{(t-\theta)} (\Omega^{(-1)})^d \right) & \xrightarrow{\text{Noncommutative Factorization}} & \left((-1)^m \Gamma_n \sum_{0 \leq i_1 < \dots < i_m} (L_{i_m} \circ \dots \circ L_{i_1})(W_0) \cdot H_{\theta, \theta q^n} \cdot \mathbf{b}_n \right)_d \\
\downarrow \delta_{1,\mathbf{z}}^M & & \downarrow \delta_{1,\mathbf{z}}^M \\
\frac{1}{D_m^d} [m]^{d-1} \mathcal{L}_{(dq^m)}^0 & \xrightarrow{\text{"Integral Shuffle"}} & (-1)^m \sum_{\underline{m}} c_{\underline{m}} \mathcal{L} \begin{pmatrix} 0 & m_1 q & \dots & m_k q^m \\ d & d(q-1) & \dots & d(q-1) q^{m-1} \end{pmatrix}
\end{array}$$

7. IDENTITIES FOR MULTIPLE ZETA VALUES AND MULTIPLE POLYLOGARITHMS AT ONE

In this section we discuss applications of our results to identities (at the finite level) for *multiple polylogarithms at one* and multiple zeta values of Thakur.

7.1. Multiple polylogarithms at one. We review notations, tools, definitions, and essential properties of multiple polylogarithms at one. We call an r -tuple $\mathbf{n} = (n_1, \dots, n_r)$ with $r \geq 1$ and with positive integers as coefficients an *array*. The integer r is called the *depth* of \mathbf{n} . We also assume by convention that there exists a unique array \emptyset of depth zero. Given an array \mathbf{n} of depth $r > 0$ as above, its *weight* is the positive integer $\sum_i n_i$. The weight of \emptyset is zero by definition. Consider an array \mathbf{n} of depth $r > 0$. We write $\mathbf{n}_1 = (n_1)$ and $\mathbf{n}_- = (n_2, \dots, n_r)$, an array of depth $r-1$. If $r=1$ we set $\mathbf{n}_- := \emptyset$. To avoid complications arrays with depth one will be identified with positive integers. The *concatenation* \mathbf{mn} of two arrays $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{n} = (n_1, \dots, n_s)$ is the array $\mathbf{mn} = (m_1, \dots, m_r, n_1, \dots, n_s)$, of depth the sum of the depths r of \mathbf{m} and s of \mathbf{n} . To avoid ambiguities and in order not to confuse it with the classical product of \mathbb{N}^* , we will sometimes denote by \cdot the concatenation operation between arrays. Define, inductively, with n a positive integer and \mathbf{n} an array of depth ≥ 1 ,

$$L_i(n) := l_i^{-n}, \quad i \geq 0, \quad L_i(n) := 0, \quad i < 0,$$

$$L_{<i}(n) := \sum_{j=0}^{i-1} L_j(n), \quad i \geq 0,$$

$$L_i(\mathbf{n}) := L_i(\mathbf{n}_1) L_{<i}(\mathbf{n}_-), \quad i \geq 0,$$

$$L_{<i}(\mathbf{n}) := \sum_{j=0}^{i-1} L_j(\mathbf{n}), \quad i \geq 0.$$

We have thus associated, to each array \mathbf{n} of depth > 0 , sequences $(L_i(\mathbf{n}))_{i \geq 0}$ and $(L_{<i}(\mathbf{n}))_{i \geq 0}$ in K . The *multiple polylogarithm at one* associated to \mathbf{n} is the well defined element of K_∞

given by

$$\text{Li}(\mathbf{n}) := \lim_{i \rightarrow \infty} L_{<i}(\mathbf{n}) = \sum_{i=0}^{\infty} L_i(\mathbf{n}).$$

We extend this formalism to handle linear combinations with coefficients in a field F containing \mathbb{F}_p . Denote by $\mathfrak{H}(F)$ the F -vector space generated by formal linear combinations

$$\sum_i h_i \mathbf{m}_i, \quad h_i \in F, \quad \mathbf{m}_i \text{ array.}$$

A basis of $\mathfrak{H}(F)$ is therefore given by the set $\{\mathbf{n} : \mathbf{n} \text{ array}\}$, which contains the empty array. Letters in fraktur font are exclusively used to designate elements of $\mathfrak{H}(F)$. Extending the concatenation F -bilinearly endows $\mathfrak{H}(F)$ with the structure of a unital noncommutative F -algebra. The unit is represented by the unique empty array that we denote by 1 . If F is a field extension of K , we have well defined F -linear maps

$$L_i, L_{<i} : \mathfrak{H}(F) \rightarrow F, \quad \text{Li} : \mathfrak{H}(K) \rightarrow K_{\infty}.$$

By a variant of Hoffman's Theorem on quasi-shuffle algebras [20] (see also [8, Theorem 1.150]) we can construct a quasi-shuffle product $*$ on $\mathfrak{H}(K)$ associated with the following *diamond operation* \diamond : $m \diamond n = m + n$ with $m, n \in \mathbb{N}^*$ (it is extended F -linearly to the F -linear span of arrays of depth zero and one). This defines a structure $(\mathfrak{H}(K), +, *)$ of unital K -algebra over $\mathfrak{H}(K)$ such that for all i and arrays \mathbf{m}, \mathbf{n} ,

$$L_{<i}(\mathbf{m} * \mathbf{n}) = L_{<i}(\mathbf{m})L_{<i}(\mathbf{n}), \quad \text{Li}(\mathbf{m} * \mathbf{n}) = \text{Li}(\mathbf{m})\text{Li}(\mathbf{n}).$$

The product, often called *shuffle product*, is defined, inductively on depths, by

$$1 * \mathbf{m} = \mathbf{m} * 1 = \mathbf{m}, \quad \mathbf{m} * \mathbf{n} = (\mathbf{m})_1(\mathbf{m}_- * \mathbf{n}) + (\mathbf{n})_1(\mathbf{m} * \mathbf{n}_-) + ((\mathbf{m})_1 \diamond (\mathbf{n})_1)(\mathbf{m}_- * \mathbf{n}_-).$$

The diamond operation \diamond itself can be extended to a product \diamond over $\mathfrak{H}(K)$ in such a way that, for all $i \geq 0$ and \mathbf{m}, \mathbf{n} arrays,

$$L_i(\mathbf{m} \diamond \mathbf{n}) = L_i(\mathbf{m})L_i(\mathbf{n}) \quad \text{Li}(\mathbf{m} \diamond \mathbf{n}) = \sum_{i \geq 0} L_i(\mathbf{m})L_i(\mathbf{n}).$$

Explicitly, one defines, inductively on depths,

$$\mathbf{m} \diamond 1 = 1 \diamond \mathbf{m} = \mathbf{m}, \quad \mathbf{m} \diamond \mathbf{n} = ((\mathbf{m})_1 \diamond (\mathbf{n})_1)(\mathbf{m}_- * \mathbf{n}_-).$$

7.2. Identities for multiple polylogarithms at one. Let us choose $d \geq 1$. Let us set:

$$(7.1) \quad \mu_{i,j} := f_d(\theta, \theta^{q^{i+1}}, \theta^{q^j}) l_i^{-d(q-1)}.$$

To begin with, we are going to show that, for all j ,

$$\mu_{i,j} = L_i(\mathfrak{k}_j), \quad i \geq 0$$

for explicitly computable elements \mathfrak{k}_j of $\mathfrak{H}(K)$, independent of $i \geq 0$. This is essential for the proof of the main result of this section, Theorem 7.6. We have:

Proposition 7.1. *For all $j \geq 1$ and for all $i \geq 0$ we have*

$$\mu_{i,j} = \sum_{k=0}^{d-1} \binom{d}{k} (\theta - \theta^q)^k f_{d-k}(\theta, \theta^q, \theta^{q^j}) L_i(\mathbf{c}_k),$$

where

$$(7.2) \quad \mathbf{c}_k = ((d-k)(q-1), (q-1)^{*k}).$$

In (7.2), \mathbf{c}_k is the concatenation of the array $(d-k)(q-1)$ (depth one) and the array $(q-1)^{*k}$ is the k -th power of the array of depth one $(q-1)$ for the multiplication $*$ (see Remark 7.3). Hence

$$\mathbf{c}_j = \sum_{k=0}^{d-1} \binom{d}{k} (\theta - \theta^q)^k f_{d-k}(\theta, \theta^q, \theta^{q^j}) \mathbf{c}_k.$$

Note that the arrays \mathbf{c}_k do not depend on j . One of the conventions of our paper is that empty products are equal to one. In conformity with this we set $\mathbf{m}^{*0} = 1$ the empty array, for each array \mathbf{m} . Therefore, if $k=0$, $\mathbf{c}_k = d(q-1)$ (depth one).

The proof of Proposition 7.1 relies on the next elementary result.

Lemma 7.2. *For indeterminates x, y, z, t and for all $d \geq 0$, we have the formula*

$$f_d(x, y, z) = \sum_{k=0}^{d-1} \binom{d}{k} (t-y)^k f_{d-k}(x, t, y).$$

Proof. From the definition:

$$\begin{aligned} f_d(x, y, z) &= \frac{(z-y)^d - (x-y)^d}{z-x} \\ &= \frac{(z-t+t-y)^d - (x-t+t-y)^d}{z-x} \\ &= \sum_{k=0}^{d-1} \binom{d}{k} (t-y)^k \frac{(z-t)^{d-k} - (x-t)^{d-k}}{z-x} \\ &= \sum_{k=0}^{d-1} \binom{d}{k} (t-y)^k f_{d-k}(x, t, z). \end{aligned}$$

□

In particular, we have the following formula:

$$(7.3) \quad f_d(x, y, z) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} \frac{x^{d-i} - z^{d-i}}{x-z} y^i.$$

Proof of Proposition 7.1. We apply the formula of Lemma 7.2 with $x = \theta, y = \theta^{q^{i+1}}, z = \theta^{q^j}, t = \theta^q$ to the definition of $\mu_{i,j}$:

$$\begin{aligned} \mu_{i,j} &= f_d(\theta, \theta^{q^{i+1}}, \theta^{q^j}) l_i^{-d(q-1)} \\ &= \sum_{k=0}^{d-1} \binom{d}{k} f_{d-k}(\theta, \theta^q, \theta^{q^j}) (\theta - \theta^{q^i})^{qk} l_i^{-d(q-1)}. \end{aligned}$$

Observe that

$$\begin{aligned} (\theta - \theta^{q^i})^{qk} l_i^{-d(q-1)} &= (\theta - \theta^{q^i})^{qk} l_i^{-(d-k)(q-1)-kq+k} \\ &= l_i^{-(d-k)(q-1)} \left(\frac{l_i}{l_{i-1}^q} \right)^k. \end{aligned}$$

One deduces, from Theorem 4.9 (case $d = 1, m = 1$) the formula

$$(7.4) \quad \frac{l_i}{l_{i-1}^q} = (\theta - \theta^q) L_{<i}(q-1), \quad i \geq 0.$$

Hence

$$l_i^{-(d-k)(q-1)} \left(\frac{l_i}{l_{i-1}^q} \right)^k = (\theta - \theta^q)^k L_i(\mathbf{c}_k)$$

which proves the formula of our statement. □

In the case $j = 1$ the formula of Proposition 7.1 is simpler:

$$(7.5) \quad \mu_{i,1} = (\theta - \theta^q)^{d-1} \sum_{k=0}^{d-1} \binom{d}{k} L_i(\mathbf{c}_k).$$

This is due to the fact that $f_s(x, z, z) = (x - z)^{s-1}$ for all s .

Remark 7.3. Given a positive integer n , the powers $(n)^{*k} \in \mathfrak{H}(\mathbb{F}_p)$ for the stuffle product can be computed explicitly by means of elementary combinatorics involving multinomial coefficients (recall our convention in the case $k = 0$). We have the formula (with multinomial coefficients):

$$(7.6) \quad (n)^{*k} = \sum_{r \geq 1} \sum_{(n_1, \dots, n_r) \in (\mathbb{N}^*)^r} \binom{k}{n_1, \dots, n_r} (nn_1, \dots, nn_r) \in \mathfrak{H}(\mathbb{F}_p),$$

where the second sum runs over the arrays $(n_1, \dots, n_r) \in (\mathbb{N}^*)^r$ such that $n_1 + \dots + n_r = k$.

Remark 7.4. Consider the K -linear map $\mathfrak{H}(K) \xrightarrow{\text{Li}} K_\infty$ which sends an array \mathbf{n} to $\text{Li}(\mathbf{n}) \in K_\infty$. Let w be a positive integer. If $\mathfrak{H}(K)_w$ denotes the K -span in $\mathfrak{H}(K)$ of the arrays of weight w , the image of the restriction Li_w of Li on $\mathfrak{H}(K)_w$ has the basis $(\text{Li}_w(\mathbf{n}))_{\mathbf{n}}$, where $\mathbf{n} = (n_1, \dots, n_r) \in \mathfrak{H}(K)_w$ is such that $q \nmid n_i$ for all $1 \leq i \leq r$. This is one of the main results in both papers [10, 21]. Let us denote by $\mathcal{B}_w^{\text{ND}}$ the set of the above arrays ⁽⁸⁾. From

⁸The exponent ND stands for *Ngo Dac* because the first occurrences of these important sets of arrays, later identified with bases of spaces of multiple zeta values, goes back to his paper [28].

(7.2) we see that for all $k < d$, $\mathbf{c}_k \in \mathfrak{H}(K)_{d(q-1)}$. Suppose that $d < q$. Then, for every $k \leq d$, every array occurring in the expansion of $\mathbf{c}_k \in \mathfrak{H}(K)_{d(q-1)}$ for $0 \leq k \leq d-1$ belongs to $\mathcal{B}_{d(q-1)}^{\text{ND}}$. In this case, the relations of Proposition 7.1 are uniquely determined by the parameters.

We are now in condition to prove our Theorem 7.6. Consider two arrays \mathbf{m} and \mathbf{n} in $\mathfrak{H}(K)$. Following [22], we introduce the *triangle product*

$$\mathbf{m} \triangleright \mathbf{n} := \mathbf{m}_1(\mathbf{m}_- * \mathbf{n}),$$

extended in the obvious way in the case when one of the arrays has depth zero. This product can be used to compress the expression of the stuffle product:

$$\mathbf{m} * \mathbf{n} = \mathbf{m} \diamond \mathbf{n} + \mathbf{m} \triangleright \mathbf{n} + \mathbf{n} \triangleright \mathbf{m}.$$

By [30, Lemma 2.1], this is an associative operation. It is fundamentally involved in the construction of the co-product of an Hopf algebra, see [22, Theorem B]. For us the main reason for which this operation is important is given by the next elementary result.

Proposition 7.5. *Consider arrays $\mathbf{k}_1, \dots, \mathbf{k}_m$ with positive depths. Then for all $k \geq 0$,*

$$L_{<k}(\mathbf{k}_1 \triangleright \mathbf{k}_2 \triangleright \dots \triangleright \mathbf{k}_{m-1} \triangleright \mathbf{k}_m) = \sum_{k > i_1 > \dots > i_m \geq 0} L_{i_1}(\mathbf{k}_1) \cdots L_{i_m}(\mathbf{k}_m).$$

Proof. We proceed by induction on $m \geq 1$. If $m = 1$ the property is obvious. Suppose that for $\mathbf{k}' = \mathbf{k}_2 \triangleright \dots \triangleright \mathbf{k}_m$ we have, for all i_1 ,

$$L_{<i_1}(\mathbf{k}') = \sum_{i_1 > i_2 > \dots > i_m \geq 0} L_{i_2}(\mathbf{k}_2) \cdots L_{i_m}(\mathbf{k}_m).$$

Then

$$\begin{aligned} \sum_{i_1=0}^{k-1} L_{i_1}(\mathbf{k}_1) \sum_{i_1 > i_2 > \dots > i_m \geq 0} L_{i_2}(\mathbf{k}_2) \cdots L_{i_m}(\mathbf{k}_m) &= \sum_{i_1=0}^{k-1} L_{i_1}(\mathbf{k}_1) L_{<i_1}(\mathbf{k}') \\ &= L_{<k}((\mathbf{k}_1)_1 \cdot ((\mathbf{k}_1)_- * \mathbf{k}')). \end{aligned}$$

□

Define:

$$(7.7) \quad \alpha_{k,j} = \binom{d}{k} (\theta - \theta^q)^k f_{d-k}(\theta, \theta^q, \theta^{q^j}) \in K^\times, \quad 0 \leq k < d, \quad j \geq 1$$

(it is easy to see that these elements of K are all non-zero).

Theorem 7.6. *For all $k \geq m \geq 1$ the following formula holds*

$$\begin{aligned} L_{k-m}(dq^m) &= \\ &= l_{k-m}^{-dq^m} = (-1)^m D_m \sum_{h_1=0}^{d-1} \cdots \sum_{h_m=0}^{d-1} \alpha_{h_1,m} \alpha_{h_2,m-1}^q \cdots \alpha_{h_r,1}^{q^{m-1}} L_k \left((d) \triangleright \mathbf{c}_{h_1} \triangleright \mathbf{c}_{h_2}^{*q} \triangleright \dots \triangleright \mathbf{c}_{h_m}^{*q^{m-1}} \right), \end{aligned}$$

where the coefficients $\alpha_{i,j}$ have been defined in (7.7) and the arrays \mathbf{c}_i have been introduced in Proposition 7.1.

Decomposing as a linear combination of arrays $(d) \triangleright \mathbf{c}_{h_1} \triangleright \mathbf{c}_{h_2}^{*q} \triangleright \cdots \triangleright \mathbf{c}_{h_m}^{*q^{m-1}} = \sum_i c_i \mathfrak{k}_i$, we have, for all i ,

$$\mathfrak{k}_i = (d, x_1(q-1), \dots, x_r(q-1))$$

where (x_1, \dots, x_r) is an array of weight $d \frac{q^m-1}{q-1}$ hence the relations of Theorem 7.6 are non-trivial. The problem of the computation of p -powers of arrays (for the stuffle product) is studied in [30, §2.3]. In their Proposition 2.7 the authors of this reference prove that, given an array $\mathbf{m} = (m_1, \dots, m_r)$, $\mathbf{m}^{*p^k} = (m_1 p^k, \dots, m_r p^k)$.

Proof of Theorem 7.6. First of all, note that, in the definition (5.9) of $\lambda_{<k}(m)$, we can collect the common factor

$$(7.8) \quad [m-1]^{-q(d-1)} [m-2]^{-q^2(d-1)} \cdots [1]^{-q^{m-1}(d-1)} = D_{m-1}^{-q(d-1)}.$$

Hence we can write, taking into account the definition (7.1),

$$(7.9) \quad \lambda_{<k}(m) = D_{m-1}^{-q(d-1)} \sum_{k > i_1 > \cdots > i_m} \mu_{i_1, m} \mu_{i_2, m-1}^q \cdots \mu_{i_m, 1}^{q^{m-1}}.$$

By Proposition 7.1 we can write

$$(7.10) \quad \mu_{i,j} = \sum_{h=0}^{d-1} \alpha_{h,j} L_i(\mathbf{c}_h),$$

where \mathbf{c}_k is defined in (7.2). Hence we have:

$$\begin{aligned} \lambda_{<k}(m) &= (D_{m-1}^q)^{-(d-1)} \sum_{h_1=0}^{d-1} \cdots \sum_{h_m=0}^{d-1} \alpha_{h_1, m} \alpha_{h_2, m-1}^q \cdots \alpha_{h_r, 1}^{q^{m-1}} \times \\ &\quad \times \sum_{k > i_1 > \cdots > i_m \geq 0} L_{i_1}(\mathbf{c}_{h_1}) L_{i_2}(\mathbf{c}_{h_2}^{*q}) \cdots L_{i_m}(\mathbf{c}_{h_m}^{*q^{m-1}}). \end{aligned}$$

Applying Proposition 7.5 we come to the next assertion, recalling (7.7). We have a decomposition:

$$\lambda_{<k}(m) = (D_{m-1}^q)^{-(d-1)} \sum_{h_1=0}^{d-1} \cdots \sum_{h_m=0}^{d-1} \alpha_{h_1, m} \alpha_{h_2, m-1}^q \cdots \alpha_{h_r, 1}^{q^{m-1}} L_{<k}(\mathbf{c}_{h_1} \triangleright \mathbf{c}_{h_2}^{*q} \triangleright \cdots \triangleright \mathbf{c}_{h_m}^{*q^{m-1}}).$$

By Theorem 5.1 we also have $\lambda_{<k}(m) = (-1)^m l_k^d [m]^{d-1} D_m^{-d} l_{k-m}^{-dq^m}$. Therefore

$$\begin{aligned} L_{k-m}(dq^m) &= l_{k-m}^{-dq^m} = \\ &= (-1)^m D_m \sum_{h_1=0}^{d-1} \cdots \sum_{h_m=0}^{d-1} \alpha_{h_1, m} \alpha_{h_2, m-1}^q \cdots \alpha_{h_r, 1}^{q^{m-1}} L_k \left((d) \triangleright \mathbf{c}_{h_1} \triangleright \mathbf{c}_{h_2}^{*q} \triangleright \cdots \triangleright \mathbf{c}_{h_m}^{*q^{m-1}} \right). \end{aligned}$$

This finishes the proof. \square

Remark 7.7. In Theorem 7.6, the explicit expansion of

$$\mathfrak{k} := (d) \triangleright \mathfrak{c}_{h_1} \triangleright \mathfrak{c}_{h_2}^{*q} \triangleright \cdots \triangleright \mathfrak{c}_{h_m}^{*q^{m-1}}$$

(depending on h_1, \dots, h_m) in the basis $\mathcal{B}_{dq^m}^{\text{ND}}$ can be reached inductively by iterated application of our formulas in a fashion similar to [10, 22], but we are unable to directly observe a closed formula.

7.3. Multiple power sums. Our results are stated for sequences of multiple sums of the type $L_i(\mathbf{m})$ or $L_{<i}(\mathbf{n})$, but after [28], their K -span, which is an \mathbb{F}_p -algebra, equals the K -span of sequences of multiple power sums. This is rooted in the notion of multiple zeta values of Thakur (see for example [29, 40]). Given $(n_1, \dots, n_r) \in (\mathbb{N}^*)^r$ the associated multiple zeta value of Thakur is:

$$\zeta_A(n_1, \dots, n_r) := \sum_{i_1 > \cdots > i_r \geq 0} \sum_{\substack{a_1, \dots, a_r \in A \\ \text{monic} \\ \deg_\theta(a_j) = i_j, \\ j=1, \dots, r}} \frac{1}{a_1^{n_1} \cdots a_r^{n_r}} \in K_\infty, \quad n_1, \dots, n_r \in \mathbb{N}^*.$$

Define, inductively, with n a positive integer and \mathbf{n} an array of depth ≥ 1 ,

$$S_i(n) := \sum_{\substack{a \in A \\ \text{monic} \\ \deg_\theta(a) = i}} a^{-n}, \quad i \geq 0, \quad S_i(n) = 0, \quad i < 0,$$

$$S_{<i}(n) := \sum_{j=0}^{i-1} S_j(n), \quad i \geq 0,$$

$$S_i(\mathbf{n}) := S_i(\mathbf{n}_1) S_{<i}(\mathbf{n}_-), \quad i \geq 0,$$

$$S_{<i}(\mathbf{n}) := \sum_{j=0}^{i-1} S_j(\mathbf{n}), \quad i \geq 0.$$

The above are called *multiple power sums*. They define sequences in K and satisfy properties similar to those of $L_i(\mathbf{n})$ and $L_{<i}(\mathbf{n})$. In particular we can extend S_i and $S_{<i}$ to linear maps $\mathfrak{H}(F) \rightarrow F$ (with F a field containing \mathbb{F}_p) and there is a unital algebra structure $(\mathfrak{H}(\mathbb{F}_p), +, \sqcup)$ such that, over $\mathfrak{H}(K)$ and for all \mathbf{m}, \mathbf{n} arrays,

$$S_{<i}(\mathbf{m} \sqcup \mathbf{n}) = S_{<i}(\mathbf{m}) S_{<i}(\mathbf{n}), \quad i \geq 1.$$

There also is a product governing the power sums S_i . We may denote it by \odot to distinguish it from \diamond . We have

$$S_i(\mathbf{m} \odot \mathbf{n}) = S_i(\mathbf{m}) S_i(\mathbf{n}), \quad i \geq 0, \quad \mathbf{m}, \mathbf{n} \text{ arrays},$$

see [22, Theorem A]. By using Ngo Dac's [28, Theorem A], Theorem 7.6 implies families of non-trivial linear relations between multiple power sums and ultimately, between Thakur's

multiple zeta values. Indeed:

$$\zeta_A(\mathbf{n}) = \lim_{i \rightarrow \infty} S_{<i}(\mathbf{n}) = \sum_{i=0}^{\infty} S_i(\mathbf{n}) \in K_{\infty}$$

for every array \mathbf{n} of depth ≥ 1 .

7.4. The case $m = 1$ and $d \leq q$ in Theorem 7.6. This can be handled in a slightly different way and leads to explicit identities as we explain here. Observe that by definition,

$$\mu_{k,1} = \frac{(\theta - \theta^{q^{k+1}})^d - (\theta^q - \theta^{q^{k+1}})^d}{\theta - \theta^q} l_k^d l_k^{-dq}.$$

Hence we can write:

$$\mu_{k,1} = \frac{1}{\theta - \theta^q} \left(\left(\frac{l_{k+1}}{l_k^q} \right)^d - \left(\frac{l_k}{l_{k-1}^q} \right)^d \right).$$

By (7.4) we can therefore write:

$$(7.11) \quad \mu_{k,1} = (\theta - \theta^q)^{d-1} \left(L_{<k+1}(q-1)^d - L_{<k}(q-1)^d \right),$$

which implies

$$(7.12) \quad \sum_{i < k} \mu_{i,1} = (\theta - \theta^q)^{d-1} L_{<k}(q-1)^d.$$

So far, we have not used the condition on d . A formula of Thakur [39, §3.2.2] implies that

$$(q-1)^{\sqcup d} = d(q-1), \quad d \leq q.$$

Coming back to (7.12), note that $L_{<i}(q-1) = S_{<i}(q-1)$. This ensures that

$$\sum_{i=0}^{k-1} \mu_{i,1} = (\theta - \theta^q)^{d-1} S_{<k}(d(q-1)).$$

Again by the fact that $d \leq q$, we have that $l_k^{-d} = S_k(d)$. Coming back to (7.9), with $m = 1$ and $d \leq q$, by Theorem 5.1:

$$l_k^{-d} \lambda_{<k}(1) = (\theta - \theta^q)^{-1} S_{k-1}(dq).$$

We have reached Thakur's formula (see [39, Theorem 5])

$$S_{k-1}(dq) = (\theta - \theta^q)^d S_k(d, d(q-1)), \quad k \geq 0, \quad 1 \leq d \leq q.$$

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